Strongly regular types

Predrag Tanović

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Minimal first-order structures, Annals of Pure and Applied Logic vol.162(2011), pp.948-957.

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$\bar{p}$ is 'free' extension of $p = \bar{p} \mid A$.

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Is there an A-invariant global s.r. extension of p?

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We operate in the monster $\bar{M}$, $\bar{p}$ is a global type. $A, B, \ldots$ are small; $M \prec \bar{M}$ is small, while $N \prec \bar{M}$ can be large.

Invariant types

- $p \in S(B)$ *does not split over* $A \subset B$ if for all $\bar{b}_1, \bar{b}_2 \in B$ and ϕ over A:
$$\bar{b}_1 \equiv \bar{b}_2 \ (A) \quad \text{implies} \quad (\phi(\bar{x}, \bar{b}_1) \leftrightarrow \phi(\bar{x}, \bar{b}_2)) \in p(\bar{x})$$
- A global type is *invariant* if it does not split over some small A (we say that it is A-invariant).
- A Morley sequence in $\bar{p}$ over (a small) A is one that satisfies
$$a_{i+1} \models \bar{p} \,|\, (A, a_0, a_1, ..., a_i) \ .$$
- If $\bar{p}$ is A-invariant then Morley sequences over A are indiscernible over A; $\bar{p}^n$ is well-defined.

There are two kinds of invariant types; let $\bar{p}$ be A-invariant and let $\bar{a}_1, \bar{a}_2 \models \bar{p}^2$.

1. (Symmetric) When $\bar{p}^2$ is symmetric:
$$\bar{a}_1\bar{a}_2 \equiv \bar{a}_2\bar{a}_1 \ (\bar{M})$$
In this case every Morley sequence in $\bar{p}$ over any $B \supset A$ is totally indiscernible.

2. (Asymmetric) When $\bar{a}_1\bar{a}_2 \not\equiv \bar{a}_2\bar{a}_1 \ (\bar{M})$

Generic stability

Definition

A non-algebraic global type $\bar{p}(\bar{x}) \in S(\bar{M})$ is *generically stable* if, for some small A, it is A-invariant and:

if $(\bar{a}_i : i < \alpha)$ (any α, not only ω) is a Morley sequence in $\bar{p}$ over A then for any formula $\phi(\bar{x})$ (with parameters from $\bar{M}$) $\{i : \models \phi(\bar{a}_i)\}$ is either finite or co-finite.

Let $\bar{p}(\bar{x}) \in S(\bar{M})$ be generically stable and A-invariant. Then:

1. $\bar{p}$ is definable over A.
2. Any Morley sequence of $\bar{p}$ over A is totally indiscernible.
3. $\bar{p}$ is the unique global nonforking extension of $p \mid A$.

Pregeometries

- (P, cl), where cl is an operation on subsets of P, is a *pregeometry* if for all $A, B \subseteq P$ and $a, b \in P$:

Monotonicity $A \subseteq B$ implies $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

Finite character $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) \mid A_0 \subseteq A \text{ finite}\}$;

Transitivity $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$;

Exchange (symmetry)

$$a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A) \quad \text{implies} \quad b \in \mathrm{cl}(A \cup \{a\}).$$

- cl is called a (finitary) *closure operator* if it satisfies the first three axioms.

Weak orthogonality

Let $p(\bar{x}), q(\bar{y}) \in S(A)$.

- p and q are *weakly orthogonal*, $p \overset{w}{\perp} q$, if
  $$p(\bar{x}) \cup q(\bar{y}) \text{ determines a complete type.}$$
- '$p(x)$ has a unique extension over $B \supset A$' is denoted by $p \vdash p \,|\, B$.
- $p \vdash p \,|\, B$ is equivalent to $p \overset{w}{\perp} \mathrm{tp}(B/A)$ and
- If $\bar{b} \models q$ then:
  $$p(\bar{x}) \overset{w}{\perp} q(\bar{y}) \quad \text{iff} \quad p(\bar{x}) \vdash p \,|\, (A, \bar{b}).$$

Strong regularity was built on strong minimality, where a natural pregeometry exists.

