Thin Projective Equivalence Relations and Inner Models

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Definition. An equivalence relation $E \subseteq \omega^\omega \times \omega^\omega$ is called *thin* if there is no perfect set of pairwise inequivalent reals.

Question. How does an inner model look like, if for any thin projective equivalence relation, every equivalence class has a representative in the inner model?

Theorem. *(Hjorth 1993) Assume* $x^{\#}$ *exists for every* $x \in \omega^\omega$*. Then the following statements are equivalent for an inner model* M*:*

1. *For all thin* $\Pi^1_2(z)$ *equivalence relations with* $z \in M$*, every equivalence class has a representative in* M
2. $\omega_1^M = \omega_1^V$ *and* $M \prec_{\Sigma^1_3} V$

Theorem. *(Hjorth, Schindler, Schlicht 2006) Assume* $Det(\underset{\sim}{\Delta}^1_{2n})$ *holds and* $M^\dagger_{2n-2}(x)$ *exists for every* $x \in \omega^\omega$. *Then the following statements are equivalent for an inner model* M*:*

1. *For all thin* $\Pi^1_{2n}(z)$ *equivalence relations with* $z \in M$, *every equivalence class has a representative in* M
2. $T^M_{2n-1} = T^V_{2n-1}$ *and* $M \prec_{\Sigma^1_{2n+1}} V$

where T_{2n-1} *is the tree from a* Π^1_{2n-1} *scale.*

We prove that (2) implies (1). Assume that $n = 2$ and E is a thin Π^1_4 equivalence relation.

Suppose $x \in \omega^\omega$. We have to find $x' \in \omega^\omega \cap M$ with $(x, x') \in E$.

Since E is Π^1_4 , its complement is $\underset{\sim}{\delta}^1_3$-Suslin via a tree computed from T_3. A theorem of Harrington and Shelah proves that there is a formula $\varphi \in \mathcal{L}_{\infty,0} \cap L_\alpha[T_3]$ with

- $\varphi(x)$
- $\forall y \ (\varphi(y) \Rightarrow (x, y) \in E)$

where α is least such that $L_\alpha[T_3] \models KP$. The language $\mathcal{L}_{\infty,0}$ is built from atomic formulas $n \in x$ and $n \notin x$ by infinitary conjunctions and disjunctions, so that $\mathcal{L}_{\infty,0}$ formulas describe a real.

Since $\varphi \in L_\alpha[T_3]$, there is $y \in \omega^\omega \cap M$ such that φ is definable from T_3, y by a term t_φ in any transitive model of KP containing T_3 and y.

Idea of proof:

Try to write $\exists x \varphi(x)$ as a Σ^1_5 statement. For this purpose, reconstruct T_3 in an iterate of $M_2^\dagger(x, y)$, so that you can compute $\varphi = t_\varphi(y, T_3)$ in the iterate. Then you can express $\varphi(x)$ in $M_2^\dagger(x, y)$.

Here $M_2^\dagger(x, y)$ is the smallest $(\omega_1 + 1)$-iterable premouse built over (x, y) with 2 Woodin cardinals and a measurable cardinal above. Let $\gamma < \delta < \kappa$ such that $M_2^\dagger(x, y) \vDash \gamma, \delta$ are Woodin cardinals and κ is measurable.

Let $\overline{V}$ be countable with $x, y \in \overline{V}$ and $\pi : \overline{V} \to_{\Sigma_{100}} V$ elementary. Let $\overline{M} = \pi^{-1}{}''M$, $\overline{T_3} = \pi^{-1}(T_3)$, etc.

By forming Skolem hulls, in $\overline{M}$ we can construct substructures $X_0 \prec X_1 \prec \ldots \prec M_2^\dagger(x, y)$ and ordinals $\gamma_0 < \gamma_1 < \ldots < \gamma$, $\delta_0 \leq \delta_1 \leq \ldots \leq \delta$, $\kappa_0 \leq \kappa_1 \leq \ldots \leq \kappa$, with

1. $V_{\gamma_i}^{X_i} = V_{\gamma_i}^{M_2^\dagger(x,y)}$ for all $i \in \omega$
2. $X_i \models \gamma_i < \delta_i$ are both Woodin cardinals and $\kappa_i > \delta_i$ is measurable
3. $\sup_{i \in \omega} \gamma_i = \gamma$

Then each X_i is ω_1-iterable.

