The constructible universe of ZFA

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WHY THE FOUNDATION AXIOM?

Axiom 1. $\in$ *is a well founded relation.*

Kunen's textbook justifies the axiom of foundation on the ground that it is an essential technical tool to develop a reasonable first order axiomatization of set theory.

There are many key properties of ZFC which rely on the axiom of foundation, two remarkable ones are:

(i) R is a well-founded relation is first order definable by a Δ_1-property in ZFC.

(ii) There is a cumulative hierarchy of the universe.

(i) follows from the fact that α is a Von Neumann ordinal is Σ_0-definable in ZFC by the formula:

α *is transitive and linearly ordered by* $\in$.

R is a well-founded relation on X can be defined by the Π_1-formula:

$\forall Y \subseteq X \, \exists z \in Y$ *which is minimal for* $R \restriction Y$

and by the Σ_1-formula:

There is a rank function from (X, R) *into the ordinals*

This is essential....

- ....to show that well-foundedness is absolute between transitive models $M \subseteq N$ of ZFC,

- ....to prove the existence and uniqueness of Mostowski's collapse (equivalent to foundation),

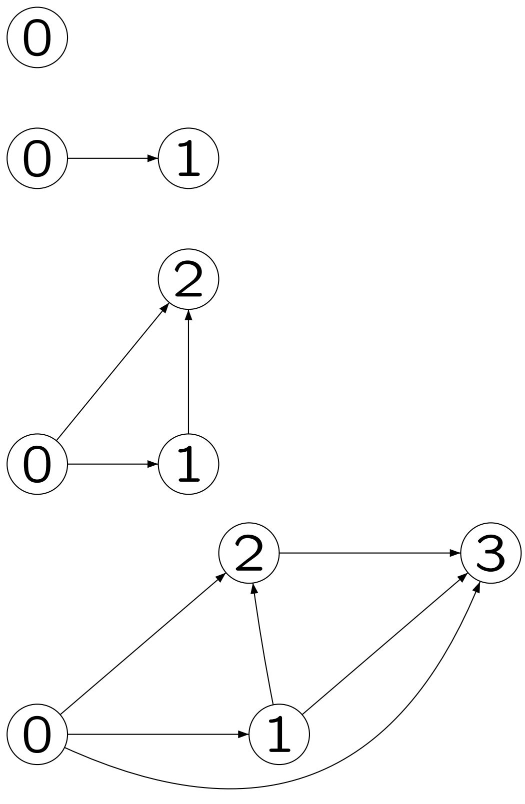
- ....to prove Shonfield Σ^1_2-absoluteness lemma,

- ....

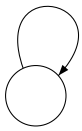
(ii) is useful for example to prove the reflection theorem.

GRAPH REPRESENTATION OF SETS

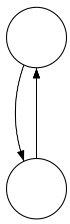
WELL FOUNDED SETS



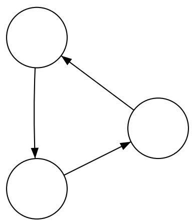
POSSIBLE GRAPHS OF ILL-FOUNDED SETS



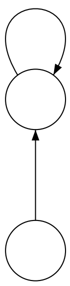
$x = \{x\}$



$x = \{y\}$ and $y = \{x\}$



$x = \{y\}$, $y = \{z\}$ and $z = \{x\}$



$x = \{0, x\}$

Antifoundation axioms try to enlarge the class of graphs which can be the realization of the transitive closure of a set.

The key problem is to get a reasonable criterion for equality.

In the well founded case two well-founded graphs are representing the same transitive set iff their Mostowski collapse is equal.

Forti and Honsel and indipendently Aczel formulated this strengthening of Mostowski's collapse:

Axiom 2 (X_1). *Every binary relation R on a set X has a unique collapse on a transitive set.*

ZFA is the theory ZFC where foundation is replaced by X_1

BISIMILARITY

Definition 3. *Two graphs* (X, R), (Y, S) *are bisimilar if there is* $B \subseteq X \times Y$ *such that:*

$$x\, B\, y \iff \forall z\, R\, x\, \exists w\, S\, y \text{ such that } z\, B\, x$$
$$\wedge \forall w\, S\, y\ \exists z\, R\, x \text{ such that } z\, B\, w$$

Examples:

If R is a well founded relation on X and π is its transitive collapse on a set Y, π is a bisimilarity between R and $\in\restriction Y$.

A one point loop is bisimilar to a two point loop, to an n-point loop.......

If ϕ is an automorphism of a graph (X, R) onto itself, and $\equiv_\phi$ is the orbit equivalence relation,

$(X, R)/\equiv_\phi$ is bisimilar to (X, R).

