Epistemic Logic with Questions

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Questions as a part of inferential structures

Inferential Erotetic Logic (A. Wiśniewski, based on classical logic)

Evocation

$$\langle \Gamma, Q \rangle$$

Erotetic implication

$$\langle \langle Q_1, \Gamma \rangle, Q_2 \rangle$$

Example of e-implication

Q_1: What is Peter graduate of: faculty of law or faculty of economy?

I can be satisfied by the answer

He is a lawyer.

even if I did not ask

Q_2: What is Peter: lawyer or economist?

The connection between Q_1 and Q_2 could be done by the following knowledge base Γ:

Someone is graduate of a faculty of law iff he/she is a lawyer.
Someone is graduate of a faculty of economy iff he/she is an economist.

One-agent propositional epistemic logic

propositional language with modality K (knowledge as "necessity") and M ($M\varphi \equiv \neg K \neg \varphi$)

semantics

- **Kripke frame** $\mathcal{F} = \langle S, R \rangle$ with a set of states (points, indices, possible worlds) S and an accessibility relation $R \subseteq S^2$.

- **Kripke model** $\mathrm{M} = \langle \mathcal{F}, \models \rangle$ where $\models$ is a satisfaction relation between states and formulas.

The satisfaction relation $\models$ is defined by a standard way:

1. For each $\varphi \in \mathcal{A}$ and $(\mathbf{M}, s)$: either $(\mathbf{M}, s) \models \varphi$ or $(\mathbf{M}, s) \not\models \varphi$.

2. $(\mathbf{M}, s) \models \neg\varphi$ iff $(\mathbf{M}, s) \not\models \varphi$

3. $(\mathbf{M}, s) \models \psi_1 \vee \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ or $(\mathbf{M}, s) \models \psi_2$

4. $(\mathbf{M}, s) \models \psi_1 \wedge \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ and $(\mathbf{M}, s) \models \psi_2$

5. $(\mathbf{M}, s) \models \psi_1 \rightarrow \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ implies $(\mathbf{M}, s) \models \psi_2$

6. $(\mathbf{M}, s) \models K\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$, for each s_1 such that sRs_1

Incorporating questions

extend epistemic language by ? and appropriate brackets

$$Q = ?\underbrace{\{\alpha_1, \ldots, \alpha_n\}}_{dQ}$$

Q requires one of the following answers:

It is the case that α_1.
$\vdots$
It is the case that α_n.

A questioner presupposes at least $(\alpha_1 \vee \ldots \vee \alpha_n)$ and maybe more.

Presuppositions

presupposition of a question Q

$\varphi \in \mathsf{Pres}Q$ iff $(\forall \mathbf{M})(\forall s)(\forall \alpha \in dQ)((\mathbf{M}, s) \models \alpha \rightarrow \varphi)$

prospective presupposition of a question Q

$\varphi \in \mathsf{PPres}Q$ iff $\varphi \in \mathsf{Pres}Q$ and $(\forall \mathbf{M})(\forall s)$

$(\mathbf{M}, s) \models \varphi$ implies $(\exists \alpha \in dQ)((\mathbf{M}, s) \models \alpha)$

- Each prospective presupposition is a maximal presupposition.
- If $\varphi, \psi \in \mathsf{PPres}Q$, then $\varphi \equiv \psi$.

Q is *sound* at $(\mathbf{M}, s)$

$$(\mathbf{M}, s) \models Q$$

iff

1. $(\forall \alpha \in dQ)$

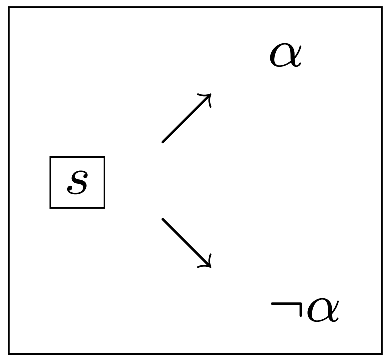
   (a) $(\mathbf{M}, s) \models M\alpha$

   (b) $(\mathbf{M}, s) \not\models K\alpha$

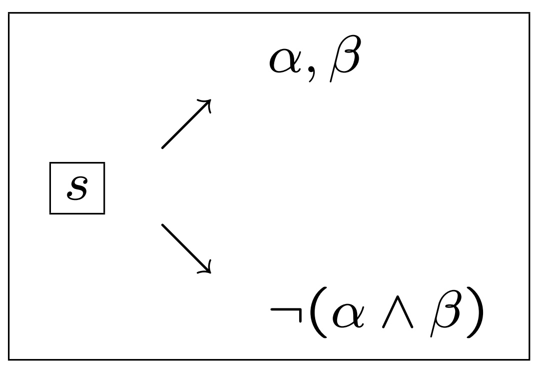
2. $(\forall \varphi \in \mathsf{Pres}Q)((\mathbf{M}, s) \models K\varphi)$

A question sound at $(\mathbf{M}, s)$ forms a partitioning on the afterset.