- $(M, \ldots)$ is *minimal* if any definable (with parameters) subset of M is either finite or co-finite.
- A complete T is *strongly minimal* if any $M \models T$ is minimal.

Theorem (Marsh)

(1) If M is minimal then (M, acl) is a pregeometry.

(2) If $M \equiv N$ are both minimal then

$$M \cong N \quad \text{iff} \quad \dim(M) = \dim(N).$$

T ω-stable

Strong regularity should induce a natural pregeometry.

- A stationary non-algebraic type $p \in S(A)$ is *strongly regular via* $\phi(x) \in p$ (or (p, ϕ) is s.r.) if for any $B \supset A$ and $q \in S(B)$ containing $\phi(x)$

$$\text{either} \quad q = p|B \qquad \text{or} \qquad p \perp q \ (p\,|\,B \overset{w}{\perp} q).$$

- There is a natural pregeometry on $\phi(\bar{M})$:

$$\text{for } X \subset \phi(\bar{M}): \quad \mathrm{cl}(X) = \{a \in \phi(\bar{M}) \mid a \not\models p\,|\,X \cup A\}$$

- If $p \in S_1(M)$ then (p, ϕ) is s.r. iff

$$p \overset{w}{\perp} \mathrm{tp}(b/M) \quad \text{for all } b \in \phi(\bar{M}) \setminus p(\bar{M});$$

Equivalently: $\quad p \vdash p \,|\, Mb \quad \text{for all } b \in \phi(\bar{M}) \setminus p(\bar{M})$

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- A non-algebraic type $p \in S_1(A)$, which is finitely satisfiable in A, is s.r. via $\phi(x)$ iff:

$$p \stackrel{w}{\perp} \mathrm{tp}(\bar{b}/A) \quad \text{for all } \bar{b} \in \phi(\bar{M}) \setminus p(\bar{M});$$

Equivalently: $\quad p \vdash p \,|\, A\bar{b} \quad \text{for all } \bar{b} \in \phi(\bar{M}) \setminus p(\bar{M})$

Definition (any T)

Let $p \in S(A)$ and $\phi(x) \in p$. We say that (p, ϕ) satisfies ${}^{w}\!\perp$-condition if:

$$p \,{}^{w}\!\perp \operatorname{tp}(\bar{b}/A) \quad \text{for all } \bar{b} \subset \phi(\bar{M}) \setminus p(\bar{M});$$

Equivalently: $\quad p \vdash p \,|\, A\bar{b} \quad \text{for all } \bar{b} \in \phi(\bar{M}) \setminus p(\bar{M})$

Definition

A non-algebraic type $p(x) \in S_1(A)$ is *locally strongly regular via* $\phi(x) \in p(x)$ if

(1) $p(x)$ is finitely satisfiable in A;

(2) $(p(x), \phi(x))$ satisfies ${}^{w}\!\perp$-condition.

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Example 1. The generic type of a minimal structure is loc.s.r via $x = x$ ('generic' is the unique nonalgebraic type in $S_1(M)$).

Example 2. The generic type of a quasiminimal structure is loc.s.r via $x = x$. (Corollary 7.1 from [PT]).

Definition

Let $\bar{p}(\bar{x})$ be a global non-algebraic type and let $\phi(\bar{x}) \in \bar{p}(\bar{x})$. We say that $(\bar{p}(\bar{x}), \phi(\bar{x}))$ is *strongly regular* if, for some small A over which $\phi(\bar{x})$ is defined:

(1) $\bar{p}$ is A-invariant; and

(2) $\bar{p} \mid B$ satisfies ${}^{w}\!\perp$-condition for all $B \supseteq A$.

- A global s.r. type is locally s.r.
- We will be mainly interested in 1-types over a model which are locally s.r. via $x = x$.

Where does the pregeometry come from?

In a minimal structure definable subsets are either 'small' (finite) or 'large' (co-finite). 'Large' ones determine a complete type: the unique non-algebraic 1-type (generic).

