Relevance logics and intuitionistic negation

TERNARY RELATIONAL SEMANTICS

STANDARD → RELEVANCE LOGICS

WITHOUT A SET OF DESIGNATED POINTS → NON-RELEVANT LOGICS

CONSTRUCTIVE NEGATION

Ternary relational semantics:

(1) $\boldsymbol{a \vDash \neg A \text{ iff } (Rabc \ \& \ c \in S) \Rightarrow b \nvDash A}$

(A formula of the form) $\neg A$ is true in point *a* iff *A* is false in all points *b* such that *Rabc* for all consistent points *c*.

(2) $\boldsymbol{a \vDash \neg A \text{ iff } Rabc \Rightarrow b \nvDash A}$

(A formula of the form) $\neg A$ is true in point *a* iff *A* is false in all points *b* such that *Rabc* for all points *c*.

Binary relational semantics:

(3) $\boldsymbol{a \vDash \neg A \text{ iff } (Rab \ \& \ b \in S) \Rightarrow b \nvDash A}$

(A formula of the form) $\neg A$ is true in point *a* iff *A* is false in all accessible consistent points. (Minimal intuitionistic clause).

(4) $\boldsymbol{a \vDash \neg A \text{ iff } Rab \Rightarrow b \nvDash A}$

(A formula of the form) $\neg A$ is true in point *a* iff *A* is false in all accessible points. (Intuitionistic clause).

D¬. $\neg A \leftrightarrow (A \rightarrow F)$

(*F* is a propositional falsity constant)

CONCEPTS OF CONSISTENCY

Let L be a logic and *a* an L-theory (a set of formulas closed under adjunction and provable entailment):

1. *a* is w-inconsistent1 iff $\neg B \in a$, B being a theorem of L.
2. *a* is w-inconsistent2 iff $B \in a$, $\neg B$ being a theorem of L.
3. *a* is negation-inconsistent iff $A \wedge \neg A \in a$, for some wff A.
4. *a* is absolutely inconsistent iff *a* contains every wff.

*(*a* is consistent iff *a* is not inconsistent).

PARADOXES

PARADOXES OF RELEVANCE:

Characteristic exemplars:

(i) $A \to (B \to A)$ (K axiom)

(ii) If $\vdash A$, then $\vdash B \to A$ (K rule)

PARADOXES OF CONSISTENCY

Characteristic exemplars:

(iii) $(A \wedge \neg A) \to B$ (ECQ axiom)

(iv) $\neg A \to (A \to B)$ (EFQ axioms)

(v) $A \to (\neg A \to B)$

THE BORDERLINES OF RELEVANCE LOGICS

EXAMPLES:

- Paradoxical, non-relevance logic **R-mingle** (Anderson et al.).
- Logic **KR** (R_+ plus a De Morgan negation together with the ECQ axiom) (Meyer and Routley).
- **CR** (R plus a Boolean negation), **CE** (E plus a Boolean negation) (Routley, Meyer and others).

OUR RESEARCH:

- R_+ and some of its extensions plus a constructive intuitionistic-type negation.

MINIMAL INTUITIONISTIC NEGATION / INTUITIONISTIC NEGATION

MINIMAL INTUITIONISTIC LOGIC:

J_+ plus:

(i) $(A \to B) \to (\neg B \to \neg A)$

(ii) $A \to \neg\neg A$

(iii) $(A \to \neg A) \to \neg A$

(iv) $\neg A \to (A \to \neg B)$

INTUITIONISTIC LOGIC:

J_+ plus (i)-(iii) and:

(v) $\neg A \to (A \to B)$

MINIMAL INTUITIONISTIC NEGATION:

S_+ plus (i)-(iv) (S_+ is a positive logic)

INTUITIONISTIC NEGATION:

S_+ plus (i)-(iii) and (v) (S_+ is a positive logic)

CHARACTERISTICS OF THE LOGICS INTRODUCED

- All of them are included in minimal or in full intuitionistic logic.
- None of them is included in Lewis' modal logic S5.
- None of them is included in R-mingle.
- They are not included in KR or CR.

