

The Urysohn sphere is oscillation stable

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Oscillation stability for Banach spaces

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Definition

Let X be an infinite dimensional Banach space.

X is **oscillation stable** when for every $f : \mathbb{S}_X \rightarrow [0, 1]$ uniformly continuous, every $\varepsilon > 0$, every $Y \subset X$ closed infinite dimensional, there $Z \subset Y$ closed infinite dimensional such that:

$$\text{osc}(f, Z) := \sup_{x, y \in \mathbb{S}_X \cap Z} |f(y) - f(x)| < \varepsilon.$$

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Theorem (Gowers, 91)

The space c_0 is oscillation stable.

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Theorem (Odell-Schlumprecht, 94)

The Hilbert space ℓ_2 is not oscillation stable.

Reformulation of the problem for ℓ_2

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Definition

Let X be a metric space.

X is *metrically oscillation stable* if for every $f : X \rightarrow [0, 1]$ uniformly continuous, $\varepsilon > 0$, there is \tilde{X} isometric to X such that:

$$\forall x, y \in \tilde{X}, \quad |f(y) - f(x)| < \varepsilon.$$

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Equivalently:

Let $X = A_1 \cup \dots \cup A_k$, $\varepsilon > 0$. There is \tilde{X} isometric to X and $i \leq k$ such that

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Theorem (Odell-Schlumprecht)

The unit sphere \mathbb{S}^∞ of ℓ_2 is not metrically oscillation stable.

Open question

Question

Is there a proof based on the intrinsic metric structure of \mathbb{S}^∞ ?

A good candidate for a better understanding

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Definition

Up to isometry, there is a unique metric space \mathbb{S} with distances in $[0, 1]$ which is:

- 1. Complete, separable.*
- 2. Ultrahomogeneous (every isometry between finite subsets of \mathbb{S} extends to an isometry of \mathbb{S} onto itself).*
- 3. Universal for the separable metric spaces with distances in $[0, 1]$.*

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Let $X = \mathbb{S}^\infty$ or \mathbb{S} , $f : X \rightarrow [0, 1]$ uniformly continuous, $\varepsilon > 0$,
 $K \subset X$ compact.
Then f ε -stabilizes on an isometric copy of K .

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- ▶ Higher dimensional Ramsey properties.
- ▶ Behaviour of $\text{iso}(\mathbb{S}^\infty)$ and $\text{iso}(\mathbb{S})$ as topological groups.

Main question

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Is \mathbb{S} metrically oscillation stable?

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Proposition

The space \mathbb{S} admits countable ultrahomogeneous dense subsets.

Question

Let $X \subset \mathbb{S}$ be countable dense ultrahomogeneous.

*Is X **indivisible**?*

ie: Let $X = A_1 \cup \dots \cup A_k$. Is there \tilde{X} isometric to X and $i \leq k$ such that

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Theorem (Delhommé-Laflamme-Pouzet-Sauer)

No.

Remark

Crucial point: The distance set of X is too rich.

Second attempt: Discretization

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Definition

Up to isometry, there is a unique metric space \mathbb{U}_m with distances in $\{1, \dots, m\}$ which is:

- 1. Countable.*
- 2. Ultrahomogeneous.*
- 3. Universal for the countable metric spaces with distances in $\{1, \dots, m\}$.*

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Theorem (López Abad - NVT)

TFAE

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Theorem (NVT - Sauer)

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Definition

Let $X \subset \mathbb{U}_m$, $f : X \longrightarrow \{1, \dots, m\}$.

f is *Katětov over X* when it defines a 1-point metric extension of X :

$$\forall x, y \in X, \quad |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$$

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Remark

If $\min f = p$, then $O(f, \mathbb{U}_m)$ is isometric to $\mathbb{U}_{\min(2p, m)}$.

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► *Elements of \mathbb{P} :*

Pairs $s = (f_s, C_s)$ where

1. $f_s : \text{dom}f_s \longrightarrow \{1, \dots, m\}$ *finite.*
2. $C_s \subset \mathbb{U}_m$ *isometric to \mathbb{U}_m .*
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► *Relations \leq and \leq_k :*

$$t \leq s \leftrightarrow (\text{dom}f_s \subset \text{dom}f_t \subset C_t \subset C_s \text{ and } f_t \upharpoonright \text{dom}f_s = f_s).$$

$$t \leq_k s \leftrightarrow (t \leq s \text{ and } \min f_t = \min f_s - k).$$

A notion of largeness

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Definition

Let $\Gamma \subset \mathbb{U}_m$, $s \in \mathbb{P}$.

Then Γ *is large relative to s* when:

- ▶ If $\min f_s = 1$:

$$\forall t \leq_0 s \quad (O(f_t, \mathbf{C}_t) \cap \Gamma \text{ is infinite}).$$

- ▶ If $\min f_s > 1$:

$$\forall t \leq_0 s \quad \exists u \leq_1 t \quad (\Gamma \text{ is large relative to } u).$$

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Lemma

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Lemma

Let $s \in \mathbb{P}$. Assume Γ is not large relative to s .
Then there is $t \leq_0 s$ such that $\mathbb{U}_m \setminus \Gamma$ is large relative to t .

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There is a unique Banach space H into which the Urysohn space \mathbb{U} embeds isometrically with dense linear span.

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