CM-triviality
and
Geometric Elimination of Imaginaries

Ikuo Yoneda
Department of Mathematics
Tokai University

18/7/2007

1 Introduction

Hrushovski gave a counterexample to the Zilber's conjecture on strongly minimal sets by Generic Relational Structures, i.e. relational countable structures constructed by amalgamating relational finite structures.

Generic relational structures are usually $\cdots$

CM-TRIVIAL.

To show the CM-triviality of generic structures, we needed two steps.

- **1st step:**
  Show weak elimination of imaginaries.
- **2nd step:**
  Working in the real sort, show CM-triviality.

Following these two steps,

I proved CM-triviality of
Herwig's weight ω small theory, and
Baldwin-Shi's stable generic structures.

A question comes up :

Is there a way to show CM-triviality without showing Weak Elimination of Imaginaries?

I find the following answer.

THE MAIN RESULT

In simple theories with elimination of hyperimaginaries,

CM-triviality in the real sort (I will define)

$\Downarrow$

Geometric elimination of imaginaries

$+$

CM-triviality in the original sense, firstly introduced by Hrushovski.

2 Set-up

From now on, let T be a simple theory with elimination of hyperimaginaries
and $\overline{\mathrm{M}}$ be a sufficiently saturated model of T.

(Hyper-)imaginary elements are equivalence classes of (type-)definable equivalence relations.

We work in $\overline{\mathrm{M}}^{\mathrm{eq}}$, the eq-structure, consisting of imaginary elements.

3 CM-triviality

Hrushovski's Definition for CM-triviality

T is CM-trivial, if for any $\mathbf{a}, \mathbf{A}, \mathbf{B} \subset \overline{\mathbf{M}}^{\mathbf{eq}}$ with $\mathbf{acl^{eq}}(\mathbf{aA}) \cap \mathbf{acl^{eq}}(\mathbf{B}) = \mathbf{acl^{eq}}(\mathbf{A})$,

$$\mathbf{Cb}(\mathbf{stp}(\mathbf{a}/\mathbf{A})) \subseteq \mathbf{acl^{eq}}(\mathbf{Cb}(\mathbf{stp}(\mathbf{a}/\mathbf{B}))).$$

- $\mathbf{acl^{eq}}(*)$ **denotes algebraic closure in** $\overline{\mathbf{M}}^{\mathbf{eq}}$

Equivalently,

for any $\quad \mathbf{a}, \quad \mathbf{A} = \mathbf{acl^{eq}}(\mathbf{A}), \quad \mathbf{B} = \mathbf{acl^{eq}}(\mathbf{B}) \subset \overline{\mathbf{M}}^{\mathbf{eq}}$,

$$\mathbf{a} \underset{\mathbf{A}}{\downarrow} \mathbf{B} \Rightarrow \mathbf{a} \underset{\mathbf{A} \cap \mathbf{acl^{eq}}(\mathbf{a}, \mathbf{B})}{\downarrow} \mathbf{B}.$$

- **We are working in the eq-structure,** <u>**not in the real sort**</u>**.**

My Definition for CM-triviality

T is CM-trivial in the real sort, if,
for any $\bar{a}$, $A = acl(A)$, $B = acl(B) \subset \overline{M}$,

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_{A} B \Rightarrow \bar{a} \mathop{\smile\!\!\!\!\!\!\;\;|}_{A \cap acl(\bar{a}, B)} B.$$

- **Notice that everything is in the real sort.**

IND/I

T has the independence over intersections (IND/I), if, for any $\bar{\mathbf{a}}$, $\mathbf{A} = \mathbf{acl}(\mathbf{A})$, $\mathbf{B} = \mathbf{acl}(\mathbf{B}) \subset \overline{\mathbf{M}}$

$$\bar{\mathbf{a}} \underset{\mathbf{A}}{\downarrow} \mathbf{B}, \quad \bar{\mathbf{a}} \underset{\mathbf{B}}{\downarrow} \mathbf{A} \Rightarrow \bar{\mathbf{a}} \underset{\mathbf{A} \cap \mathbf{B}}{\downarrow} \mathbf{AB}.$$

- **Notice that everything is in the real sort.**

Proposition A

CM-triviality in the real sort $\Rightarrow$ IND/I.

The key point of the proof: Assume
$\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_A B$, $\quad \bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_B A$, $\quad A = \mathrm{acl}(A)$, $\quad B = \mathrm{acl}(B)$.

By $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_B AB$, $\quad \mathrm{acl}(\bar{a}, B) \cap AB = B$ **follows.**
So we have
$B \cap A \subseteq \mathrm{acl}(\bar{a}, B) \cap A \subseteq (\mathrm{acl}(\bar{a}, B) \cap AB) \cap A \subseteq B \cap A$.

