On Cardinals in Set Theory without Choice and Regularity

Denis I. Saveliev

18 July 2007, Wrocław

I shall discuss cardinals in set theory minus Choice and Regularity. There are (at least) two reasons to do this:

The first reason is to understand (in reverse mathematics style) which axioms are really necessary to develop certain parts of set theory.

The second reason is to check that we add various axioms "without loss of generality", in the sense which let me explain via an example:

Example. "All sets are constructible"

$$V = L$$

is a nice axiom because it looks "empirically complete", but we don't believe it because it "too restricts possible objects". More precisely: because it *restricts the interpretability strength*. May be so is "All sets are well-founded"

$$V = \bigcup_{\alpha} V_{\alpha}$$

or not?

Both reasons lead to such an investigation.

In this talk, I'll speak only about problems concerning singularity of cardinals.

Let me start with the basic definition.

Definition. Given a set X, its *cardinal* number

$$|X|$$

is the class of all sets of *the same size* that X, i.e., admitting a one-to-one map onto X.

Thus

$$|X| = |Y|$$

means "There is a bijection of X onto Y".

Cardinals of nonempty sets are proper classes; so, we have a little technical obstacle:

How quantify cardinals?

In some happy cases we can represent them by sets:

If $|X|$ is a *well-ordered cardinal*, i.e., meets the class of (von Neumann's) ordinals, we can identify it with the least such ordinal (an *initial* ordinal).

If $|X|$ is a *well-founded cardinal*, i.e., meets the class of well-founded sets, we can identify it with the lower level of the intersection (so-called Scott's trick).

What is in general? The answer is

No matter

because instead of cardinals, we can say about sets and bijections.

Notations:

The German letters

$$\mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \ldots$$

denote arbitrary cardinals. The Greek letters

$$\lambda, \mu, \nu, \ldots$$

denote well-ordered ones (i.e., initial ordinals). The first Greek letters

$$\alpha, \beta, \gamma, \ldots$$

denote ordinals.

There are two basic relations on cardinals, which are *dual* in a sense:

$$|X| \leq |Y|$$

means "X is empty or there is an injection of X into Y", while

$$|X| \leq^* |Y|$$

means "X is empty or there is an surjection of Y onto X".

Equivalently:

$|X| \leq |Y|$ means "There is a subset of Y of size $|X|$", and

$|X| \leq^* |Y|$ means "X is empty or there is a partition of Y into $|X|$ pieces".

Clearly:

Both relations are reflexive and transitive.

The first is antisymmetric (Dedekind, independently Bernstein), the second is not in general.

The relations coincide on well-ordered cardinals.

Two important functions on cardinals (Hartogs and Lindenbaum resp.):

$$\aleph(\mathfrak{n}) = \{\alpha : |\alpha| \leq \mathfrak{n}\},$$

$$\aleph^*(\mathfrak{n}) = \{\alpha : |\alpha| \leq^* \mathfrak{n}\}.$$

Equivalently,

$\aleph(\mathfrak{n})$ is the least α such that on a set of size $\mathfrak{n}$ there is no well-ordering of length α.

$\aleph^*(\mathfrak{n})$ is the least α such that on a set of size $\mathfrak{n}$ there is no pre-well-ordering of length α.

Customarily, ν^+ denotes $\aleph(\nu)$ for ν well-ordered.

Clearly:

$\aleph(\mathfrak{n})$ and $\aleph^*(\mathfrak{n})$ are well-ordered cardinals.

$\aleph(\mathfrak{n}) \not\leq \mathfrak{n}$ and $\aleph^*(\mathfrak{n}) \not\leq^* \mathfrak{n}$.

$\aleph(\mathfrak{n}) \leq \aleph^*(\mathfrak{n})$, and both operations coincide on well-ordered cardinals. On other cardinals, the gap can be very large:

Example. Assume AD. Then $\aleph(2^{\aleph_0}) = \aleph_1$ while $\aleph^*(2^{\aleph_0})$ is a very large cardinal (customarily denoted Θ).