[(iv) $\neg A \rightarrow (A \rightarrow \neg B)$ is a theorem of B_{jm} (Routley and Meyer's B_+ plus minimal intuitionistic negation)].

- They provide an unexplored perspective on the borderlines between relevance and non-relevance logics.
- The K rule :

  If $\vdash A$, then $\vdash B \rightarrow A$

  and so, the K axiom :

  $A \rightarrow (B \rightarrow A)$

  are not provable in any of them.
- They have paradoxes of consistency but they do not have paradoxes of relevance, in general.
- They are an interesting class of subintuitionistic logics with intuitionistic negation but without the K axiom characteristic of intuitionistic logic or the K rule characteristic of some modal logics.

THE LOGIC B_{jm}

B_+ :

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B)$ / $B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C)]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules of derivation:

Modus ponens: if $\vdash A$ and $\vdash A \to B$, then $\vdash B$

Adjunction: if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$

Suffixing: if $\vdash A \to B$, then $\vdash (B \to C) \to (A \to C)$

Prefixing: if $\vdash B \to C$, then $\vdash (A \to B) \to (A \to C)$

B_{jm}:

We add to the sentential language of B_+ the propositional falsity constant F together with the definition:

$\neg A =_{df} A \to F$

B_{jm} is axiomatized by adding to B_+ the following axioms:

A7. $[A \to (B \to F)] \to [B \to (A \to F)]$

A8. $(B \to F) \to [(A \to B) \to (A \to F)]$

A9. $[A \to [A \to (B \to F)]] \to [A \to (B \to F)]$

A10. $F \to (A \to F)$

THEOREMS OF B_{jm}:

T1. $[(A \vee B) \to F] \leftrightarrow [(A \to F) \wedge (B \to F)]$
$\neg (A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

T2. $[(A \to F) \vee (B \to F)] \to [(A \wedge B) \to F]$
$(\neg A \vee \neg B) \to \neg (A \wedge B)$

T3. $F \to F$
$\neg F$

T4. $\boldsymbol{A \to [(A \to F) \to F]}$
$\boldsymbol{A \to \neg\neg A}$

T5. $\boldsymbol{(A \to B) \to [(B \to F) \to (A \to F)]}$
$\boldsymbol{(A \to B) \to \neg B \to \neg A}$

T6. $\boldsymbol{B \to [[A \to (B \to F)] \to (A \to F)]}$
$\boldsymbol{B \to [(A \to \neg B) \to \neg A]}$

T7. $A \to [[A \to (B \to F)] \to (B \to F)]$
$A \to [(A \to \neg B) \to \neg B]$

T8. $(A \to F) \to [A \to (B \to F)]$
$\neg A \to (A \to \neg B)$

T9. $A \to [(A \to F) \to (B \to F)]$
$A \to (\neg A \to \neg B)$

T10. $A \to (F \to F)$
$A \to \neg F$

T11. $(B \to F) \to [A \to (B \to F)]$
$\neg B \to (A \to \neg B)$

T12. $B \to [(A \to F) \to (A \to F)]$
$B \to (\neg A \to \neg A)$

T13. $\boldsymbol{[A \to (A \to F)] \to (A \to F)}$
$\boldsymbol{(A \to \neg A) \to \neg A}$

T14. $\boldsymbol{[A \to (B \to F)] \to [(A \to B) \to (A \to F)]}$
$\boldsymbol{(A \to \neg B) \to [(A \to B) \to \neg A]}$

T15. $\boldsymbol{(A \to B) \to [[A \to (B \to F)] \to (A \to F)]}$
$\boldsymbol{(A \to B) \to [(A \to \neg B) \to \neg A]}$

T16. $\boldsymbol{[A \wedge (A \to F)] \to F}$
$\boldsymbol{\neg(A \wedge \neg A)}$

T17. $\boldsymbol{[A \wedge (A \to F)] \to (B \to F)}$
$\boldsymbol{(A \wedge \neg A) \to \neg B}$

T18. $(A \vee B) \to [[(A \to F) \wedge (B \to F)] \to F]$
$(A \vee B) \to \neg (\neg A \wedge \neg B)$