Let $\omega^\omega \cap \overline{V} = \{y_i : i \in \omega\}$. We can now iterate $M_2^\dagger(x, y) \longrightarrow N_0 \longrightarrow N_1 \longrightarrow ... \longrightarrow N_i \longrightarrow ...$ so that y_i is $Col(\omega, \pi_{0i}(\gamma_i))$-generic over $\pi_{0i}(X_i)$, by Woodin's genericity iteration. Let $\pi_{ij} : N_i \longrightarrow N_j$ denote the iteration maps. Let $N_\omega = dirlim_{i \rightarrow \omega} N_i$.

Then y_i is still $Col(\omega, \pi_{0i}(\gamma_i))$-generic over $\pi_{0j}(X_i)$ for all j with $i \leq j \leq \omega$.

Note that $\sup_{i \in \omega} \pi_{0i}(\gamma_i) = \omega_1^{\overline{V}}$. Let G be a $Col(\omega, < \omega_1^{\overline{V}})$-generic filter over N_ω in V such that $\omega^\omega \cap N_\omega[G] \subseteq \omega^\omega \cap \overline{V}$.

Claim. $T_3^{\overline{V}} = T_3^{N_\omega[G]}$

Proof. It is sufficient to prove that for any Π^1_3 rank and every $y_i \in \omega^\omega \cap \overline{V}$, there is $z \in \omega^\omega \cap N_\omega[G]$ of the same rank. Note that $N_\omega[G] \prec_{\Sigma^1_3} V$ since $N_\omega[G]$ has a Woodin cardinal and a measurable above it, and is iterable.

To prove this, fix y_i. Suppose G_i is $Col(\omega, \pi_{0i}(\gamma_i))$-generic over $\pi_{0\omega}(X_i)$ with $y_i \in \pi_{0\omega}(X_i)[G_i]$. Let $\dot{x}$ be a name with $\dot{x}^{G_i} = y_i$. Let $\dot{x}_0, \dot{x}_1$ be the corresponding names for left and right generic. We can now find a condition $p \in G_i$ such that $(p, p) \Vdash$"$\dot{x}_0$ and $\dot{x}_1$ have the same rank".

Let $H \in N_\omega[G]$ generic below p over $\pi_{0\omega}(X_i)$ and $z = \dot{x}^H$. Find H' generic below p over both $\pi_{0\omega}(X_i)[G_i]$ and $\pi_{0\omega}(X_i)[H]$. Since $\pi_{0\omega}(X_i)[G_i, H']$ and $\pi_{0\omega}(X_i)[H, H']$ are iterable and have a Woodin cardinal and a measurable above it, we get $N_\omega[G_i, H'] \prec_{\Sigma^1_3} V$ and $\pi_{0\omega}(X_i)[G_i, H'] \prec_{\Sigma^1_3} V$. So these models compute the rank correctly. Hence $y_i, z, \dot{x}^{H'}$ all have the same rank. $\square$

Since $Col(\omega, < \sup \gamma_i)$ is homogeneous, we now have in $\overline{V}$:

- there is $x \in \omega^\omega$ such that $\Vdash^{M_2^\dagger(x,y)}_{Col(\omega, < \sup \gamma_i)} t_\varphi(y, T_3)(x)$

Since $M_2^\dagger(x,y)$ is coded by a $\Pi^1_4(x,y)$ real, this is a Σ^1_5 statement. Hence this is true in $\overline{M}$, let $x' \in \omega^\omega \cap \overline{M} \subseteq M$ witness this. Since we can again iterate $M_2^\dagger(x,y)$ to some N'_ω to make the reals of $\overline{M}$ generic, we have

$$T_3^{{N'}_\omega^{Col(\omega, < \omega_1^{\overline{M}})}} = T_3^{\overline{M}} = T_3^{\overline{V}}$$

Since $\varphi = t_\varphi(y, T_3^{\overline{V}})$, then $\varphi(x')$ holds. Hence $(x, x') \in E$.

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