Composition of bisimilarities is a bisimilarity.

Fact 1. *Bisimilarity is an equivalence relation between graphs.*

Fact 2. *TFAE:*

- (X_1)
- *Two graphs have the same transitive collapse iff they are bisimilar.*

WHY SHOULD WE CARE?

Theorem 4 (Forti, Honsel). *Assume* (M, E_M) *and* (N, E_N) *are models of* ZFA. *Then*

$$(M, E_M) \cong (N, E_N) \Leftrightarrow (M^{WF}, E_M) \cong (N^{WF}, E_N).$$

Theorem 5. *There is a natural cumulative hierarchy for models of* ZFA.

Fact 3. *Well-foundedness is a* Δ_1*-property of* ZFA

ZFA is an extension of ZFC whose transitive models are determined by their well-founded part and admit a variety of ill-founded sets.

How to generate the constructible universe of ZFA?

By the previous results, there is a unique transitive model L_{X_1} such that:

- $L_{X_1} \cap WF = L$,
- every transitive model M of ZFA contains L_{X_1}.

We want a simple recipe to build it.

Back to well-foundedness

We first prove that if α is transitive and linearly ordered by $\in$, then $(\alpha, \in)$ is a well order.

This is enough to have that well-foundedness is a Δ_1-property in ZFA.

Proof. Let X be an ill-founded transitive set linearly ordered by $\in$.

Let α be the supremum of the well-founded initial segments of X.

Set $Z = \alpha \cup \{Z\}$.

- Z is a transitive set (provided that it exists, this requires a little argument).

- If $\pi(x) = Z$ for all $x \in X \setminus \alpha$ and $\pi(\xi) = \xi$ for all $\xi \in \alpha$, π is a transitive collapse of $(X, \in)$ over $(Z, \in)$.

There is only one transive collapse of $(X, \in)$ on a transitive set and the identity is such. Thus π is the identity and $X = Z$.

But Z is not linearly ordered by $\in$. Contradiction. $\square$

Gödel operations

We want a simple list of Gödel operations such that the least transitive class containing the ordinals and closed under these operations is the constructible universe of ZFA.

We add to the list of Gödel operations appearing in Jech's book ($G_0(X,Y) = X \times Y$, $G_1(X) = \bigcup X$,...) the following operation:

Definition 6. *$\pi(a, R) = x$ iff R is a relation, a is in the extension of R and x is assigned to a by the transitive collapse of R provided by X_1.*

There are two simple facts to check:

Fact 4. *The operation* $\pi(a, R) = x$ *is absolute between transitive models* $M \subseteq N$ *of* ZFA.

Fact 5. *Any transitive class* M *which is closed under the standard Gödel operation and the operation* π *is a model of* ZFA.

Proof of fact 4. Let R be a relation, A be its extension, $a \in A$, $(X, \in)$ be the transitive collapse of (A, R) and π_R be the collapsing map.

$\pi(a, R) = x$ holds iff:

- π_R is a function
- $\mathrm{dom}(\pi_R) = \mathrm{ext}(R) = A$
- $\mathrm{im}(\pi_R)$ is a transitive set
- for all $b, c \in \mathrm{ext}(R)$, $b\,R\,c$ iff $\pi_R(b) \in \pi_R(c)$
- $\pi_R(a) = x$

Thus $\pi(a, R) = x$ is defined by a Σ_0-formula in the parameters π_R, a, x.

The unique "*delicate*" point is to show that if $R \in M \subseteq N$ is a relation and M, N are transitive models of ZFA, $(\pi_R)^M = (\pi_R)^N$.

Let $\pi_0 = (\pi_R)^M$, $\pi_1 = (\pi_R)^N$.

Now let $A = \text{im}(\pi_0)$ and $B = \text{im}(\pi_1)$.

A and B are transitive set and $\pi_1 \circ \pi_0^{-1} \subseteq A \times B$ is a bisimilarity between A and B, since it is the composition of two bisimilarities.

Thus A and B must be equal. So $\pi_0 = \pi_1$.
$\square$

We have sketched a proof of the following:

Theorem 7. *Assume* ZFA *and let* L_{X_1} *be the closure of the class of ordinals under the standard Gödel operations and* π. *Then* L_{X_1} *is the constructible universe of* ZFA

Why a mathematician should not care about ZFA?

All interesting mathematical theories can be coded in ZFC and ZFA does not add any clarity to the solution of these problems....

Why a Computer scientist should care about ZFA?

Many interesting data structures can be represented by sets.

For example we may have an infinite set $X = X^2$.

This is clearly impossible in ZFC.