Definition (T any complete first-order theory)

For $p(x) \in S_1(N)$ and $X \subset N$ define

$$\mathrm{cl}_p(X) = \bigcup\{\phi(N) \mid \phi(x) \text{ is over } X \text{ and } \phi(x) \notin p(x)\}$$

- Also, define $\mathrm{cl}_p^A(X) = \mathrm{cl}_p(X \cup A)$ for any $A \subset N$ and $X \subseteq N$.
- $\mathrm{cl}_p(X) = \{a \mid a \not\models p \mid X\}$.
- In minimal structures: $\mathrm{acl}(X) = \mathrm{cl}_p(X)$.

Proposition

Let $\bar{p}(x)$ be an A-invariant global type. Then $(\bar{p}(x), x = x)$ is strongly regular if and only if $\mathrm{cl}_{\bar{p}}^{A}$ is a closure operator.

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Example $(\omega, <)$ is minimal; let p be the generic.

- The pregeometry is trivial $\mathrm{acl}(\emptyset) = \mathrm{cl}_p(\emptyset) = \omega$.
- p is definable and if $\bar{p}$ is the global heir, then $(\bar{p}, x = x)$ is s.r.
- $(\bar{M}, \mathrm{cl}_{\bar{p}})$ is not a pregeometry; it is an 'infinite-dimensional' closure operator ($\bar{M}$ is not the closure of a finite subset).
- Any model of $\mathrm{Th}(\omega, <)$ uniquely determines a linear order with minimal element(after factoring out 'x, y are at finite distance').
- p has unique global coheir $\bar{q}$ and $(\bar{q}, x = x)$ is s.r.

Homogeneous pregeometries

Reformulated Theorem 4.1 from [PT].

Theorem

(1) Suppose that $A \subset N$ and $p \in S_1(N)$. If (N, cl_p^A) is an infinite-dimensional pregeometry then p is definable over A, its global heir $\bar{p}$ is generically stable and $(\bar{p}(x), x = x)$ is strongly regular (in particular, it is A-invariant and symmetric).

(2) Let $\bar{p}(x)$ be a global, A-invariant, symmetric type such that $(\bar{p}(x), x = x)$ is strongly regular. Then $\bar{p}$ is generically stable and $(\bar{M}, \mathrm{cl}_{\bar{p}}^A)$ is an infinite-dimensional pregeometry.

Note: In (1) we do *not* assume that p is A-invariant!

Hytinnen,Lessmann, Shelah, *Interpreting groups and fields in some nonelementary classes*, Journal of Mathematical Logic 5(1), 2005

Reformulated Theorem 3.1 from [PT]

Theorem (Dichotomy of global, invariant s.r. types)

Let $(\bar{p}(x), x = x)$ be strongly regular, where $\bar{p}(x)$ is A-invariant. Then $(\bar{M}, \mathrm{cl}_{\bar{p}}^{A})$ is a closure operator and:

(1) If $\bar{p}$ is symmetric then $\bar{p}$ is generically stable and $(\bar{M}, \mathrm{cl}_{\bar{p}}^{A})$ is an infinite-dimensional pregeometry.

(2) If $\bar{p}$ is asymmetric then there is a finite extension A_0 of A and an A_0-definable partial order $\leq$ such that every Morley sequence in $\bar{p}$ over A_0 is strictly increasing. Moreover, if $A_0 \subset M$ and $C_1, C_2 \subset M$ are maximal (under inclusion) Morley sequences in $\bar{p}$ over A_0 then $(C_1, \leq) \cong (C_1, \leq)$.

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Remark: in the asymmetric case $\bar{p}$ may not be definable (the coheir in $(\omega, <)$).

General dichotomy

Let $p \in S(N)$. Consider cl_p (or any other monotone, finitary operation on subsets of N). Define:

- N is finitely cl_p-generated over A if $N = \mathrm{cl}_p(A\bar{n})$ for some $\bar{n} \in N$.
- A sequence $\{a_i \mid i \in \omega\} \subset N$ is cl_p-*free over* $A \subset N$ if

$$\text{for all } n \quad a_{n+1} \notin \mathrm{cl}_p(A, a_0, ..., a_n)$$

- Similarly we define cl_p-free sequences indexed by a linear order.
- If N is not finitely cl_p-generated over A then infinite cl_p-free sequences over A exist.