T19. $(A \wedge B) \to [[(A \to F) \vee (B \to F)] \to F]$
$(A \wedge B) \to \neg (\neg A \vee \neg B)$

T20. $[A \vee (B \to F)] \to [(A \to F) \to (B \to F)]$
$(A \vee \neg B) \to (\neg A \to \neg B)$

T21. $[(A \to F) \vee (B \to F)] \to [(A \to (B \to F)]$
$(\neg A \vee \neg B) \to (A \to \neg B)$

T22. $(A \to B) \to [[(A \wedge (B \to F)] \to F]$
$(A \to B) \to \neg (A \wedge \neg B)$

T23. $(A \wedge B) \to [[(A \to (B \to F)] \to F]$
$(A \wedge B) \to \neg (A \to \neg B)$

T24. $[[(A \to F) \to F)] \to F] \to [(A \to F) \to F)]$
$\neg\neg\neg A \to \neg\neg A$

T25. $[[A \vee (A \to F)] \to F] \to F$
$\neg\neg (A \vee \neg A)$

B_{jm} MODELS

A B_{jm} *model* is a quintuple $< K, O, S, R, \models >$ where K is a set, **O and S are subsets of K such that $O \cap S \neq \varnothing$** and R is a ternary relation on K subject to the following definitions and conditions for all $a, b, c, d \in K$:

d1. $a \leq b =_{\text{df}} (\exists x \in O)\, Rxab$

d2. $R^2abcd =_{\text{df}} (\exists x \in K)\, [Rabx \,\&\, Rxcd]$

d3. $R^3abcde =_{\text{df}} (\exists x \in K)\, (\exists y \in K)\, [Rabx \,\&\, Rxcy \,\&\, Ryde]$

P1. $a \leq a$

P2. $(a \leq b \,\&\, Rbcd) \Rightarrow Racd$

P3. $\boldsymbol{(R^2abcd \,\&\, d \in S) \Rightarrow (\exists x \in S)\, R^2acbx}$

P4. $\boldsymbol{(R^2abcd \,\&\, d \in S) \Rightarrow (\exists x \in S)\, R^2bcax}$

P5. $\boldsymbol{(a \in S) \Rightarrow (\exists x \in S)\, Raax}$

P6. $\boldsymbol{(Rabc \,\&\, c \in S) \Rightarrow (a \in S \,\&\, b \in S)}$

$\models$ is a valuation relation from K to the sentences of B_{jm} satisfying the following conditions for all propositional variables p, wffs A, B and $a \in K$

(i) $(a \models p \,\&\, a \leq b) \Rightarrow b \models p$

(ii) $a \models A \vee B$ iff $a \models A$ or $a \models B$

(iii) $a \models A \wedge B$ iff $a \models A$ and $a \models B$

(iv) $a \models A \rightarrow B$ iff for all $b, c \in K$ $(Rabc \,\&\, b \models A) \Rightarrow c \models B$

(v) $\boldsymbol{a \models F}$ **iff** $\boldsymbol{a \notin S}$

A formula is *valid* ($\boldsymbol{\models_{B_{jm}} A}$) iff $\boldsymbol{a \models A}$ for all $\boldsymbol{a \in O}$ in all $\boldsymbol{B_{jm}}$ models.

B_{jm} CANONICAL MODEL:

The B_{jm} canonical model is the structure

$$<K^C, O^C, S^C, R^C, \models^C>$$

(Let K^T be the set of all theories) R^T = for all formulas A, B and a, b, $c \in K^T$, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$.

K^C = the set of all prime **non-null** theories
O^C = the set of all prime regular theories
S^C = the set of all prime non-null consistent theories.
R^C = the restriction of R^T to K^C
$\models^C$ = for any wff A and $a \in K^C$, $a \models^C A$ iff $A \in a$.

(A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all theorems of B_{jm} belong to a; a is *null* iff no wff belong to a. Finally, **a theory *a* is *inconsistent* iff $F \in a$**).

Proposition: Let $a \in K^T$, a is inconsistent ($F \in a$) iff $B \in a$ ($\neg B$ being a theorem) iff $\neg C \in a$ (C being a theorem) iff $B \wedge \neg B \in a$ (B is a wff).

