Definable Well-Ordering, the GCH, and Large Cardinals

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The existence of a definable (without parameters) well-ordering of the universe (in other words, V=HOD) and the GCH are nice properties of Gödel's constructible universe L.

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- McAloon has shown how to force V=HOD while preserving measurables, but he uses the continuum function as a coding oracle.

- Friedman has shown how one can preserve an n-superstrong or hyperstrong cardinal while forcing V=HOD + GCH, but the technique involves coding the universe into a subset of a greater cardinal, and so won't work for proper class preservation.

Our coding oracle

We want to get our well order by encoding information into whether or not some combinatorial principle holds at κ for a proper class of cardinals κ.

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But $\diamondsuit^*$ fits the bill too!

It is known that one can force $\diamondsuit^*_\lambda$ to hold or fail while preserving the GCH and without collapsing cardinals, for any successor cardinal λ. In fact, this can be done with a λ-closed, λ^+-cc forcing in each case.

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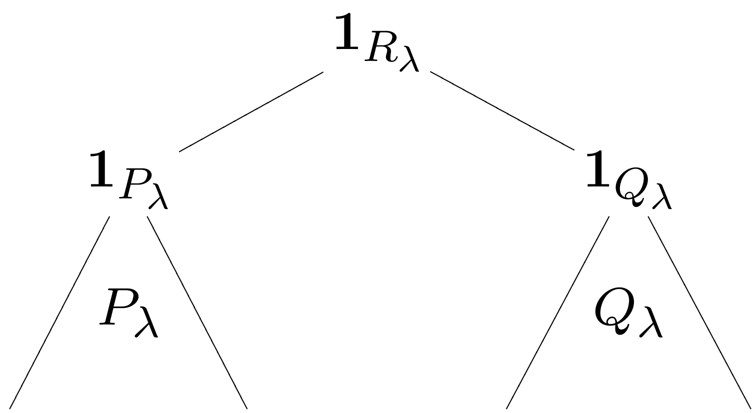
- The coding will automatically be very resilient — not having all of the information will not pose a problem. That is, the class of points at which we code need not be fully absolute.

For each successor cardinal λ, let P_λ be the forcing which produces a $\diamondsuit^*_\lambda$ sequence, let Q_λ be the forcing which makes $\diamondsuit^*_\lambda$ fail, and let R_λ be the sum of P_λ and Q_λ, that is, the partial order combining them below a new maximum element.



Theorem 1. *Suppose $V \models$ GCH, and let S be the reverse Easton iteration (that is, iteration with direct limits taken at regular stages and inverse limits elsewhere) which has as the iterand at stage α*

- *$\dot{R}_\alpha$ if α is a successor cardinal, and*
- *the trivial forcing otherwise.*

Then forcing with S yields a model of ZFC + GCH + V=HOD.

Proof (sketch). Since the tail of the forcing becomes successively more closed, every set X that is added appears by some stage.

Once X has appeared, it is dense for it to be encoded in the choices made between 1_{P_λ} and 1_{Q_λ} — for example encode the $\in$ relation on $\mathrm{trcl}(X)$ as a subset of $|\mathrm{trcl}(X)|$. $\dashv$

Preserving large cardinals

We want to preserve large cardinals from the ground model, by lifting the witnessing elementary embeddings. This is fairly standard, but generally requires us to choose generics containing a *master condition*. If we want to preserve many large cardinals, this becomes untenable.

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In other contexts, this issue can be resolved by making the forcing partial order homogeneous, so that the generic can be "twisted" to lie below the master condition "after the fact".

But our partial order is inherently inhomogeneous; and indeed, no homogeneous forcing can ever force $V = \text{HOD}$, as HOD of the extension will be contained in the ground model.

Instead, we arrange that the master conditions are trivial, by making the forcing iterands trivial.

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We only need to make the forcing trivial in a certain relevant range for each large cardinal. As long as

- the large cardinals are witnessed by boundedly many embeddings, and

- the class of large cardinals we want to preserve isn't too large (more or less, is non-stationary in ORD)

then there is still plenty of room in which to perform our coding.

So we can't necessarily preserve *all* cardinals of a given kind, but we can preserve a proper class of them, and there is much flexibility in which proper class.

For example:

Theorem 2. *Suppose that the GCH holds and there is a proper class of ω-superstrong cardinals, and let δ be an arbitrary ordinal. Then a definable well-order of the universe may be forced while preserving the GCH and all*

measurable,
η-strong for $\eta < \delta$,
Woodin,
n-superstrong for $n \in \omega + 1$,
hyperstrong,
$\kappa^{+\eta}$-supercompact for $\eta < \delta$,
η-extendible for $\eta < \delta$, and
m-huge for $m \in \omega$

cardinals that are not limits of ω-superstrong cardinals.