For $p \in S_1(N)$ we can define the smallest closure operator extending cl_p:

$$\mathrm{cl}_p^0(X) = X \quad \text{and} \quad \mathrm{cl}_p^{n+1}(X) = \mathrm{cl}_p(\mathrm{cl}_p^n(X))$$

$$\mathrm{Cl}_p(X) = \bigcup_{n\in\omega} \mathrm{cl}_p^n(X)$$

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If N is quasiminimal and p the generic type, then Cl_p is Zilber's ccl-operator.

Definition

$p \in S_1(N)$ is *based on* $A \subset N$ if p does not split over A, and N is not finitely Cl_p-generated over A.

Theorem 6.1 from [PT].

Theorem

Suppose that $p(x)$ is based on $A \subset N$. Then cl_p^A is a closure operator and we have two cases:

(1) p^2 is symmetric; cl_p^A is a pregeometry operator on N, p is definable, $\bar{p}$ (its unique global heir) is generically stable and $(\bar{p}(x), x = x)$ is strongly regular.

(2) p^2 is asymmetric; there is a finite extension A_0 of A and an A_0-definable partial order $\leq$ such that every cl_p-free sequence over A_0 is strictly increasing.

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Question If M is asymmetric, quasiminimal must there exist a definable partial order on singletons, which has uncountable increasing chain?

Local regularity

In the definable case the dichotomy applies (Proposition 7.1 from [PT]).

Proposition

Suppose that $p(x) \in S_1(A)$ is definable and locally strongly regular via $\phi(x) \in p(x)$. Let $\bar{p}(x)$ be its global heir. Then $(\bar{p}(x), \phi(x))$ is strongly regular (and of course definable).

What is a ‘free‘ A-invariant extension of a loc.s.r $p \in S_1(A)$?

- By an *A-type* we mean a non-algebraic type which is f.s. in A.
- $\{\bar{a}_i \mid i \in \omega\}$ is an *A-sequence over B* if $\mathrm{tp}(\bar{a}_n/(B, \bar{a}_0, ..., \bar{a}_{n-1}))$ is an A-type for all n.
- ‘Semiisolated’ is opposite to ‘being a coheir’: $a \in \mathrm{Sem}_p(X)$ iff:

$$\text{there is } \theta(x) \in \mathrm{tp}(a/X \cup A) \text{ such that } \theta(x) \vdash p)x)$$

- if $\bar{p}$ is f.s. in A then $\mathrm{Sem}_p(X) \subseteq \mathrm{cl}^A_{\bar{p}}(X)$; $=$ holds in the gen.stable s.r. case.

Definition

Let $p \in S(A)$ be locally s.r. via $x = x$. We say that p is *symmetric* if there exists an infinite, totally indiscernible A-sequence of realizations of p. Otherwise p is *asymmetric*.

Proposition 7.2 from [PT]:

Proposition

Suppose that $p(x) \in S_1(A)$ is locally strongly regular via $x = x$. Then p has a global A-invariant extension $\bar{p}$ such that $(\bar{p}(x), x = x)$ is strongly regular and generically stable if and only if p is symmetric

Minimal structures

Let $(M, \ldots)$ be a minimal first-order structure and let $p \in S_1(M)$ be the unique non-algebraic 1-type.

Dichotomy of minimal structures

There are two types of minimal structures:

Symmetric p is definable, its unique global heir $\bar{p}$ is generically stable and $(\bar{p}(x), x = x)$ is strongly regular.

Asymmetric There is an infinite $C \subseteq M$ directing a type over some finite $A \subset \bar{M}$.

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Asymmetric There is an infinite $C \subseteq M$ directing a type over some finite $A \subset \bar{M}$.

In the asymmetric case we do not claim the existence of a global, invariant, strongly regular extension of p

- We say that an infinite set $C \subset \mathrm{dcl}(A)$ *directs a type over* A if there is an A-definable partial ordering $\leq$ on $\bar{M}$ such that:

  (1) $\{x \in C \mid c \leq x\}$ is a co-finite subset of C (for all $c \in C$),

  (2) $(C, \leq)$ is an initial part of $(\bar{M}, \leq)$.