THE LOGIC B_j

We add to the sentential language of B_+ the propositional falsity constant F together with the definition:

$\neg A =_{df} A \to F$

B_j is axiomatized by adding to B_+ the following axioms:

A7. $[A \to (B \to F)] \to [B \to (A \to F)]$

A8. $(B \to F) \to [(A \to B) \to (A \to F)]$

A9. $[A \to [A \to (B \to F)]] \to [A \to (B \to F)]$

A10. $\boldsymbol{F \to A}$

THEOREMS OF B_j:

T26. $\boldsymbol{(A \to F) \to (A \to B)}$

$\boldsymbol{\neg A \to (A \to B)}$

T27. $\boldsymbol{A \to [(A \to F) \to B]}$

$\boldsymbol{A \to (\neg A \to B)}$

T28. $\boldsymbol{[A \wedge (A \to F)] \to B}$

$\boldsymbol{(A \wedge \neg A) \to B}$

T29. $A \to [B \to [(A \to F) \to F]]$

$A \to (B \to \neg\neg A)$

T30. $(A \vee B) \to [(A \to F) \to [(B \to F) \to F]]$

$(A \vee B) \to (\neg A \to \neg\neg B)$

T31. $[(A \to F) \vee B] \to [A \to [(B \to F) \to F]]$

$(\neg A \vee B) \to (A \to \neg\neg B)$

B_j MODELS

A B_j *model* is a quadruple **<*K*, *O*, *R*, ⊨ >** where *K* is a non-empty set, *O* is a subset of *K* and *R* and ⊨ are defined (similarly) as in B_{jm} models, except that clause (v) is now substituted for:

(v'). $a \nvDash F$ **for all** $a \in K$

A is valid ($\vDash_{Bj} A$) iff $a \vDash A$ for all $A \in O$ in all B_j models.

B_j CANONICAL MODEL

The canonical model is the quadruple $<K^C, O^C, R^C, \vDash^C>$ where **K^C is the set of all non-null consistent prime theories**, and O^C, R^C and $\vDash^C$ are defined as in the B_{jm} canonical model, its items now being referred to B_j theories.

Proposition: Let $a \in K^T$, a is inconsistent ($F \in a$) iff $B \in a$ ($\neg B$ being a theorem) iff $\neg C \in a$ (C being a theorem) iff $B \wedge \neg B \in a$ (B is a wff) iff a contains every well formed formula.

EXTENSIONS OF B_{jm} AND B_j

AXIOMS:

A12. $(B \to C) \to [(A \to B) \to (A \to C)]$
A13. $(A \to B) \to [(B \to C) \to (A \to C)]$
A14. $[A \to (A \to B)] \to (A \to B)$
A15. If $\vdash A$, then $\vdash (A \to B) \to B$
A16. $A \to [(A \to B) \to B]$
A17. $A \to (A \to A)$

- TW_+ ("Contractionless positive Ticket Entailment") = B_+ plus A12 & A13
- T_+ ("Positive Ticket Entailment") = TW_+ plus A14.
- E_+ ("Positive Entailment Logic") = T_+ plus A15.
- R_+ = E_+ plus A16.
- RMO_+ = R_+ plus A17.

POSTULATES:

PA12. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \,\&\, Raxd)$
PA13. $R^2abcd \Rightarrow (\exists x \in K)(Racx \,\&\, Rbxd)$
PA14. $Rabc \Rightarrow R^2abbc$
PA15. $(\exists x \in O)\, Raxa$
PA16. $Rabc \Rightarrow Rbac$
PA17. $Rabc \Rightarrow (a \leq b$ or $b \leq c)$

EXTENSIONS OF B_{jm} AND B_j

MATRICES:

The K rule (and therefore, the K axiom) is not derivable in B_j plus A12-A17:

$\to$	0	1	2	3
0	3	3	3	3
1	0	1	2	3
2	0	0	2	3
3	0	0	0	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	0	2	2
3	0	0	0	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

- Designated values: 1, 2, 3
- *F* is assigned the value 0
- This set of matrices satisfies the axioms of B_j and A12-A17 and falsifies K when $v(A) = 1$ and $v(B) = 2$