Notations:

$$\mathrm{Cov}(\mathfrak{l}, \mathfrak{m}, \mathfrak{n})$$

means "A set of size $\mathfrak{n}$ can be covered by $\mathfrak{m}$ sets of size $\mathfrak{l}$".

$\mathrm{Cov}({<}\,\mathfrak{l}, \mathfrak{m}, \mathfrak{n})$ and $\mathrm{Cov}(\mathfrak{L}, \mathfrak{m}, \mathfrak{n})$ (where $\mathfrak{L}$ is a class of cardinals) have the appropriate meanings.

Definition. A cardinal $\mathfrak{n}$ is *singular* iff $\mathrm{Cov}({<}\,\mathfrak{n}, {<}\,\mathfrak{n}, \mathfrak{n})$, and *regular* otherwise.

Fact. *Assume* AC. *Then* $\mathrm{Cov}(\mathfrak{l}, \mathfrak{m}, \mathfrak{n})$ *implies* $\mathfrak{n} \leq \mathfrak{l} \cdot \mathfrak{m}$.

Corollary. *Assume* AC. *Then all the successor alephs are regular.*

What happens without AC?

Theorem (Feferman and Lévy). *$\aleph_1$ can be singular.*

Moreover, under a large cardinal hypothesis, so can be all uncountable alephs:

Theorem (Gitik). *All uncountable alephs can be singular.*

It clearly follows $\mathrm{Cov}(<\aleph_\alpha, \aleph_0, \aleph_\alpha)$ for all α simultaneously.

Remark. What is the consistency strength?

Without successive singular alephs:
The same as of ZFC.

With λ, λ^+ both singular:
Between 1 Woodin cardinal (Schindler improving Mitchell) and ω Woodin cardinals (Martin Steel Woodin).

So, in general case:
A proper class of Woodins.

Specker posed the following problem:

Is $\mathrm{Cov}(\aleph_\alpha, \aleph_0, 2^{\aleph_\alpha})$ consistent for all α simultaneously?

Partial answer:

Theorem (Apter Gitik). *Let $A \subseteq Ord$ consist either*

(i) of all successor ordinals; or

(ii) of all limit ordinals and all successor ordinals of form $\alpha = 3n, 3n+1, \gamma+3n$, or $\gamma+3n+2$, where γ is a limit ordinal.

Then

$$(\forall \alpha \in A)\, \mathrm{Cov}(\aleph_\alpha, \aleph_0, 2^{\aleph_\alpha})$$

is consistent (modulo large cardinals).

In general, the problem remains open.

Question: How singular can cardinals be without AC? in the following sense:

How small are $\mathfrak{l} \le \mathfrak{n}$ satisfying $\mathrm{Cov}(<\mathfrak{l}, <\mathfrak{n}, \mathfrak{n})$? The answer for well-ordered $\mathfrak{n}$:

$$\mathfrak{l} < \mathfrak{n} \text{ is impossible}$$

Theorem 1. $\mathrm{Cov}(<\lambda, \mathfrak{m}, \nu)$ *implies* $\nu \le^* \lambda \cdot \mathfrak{m}$, *and so*

$$\nu^+ \le \aleph^*(\lambda \cdot \mathfrak{m}).$$

Corollary. $\neg\mathrm{Cov}(<\lambda, \lambda, \lambda^+)$ *for all* $\lambda \ge \aleph_0$.

Since $\mathrm{Cov}(\lambda, \lambda, \lambda^+)$ is consistent, the result is exact.

Remark. $\mathrm{Cov}(\aleph_0, \aleph_0, \aleph_2)$ is an old result of Jech. (I'm indebted to Andreas Blass who informed me.) By Corollary, really $\mathrm{Cov}(\aleph_0, \aleph_1, \aleph_2)$.

