General reducibilities for sets of reals

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Definition (W.W.Wadge, 1972)

A is (continuously) reducible to B just in case there is a continuous function f such that

$$x \in A \iff f(x) \in B$$

for every real x.

Reducibilities for sets of reals

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- **Selfdual degrees**: $[A]_{\mathcal{F}}$ such that $A \leq_{\mathcal{F}} \neg A$
- **Nonselfdual pairs**: $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ such that $A \nleq_{\mathcal{F}} \neg A$

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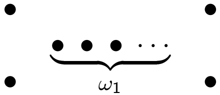

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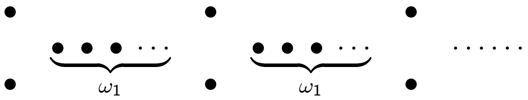

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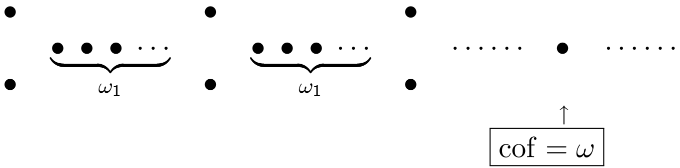

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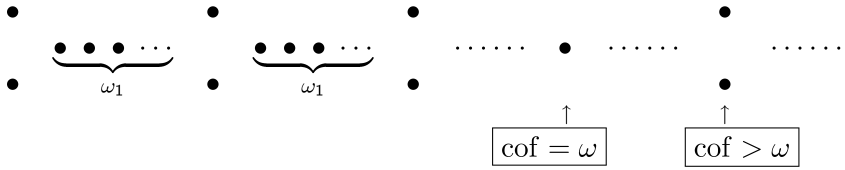

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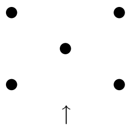

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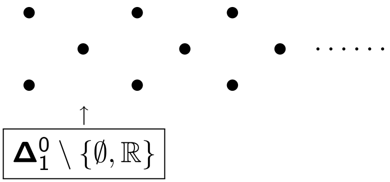

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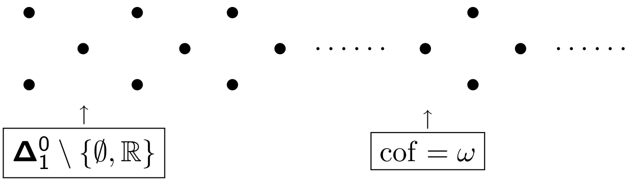

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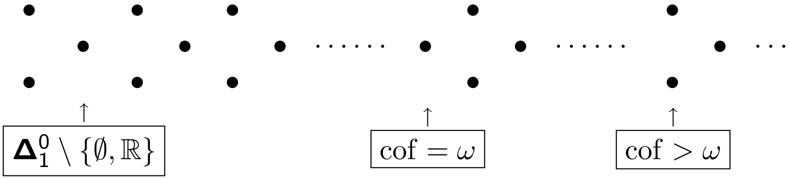

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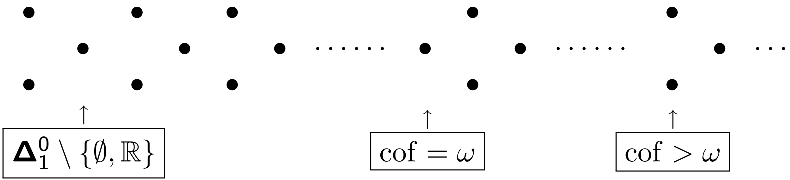

The length of the Wadge hierarchy (and of the Lipschitz one) is exactly $\Theta = \sup\{\alpha \mid \exists f (f : \mathbb{R} \twoheadrightarrow \alpha)\}$.

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Problem 1: Can we determine the degree-structure of any "reasonable" $\mathcal{F}$?