By CM-triviality in the real sort, we have

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(\bar{a},B)\cap A} B.$$

$\underline{\textbf{Proposition B}}$ **IND/I $\Leftrightarrow$ GEI.**

Geometric Elimination of Imaginaries
means that
for any $\mathbf{i} \in \overline{\mathbf{M}}^{\mathbf{eq}}$, there exists $\bar{\mathbf{a}} \subset \overline{\mathbf{M}}$ such that

$$\mathbf{i} \in \mathbf{acl}^{\mathbf{eq}}(\bar{\mathbf{a}}),$$

$$\bar{\mathbf{a}} \in \mathbf{acl}^{\mathbf{eq}}(\mathbf{i}).$$

The proof of "IND/I ⇒ GEI."

Fix i $= \bar{a}_E$. Take $\bar{b}, \bar{c}$ such that $\bar{b}, \bar{c} \models tp(\bar{a}/i)$ and $\bar{a}, \bar{b}, \bar{c}$ are independent over i.

By $\bar{a} \downarrow_{\bar{b}} \bar{c}, \bar{a} \downarrow_{\bar{c}} \bar{b}$ and IND/I, we have

$$\bar{a} \underset{acl(\bar{b}) \cap acl(\bar{c})}{\downarrow} \bar{b}, \bar{c}.$$

Let A $= acl(\bar{b}) \cap acl(\bar{c})$.
As i $\in dcl^{eq}(\bar{a})$, we have "i $\in acl^{eq}(A)$".
By $\bar{b} \downarrow_{i} \bar{c}$, we see "A $\subseteq acl^{eq}(i)$".

Under GEI,
CM-triviality in the real sort=CM-triviality.

Main Theorem

CM-triviality in the real sort
$\Downarrow$
GEI+CM-triviality.

Two Remarks

(1) Simple generic structures have the following NICE characterization of non-forking;

$$\mathbf{A} \underset{\mathbf{A}\cap\mathbf{B}}{\downarrow} \mathbf{B} \Leftrightarrow \mathbf{A} \otimes_{\mathbf{A}\cap\mathbf{B}} \mathbf{B} = \mathbf{A} \cup \mathbf{B} = \mathbf{cl}_{\overline{\mathbf{M}}}(\mathbf{A} \cup \mathbf{B})$$

for any $\mathbf{A} = \mathbf{acl}(\mathbf{A}), \mathbf{B} = \mathbf{acl}(\mathbf{B}) \subset \overline{\mathbf{M}}$**.**

From this, we can check that simple generic structures are CM-trivial in the real sort.

Main Theorem directly shows the CM-triviality of simple generic structures.

(2)

$$\text{CM-triviality in the real sort} \not\Uparrow \text{CM-triviality in the original sense.}$$

In [E], D.Evans gave an ω-categorical SU $= 1$ CM-trivial structure $\mathfrak{C}$ without WEI interpreted in an ω-categorical SU $= 2$ generic binary graph. I checked $\mathfrak{C}$ does not have GEI.

Remark on IND/I

In pregeometric surgical theories, IND/I⇒GEI.

In O-minimal case, "IND/I⇒EI" only holds.

ENDING : 4 Problems on CM-triviality

(1) In stable theories, does CM-triviality imply CM-triviality in the real sort?

(2) Is any superstable CM-trivial theory ω-stable?

This is a generalization of Baldwin's Problem: Is any superstable ω-saturated generic structure ω-stable?

(3) Recall that n-ampleness is defined in the eq-structures.

non-1-amlpleness $\Leftrightarrow$ One-basedness $\Rightarrow$ non-2-ampleness $\Leftrightarrow$ CM-triviality $\Rightarrow$ non-3-ampleness $\Rightarrow$ non-4-ampleness $\Rightarrow \cdots$

- **Define non-3-ampleness in the real sort. And does it imply GEI?**

(4) In Zariski geometries, local modularity is equivalent to CM-triviality.

- **In O-minimal theories, is local modularity equivalent to CM-triviality?**

References

[E] D.M.Evans, $\aleph_0$-categorical structures with a predimension, Annals of Pure and Applied Logic 116 (2002), 157-186.

[G] Jerry Gagelman, Stability in geometric theories, Annals of Pure and Applied Logic 132 (2005), 313-326.

[H] Ehud Hrushovski, A new strongly minimal sets, Annals of Pure and Applied Logic 88 (1993), 147-166.

[Y] Ikuo Yoneda, Forking and some eliminations of imaginaries, submitted, October 2006.