Examples 1

- $(\mathbf{M}, s) \models\ ?\alpha$ means 

  The same is for $(\mathbf{M}, s) \models\ ?\neg\alpha$.

- $(\mathbf{M}, s) \models\ ?(\alpha \wedge \beta)$ 

  Analogously for $?(\alpha \vee \beta)$.

Examples 2

- $?|\alpha, \beta|$ is equal to $?\{(\alpha \wedge \beta), (\neg\alpha \wedge \beta), (\alpha \wedge \neg\beta), (\neg\alpha \wedge \neg\beta)\}$.

$$\begin{tikzcd}
 & \alpha, \beta \\
\boxed{s} \arrow[ur] \arrow[r] \arrow[dr] \arrow[d] & \alpha, \neg\beta \\
\neg\alpha, \neg\beta & \neg\alpha, \beta
\end{tikzcd}$$

- $(\mathbf{M}, s) \models ?\{\alpha, \beta\}$, then $(\mathbf{M}, s) \models K(\alpha \vee \beta)$

$$\begin{tikzcd}
 & \alpha, \neg\beta, (\alpha \vee \beta) \\
\boxed{s} \arrow[ur] \arrow[dr] & \\
 & \neg\alpha, \beta, (\alpha \vee \beta)
\end{tikzcd}$$

Evocation

$$(\mathbf{M}, s) \models \Gamma \xrightarrow{i} Q$$

iff

$$(\mathbf{M}, s) \models K\Gamma$$

and

$$(\mathbf{M}, s) \models Q$$

coincides with *question in an information set* (J. Groenendijk, M. Stokhof)

E-implication

$$(\mathbf{M}, s) \models (\Gamma, Q_1) \Rightarrow Q_2$$

iff

$$((\mathbf{M}, s) \models K\Gamma \text{ and } (\mathbf{M}, s) \models Q_1)$$

implies

$$(\mathbf{M}, s) \models Q_2$$

Pure e-implication ($\Gamma = \emptyset$)

$$(\mathbf{M}, s) \models Q_1 \rightarrow Q_2$$

iff

$$(\mathbf{M}, s) \models Q_1 \text{ implies } (\mathbf{M}, s) \models Q_2$$

Examples of pure e-implication

- $\models ?\alpha \rightarrow ?\neg\alpha$ as well as $\models ?\alpha \leftarrow ?\neg\alpha$
- $\models ?(\alpha\wedge\beta) \leftarrow ?|\alpha,\beta|$, the same for $\vee$ instead of $\wedge$
- $\models ?|\alpha,\beta| \rightarrow ?\alpha$ and $\models ?|\alpha,\beta| \rightarrow ?\beta$
- $\models ?|\alpha,\beta| \rightarrow ?(\alpha|\beta)$
- $\models ?|\alpha,\beta| \rightarrow ?\{\alpha,\beta,(\neg\alpha\wedge\neg\beta)\}$
- $\models ?\{(\alpha\vee\beta),\alpha\} \rightarrow ?\{\alpha,\beta\}$
- $\models ?\{\alpha,\beta,\gamma\} \rightarrow ?\alpha$ (as well as $?\beta$ and $?\gamma$)

Example of e-implication

$\Gamma = \{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}$

$$\models (\Gamma,\ ?\{\beta_1, \beta_2\}) \Rightarrow ?\{\alpha_1, \alpha_2\}$$

as well as

$$\models (\Gamma,\ ?\{\alpha_1, \alpha_2\}) \Rightarrow ?\{\beta_1, \beta_2\}$$

both questions are equal with respect to Γ

Answerhood

An agent *gets* a **complete answer** φ to a question Q at $(\mathbf{M}, s)$ iff $(\mathbf{M}, s) \models K\varphi$ such that $\varphi \models \alpha$ for some $\alpha \in dQ$.

An agent *gets* a **partial answer** to a question Q at $(\mathbf{M}, s)$ iff she gets a complete answer to a question $?\varphi$ at $(\mathbf{M}, s)$ such that $Q \rightarrow ?\varphi$.

α *is* a partial answer to $?|\alpha, \beta|$

$$\begin{tikzcd}
 & \alpha, \beta \\
\boxed{s} \arrow[ur] \arrow[r] \arrow[dr] \arrow[d] & \alpha, \neg\beta \\
{\color{red}\neg\alpha, \neg\beta} & {\color{red}\neg\alpha, \beta}
\end{tikzcd}$$

Basic references

D. Harrah. The logic of questions. In D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*. Kluwer, 2002. Volume 8, pages 1–60.

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J. Groenendijk and M. Stokhof. Questions. In J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*. Elsevier, 1997. Pages 1055–1125.