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Given any set of reductions $\mathcal{F}$, we can define its *characteristic set*

$$\Delta_{\mathcal{F}} = \left\{ A \subseteq \mathbb{R} \mid A \leq_{\mathcal{F}} \mathbf{N}_{\langle 0 \rangle} \right\}.$$

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A set of reductions $\mathrm{Lip} \subseteq \mathcal{F} \subseteq \mathrm{Bor}$ is *Borel-amenable* if

$$f = \bigcup_n (f_n \restriction D_n) \in \mathcal{F}$$

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A set of reductions $\mathrm{Lip} \subseteq \mathcal{F} \subseteq \mathrm{Bor}$ is *Borel-amenable* if

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A set of reductions $\mathcal{F}$ has the *decomposition property* (**DP**) if for every $A \leq_{\mathcal{F}} \neg A \notin \Delta_{\mathcal{F}}$ there is a countable $\Delta_{\mathcal{F}}$-partition $\langle D_n \mid n \in \omega \rangle$ of $\mathbb{R}$ such that $A \cap D_n <_{\mathcal{F}} A$ for every n.

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Assume AD. *If $\mathcal{F}$ is Borel-amenable and has the* **DP** *then both after a selfdual degree and at limit levels of uncountable cofinality there is a nonselfdual pair.*

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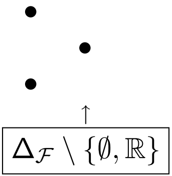

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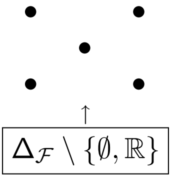

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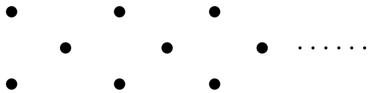

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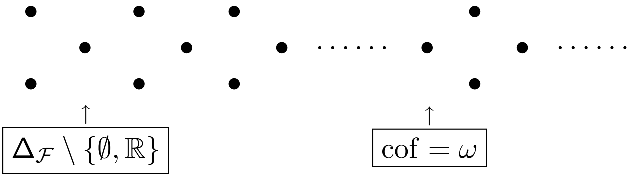

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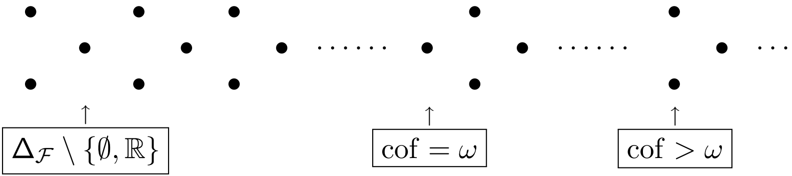

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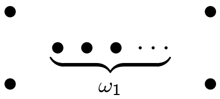

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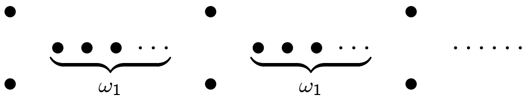

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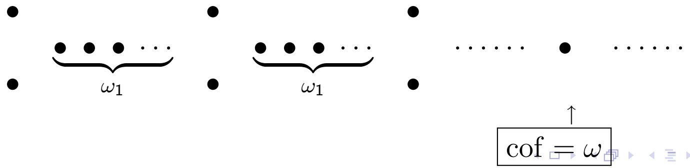

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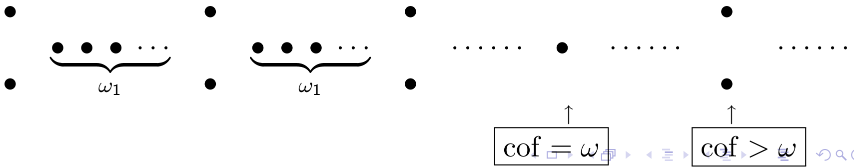

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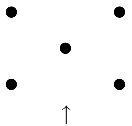

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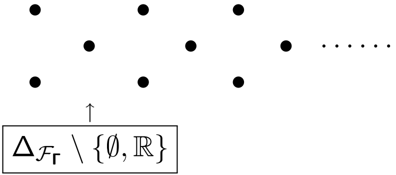

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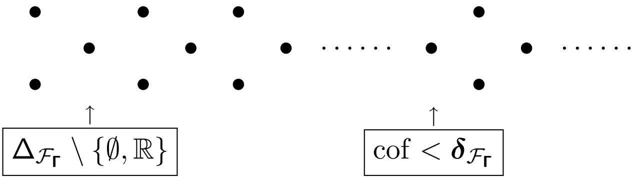

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