Next question: Let $\mathrm{Cov}(\mathfrak{l}, \mathfrak{m}, \mathfrak{n})$, is $\mathfrak{n}$ estimated via $\mathfrak{l}$ and $\mathfrak{m}$? (when $\mathfrak{n}$ is not well-ordered). The answer is

No

Even in the simplest case $\mathfrak{l} = 2$ and $\mathfrak{m} = \aleph_0$ such an estimation of $\mathfrak{n}$ is not provable:

Theorem 2. *It is consistent that for any $\mathfrak{p}$ there exists $\mathfrak{n} \not\leq \mathfrak{p}$ such that* $\mathrm{Cov}(2, \aleph_0, \mathfrak{n})$.

The proof uses a generalization of permutation model technique to the case of a proper class of atoms. We use non-well-founded sets instead of atoms. So, Regularity can influence on cardinals.

On the other hand, $\aleph(\mathfrak{n})$ and $\aleph^*(\mathfrak{n})$ are estimated via $\aleph(\mathfrak{l})$, $\aleph^*(\mathfrak{l})$, and $\aleph^*(\mathfrak{m})$:

Theorem 3.
$\mathrm{Cov}(\mathfrak{L}, \mathfrak{m}, \mathfrak{n})$ *implies*

$$\aleph(\mathfrak{n}) \leq \aleph^*(\sup_{\mathfrak{l}\in\mathfrak{L}} \aleph(\mathfrak{l}) \cdot \mathfrak{m})$$

and

$$\aleph^*(\mathfrak{n}) \leq \aleph^*(\sup_{\mathfrak{l}\in\mathfrak{L}} \aleph^*(\mathfrak{l}) \cdot \mathfrak{m}).$$

Corollary.
$\neg\mathrm{Cov}({<}\,\lambda, \lambda, 2^\lambda)$ *and* $\neg\mathrm{Cov}(\mathfrak{n}, 2^{\mathfrak{n}^2}, 2^{2^{\mathfrak{n}^{2\cdot 2}}})$.

In particular:

$\neg\mathrm{Cov}(\lambda, 2^\lambda, 2^{2^\lambda})$ and $\neg\mathrm{Cov}(\beth_\alpha, \beth_{\alpha+1}, \beth_{\alpha+2})$.

Since $\mathrm{Cov}(\mathfrak{n}, \mathfrak{n}, 2^{\mathfrak{n}})$ is consistent, the result is near optimal.

Another corollary is that Specker's request, even in a weaker form, gives the least possible evaluation of $\aleph^*(2^\lambda)$ (which is λ^{++}):

Corollary. $\mathrm{Cov}(\lambda, \lambda^+, 2^\lambda)$ *implies*

$$\aleph^*(2^\lambda) = \aleph(2^\lambda) = \lambda^{++}.$$

So, if there exists a model which gives the positive answer to Specker's problem, then in it, all the cardinals $\aleph^*(2^\lambda)$ have the least possible values.

Analogy:

$$\frac{\text{well-ordered}}{\text{non-well-ordered}} \approx \frac{\text{well-founded}}{\text{non-well-founded}}$$

Do not these results on well-ordered cardinals have some analogs for well-founded cardinals? The answer is

Conditionally Yes:

Instead of arbitrary coverings, we have to consider only coverings by extensional sets.

Let

$$\mathrm{Cov}_{\mathrm{ext}}(\mathfrak{l}, \mathfrak{m}, \mathfrak{n})$$

mean "A set of size $\mathfrak{n}$ can be covered by $\mathfrak{m}$ extensional sets of size $\mathfrak{l}$" and likewise for variants of this notation.

Let

$$\mathfrak{U}(\mathfrak{n}) \quad \text{and} \quad \mathfrak{U}^*(\mathfrak{n})$$

be the cardinals of the sets

$U(\mathfrak{n}) = \{S : S$ is a transitive well-founded set of size $< \mathfrak{n}\}$ and

$U^*(\mathfrak{n}) = \{S : S$ is a transitive well-founded set of size $<^* \mathfrak{n}\}$ resp.