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  (1) $\{x \in C \mid c \leq x\}$ is a co-finite subset of C (for all $c \in C$),
  (2) $(C, \leq)$ is an initial part of $(\bar{M}, \leq)$.
- Condition (1) can be replaced by:
  $(D1)$ Any C-sequence over A is decreasing:
  if (a, b) is a C-sequence over A then $b \leq a$;
- Let $p_C(x)$ be the set of all nonalgebraic formulas over A which are satisfied by co-finitely many $c \in C$. We say that $p_C(x)$ is *C-directed over A*.
- Any C-type over A extends $p_C(\bar{M})$.

Theorem (T small)

Suppose that C directs a type over A. Then there is an infinite $C_0 \subset C$ which directs a type over a finite extension $A_0 \supset A$ and such that for all $M \supset A_0$ the (linear) order type of a maximal C_0-sequence over A_0 in M is uniquely determined.

Asymmetric local regularity

Main Question. If p is asymmetric loc.s.r via $x = x$ must there exist a specific partial order 'directing' in some sense a type?

Fix a countable M and $p \in S_1(M)$ which is loc.s.r. via $x = x$.

Two-step strategy

Try to prove

1. p^2 is well defined and symmetric.
2. $p \mid Ma$ is loc.s.r. via x=x for any a realizing p.

Hope that any obstruction will be caused by some partial order.

The first step is successful

p^2 should be the type of a coheir sequence (a, b).

Proposition

Suppose that there are two distinct types of coheir pairs. Then there is a directed type over Me for any $e \models p$.

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Let $\Psi(x) = \bigvee_{n\in\omega} \psi_n(x)$ be over A and let $q(x)$ defines the topological boundary of $\bigcup[\psi_n(x)]$.

We say that Ψ directs q iff:

(1) $\Psi(\bar{M}) < a$ for all $a \models q$;

(2) If (a, b) is a $\Psi(\bar{M})$-sequence of realizations of q then $b \leq a$; and

(3) If $a \models q$ and $b \leq a$ then either $b \in \Psi(\bar{M})$ or $b \models q$.

The second step

Assume that p^2 is symmetric, $a \models p$. We divide this step into two sub-steps:

1. Eliminate
$$p^2 \not\perp^w \operatorname{tp}(\bar{b}/M) \ \text{ for some } \bar{b} \subset \bar{M} \setminus p(\bar{M});$$
equivalently
$$p \,|\, Ma \not\perp^w \operatorname{tp}(\bar{b}/M) \ \text{ for some } \bar{b} \subset \bar{M} \setminus p(\bar{M}).$$
2. Assuming that the previous is eliminated, eliminate:
$$p \,|\, Ma \not\perp^w \operatorname{tp}(\bar{b}/M) \ \text{ for some } \bar{b} \subset \bar{M} \setminus (p|Ma)(\bar{M}).$$

The first elimination is probably not possible in general, but the second is: there must be a directed type around.

Theorem

Suppose that $p \in S_1(M)$ is an asymmetric loc.s.r. via $x = x$ type and that there is a coheir sequence $\{a_i \mid i \in \omega\}$ such that

$$\mathrm{tp}(a_0 \ldots a_n/M) \overset{w}{\perp} \mathrm{tp}(\bar{b}/M) \quad \text{for all } \bar{b} \subset \bar{M} \setminus p(\bar{M})$$

Then there is a directed type around.

Countable models of theories with Skolem functions

Assumption: T has built-in Skolem functions and $I(\aleph_0, T) < 2^{\aleph_0}$.
In particular, T is small.

Proposition

There are no directed types around.

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Proposition

There are no directed types around.

The first elimination of the second step becomes possible.

Proposition

Every locally strongly regular type (via $x = x$) is symmetric.

Question Is every minimal (Cantor-Bendixson rank 1) type definable?

Theorem

Assume T is NIP or NSOP, has built-in Skolem functions and $I(\aleph_0, T) < 2^{\aleph_0}$. Then T is ω-stable, (nmd and) finite-dimensional, $I(\aleph_0, T) = \aleph_0$.

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Assume T is NIP or NSOP, has built-in Skolem functions and $I(\aleph_0, T) < 2^{\aleph_0}$. Then T is ω-stable, (nmd and) finite-dimensional, $I(\aleph_0, T) = \aleph_0$.

THANK YOU