They take the same place here as

$$\aleph(\mathfrak{n}) \quad \text{and} \quad \aleph^*(\mathfrak{n})$$

before.

An analog of Theorem 1 is the following result:

Theorem 4. *Let* $\mathfrak{n}$ *be well-founded. Then*

$$\mathrm{Cov}_{\mathrm{ext}}(<\mathfrak{p}, \mathfrak{m}, \mathfrak{n}) \ \textit{ implies } \ \mathfrak{n} \leq^* \mathfrak{U}(\mathfrak{p} \cdot \mathfrak{m})$$

and

$$\mathrm{Cov}_{\mathrm{ext}}(<^*\mathfrak{p}, \mathfrak{m}, \mathfrak{n}) \ \textit{ implies } \ \mathfrak{n} \leq^* \mathfrak{U}^*(\mathfrak{p} \cdot \mathfrak{m}).$$

As a corollary, we can get, e.g., an estimation of beths:

Corollary. $\mathrm{Cov}_{\mathrm{ext}}(<\beth_\alpha, \beth_\alpha, \beth_{\alpha+1})$.

Let me conclude with some problems.

Problem 1. *Is* $\neg\mathrm{Cov}(\mathfrak{n}, 2^{\mathfrak{n}}, 2^{2^{\mathfrak{n}}})$ *true for all* $\mathfrak{n}$*?*

As we known, that holds if $\mathfrak{n} = \mathfrak{n}^2$.

Problem 2. *Is* $\neg\mathrm{Cov}(< \beth_\alpha, \beth_\alpha, \beth_{\alpha+1})$ *true for all* α*?*

That holds for all successor α's while the weaker extensional version holds for all α.

Problem 3. *Is* $\mathrm{Cov}(\mathfrak{n}, \aleph_0, 2^{\mathfrak{n}^2})$ *consistent for all* $\mathfrak{n}$ *simultaneously?*

This sharps Specker's problem of course.

Problem 4. *Is it true that on successor alephs the cofinality can behave anyhow, in the following sense: Let F be any function such that*

$$F : SuccOrd \to SuccOrd \cup \{0\}$$

and F satisfies

$$\begin{array}{rl} \text{(i)} & F(\alpha) \leq \alpha \ \ \textit{and} \\ \text{(ii)} & F(F(\alpha)) = F(\alpha) \end{array}$$

for all successor α. Is it consistent

$$\mathrm{cf}\, \aleph_\alpha = \aleph_{F(\alpha)}$$

for all successor α?

Perhaps if F makes no successive cardinals singular, it is rather easy; otherwise very hard.

I said only about coverings and singular cardinals.

Really, without Choice and Regularity, one can develop a more or less powerful cardinal theory (partly by using these "positive" results) and recover, or almost recover, a part of standard cardinal arithmetic.

This is beyond my talk.

[1] Solomon Feferman and Azriel Lévy. *Independences results in set theory by Cohen's method, II.* Notices of the American Mathematical Society, 10 (1963), 593. Abstract.

[2] Moti Gitik. *All uncountable cardinals can be singular*. Israel Journal of Mathematics, 35, 1–2 (1980), 61–88.

[3] Moti Gitik. *Regular cardinals in models of* ZF. Transactions of the American Mathematical Society, 290, 1 (1985), 41–68.

[4] Ralf Dieter Schindler. *Successive weakly compact or singular cardinals*. Journal of Symbolic Logic, 64 (1999), 139–146.

[5] Ernst P. Specker. *Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom)*. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 3, 3 (1957), 173–210.

[6] Arthur W. Apter and Moti Gitik. *Some results on Specker's problem*. Pacific Journal of Mathematics, 134, 2 (1988), 227–249.