Integration of semialgebraic functions and antiderivatives of Nash functions

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1. Motivation

In
Kontsevich, Zagier: Periods. Mathematics unlimited–2001 and beyond, Springer, Berlin, 2001
we find the following definition.

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Applications:

- arithmetic geometry
- differential equations
- algebraic topology
- differential topology

Period = Integral of a semialgebraic function on $\mathbb{R}^n$ (defined over $\mathbb{Q}$)

$\longrightarrow$ Families of periods

We answer the following important question. What does one get when one integrates parameterized families of semialgebraic functions?

One has to leave the semialgebraic setting! Immediately:

1) global logarithm
$$\log x = \int_1^x \frac{dt}{t}.$$
2) (iterated) antiderivatives of germs of Nash functions at the origin as
$$\arctan x = \int_0^x \frac{dt}{1+t^2}.$$

We show that 1) and 2) are enough to get a complete picture of our question!

We introduce the setting.

2. Setting

Definition

Let $m, n \in \mathbb{N}$ and let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}, (x,t) \mapsto f(x,t)$, be a semialgebraic function. We set

$$\infty(f) := \{x \in \mathbb{R}^m \mid f(x,-) \text{ not integrable}\}$$

and

$$\mathrm{Int}(f) : \mathbb{R}^m \setminus \infty(f) \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^n} f(x,t)dt.$$

Theorem

$\infty(f)$ is semialgebraic!

Goal:

Explicit description of $\mathrm{Int}(f)$ for $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ semialgebraic.

We start with 2)

3. Integrated algebraic power series

For $n \in \mathbb{N}$

$$\underbrace{\mathcal{N}_n}_{\text{algebraic power series}} \subset \underbrace{\mathcal{O}_n = \mathbb{R}\{X_1, \ldots, X_n\}}_{\text{convergent power series}} \subset \underbrace{\mathbb{R}[[X_1, \ldots, X_n]]}_{\text{formal power series}}$$

$\mathcal{N}_n$ = germs of Nash (=analytic & semialg.) functions at $0 \in \mathbb{R}^n$

$\mathcal{O}_n$ = germs of analytic functions at $0 \in \mathbb{R}^n$

Our goal is to enlarge the rings $\mathcal{N}_n$ to rings $\mathcal{IN}_n$ in such a way that the following holds:

i) $\mathcal{IN}_n$ is defined from $\mathcal{N}_n$ by taking antiderivatives and 'innocent' algebraic operations.

ii) $\mathcal{IN}_n$ has 'good' properties.

Notation

a) By $\mathrm{Int}_n(f)$ we denote the antiderivative of a power series $f(X) = f(X', X_n)$ with respect to the variable X_n such that $\mathrm{Int}_n(f)(X', 0) = 0$. For example $\mathrm{Int}_2(X_1 + X_2) = X_1X_2 + 1/2X_2^2$.

b) For $R = (R_1, \ldots, R_n) \in \mathbb{R}^n_{>0}$ let $D^n_{\mathbb{R}}(R) := \prod_{1 \le j \le n}]-R_j, R_j[$.

c) Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we define $f_a(X_1, \ldots, X_n) := f(a_1 + X_1, \ldots, a_n + X_n)$ for a function defined at a.

The following definition does the job.

Definition

We define by $\mathcal{IN} = (\mathcal{IN})_{n\in\mathbb{N}}$ the smallest class of subrings $\mathcal{IN}_n$ of $\mathbb{R}[[X_1,\ldots,X_n]]$ such that the following properties hold.

($\mathcal{IN}$1) $\mathcal{N}_n \subset \mathcal{IN}_n$ for all $n \in \mathbb{N}$.

($\mathcal{IN}$2) If $f \in \mathcal{IN}_n$ with $f(0) \neq 0$ then $1/f \in \mathcal{IN}_n$.

($\mathcal{IN}$3) If $f \in \mathcal{IN}_n$ and $h \in (\mathbb{R}[X_1,\ldots,X_k])^n$ with $h(0) = 0$ then $f \circ h \in \mathcal{IN}_k$.

($\mathcal{IN}$4) If $f \in \mathcal{IN}_n$ and $R \in \mathbb{R}^n_{>0}$ is a radius of convergence for f then $f_a \in \mathcal{IN}_n$ for all $a \in D^n_{\mathbb{R}}(R)$.

($\mathcal{IN}$5) If $f \in \mathcal{IN}_n$ then $\mathrm{Int}_j(f) \in \mathcal{IN}_n$ for all $1 \leq j \leq n$.

We call $\mathcal{IN}_n$ the *ring of integrated algebraic power series* in n variables.

Elementary properties

a) $\mathcal{N}_n \subset \mathcal{IN}_n \subset \mathcal{O}_n$

b) $\mathcal{IN}_n$ is a local ring with $(\mathcal{IN}_n)^* = \{f \in \mathcal{IN}_n \mid f(0) \neq 0\}$.

To show that a property $(*)$ holds for all $f \in \mathcal{IN}_n$ and all $n \in \mathbb{N}$, it is enough, by the defining axioms ($\mathcal{IN}$1) - ($\mathcal{IN}$5), to show the following steps.

$S1_*$ All elements of $\mathcal{N}_n$ have property $(*)$.

$S2_*$ If $f, g \in \mathcal{IN}_n$ have property $(*)$ then $f+g$, fg and, for $f(0) \neq 0$, $1/f$ have property $(*)$.

$S3_*$ If $f \in \mathcal{IN}_n$ has property $(*)$ and $h \in (\mathbb{R}[X_1, \ldots, X_k])^n$ with $h(0) = 0$ then $f \circ h \in \mathcal{IN}_k$ has property $(*)$.

$S4_*$ If $f \in \mathcal{IN}_n$ has property $(*)$ and $R \in \mathbb{R}^n_{>0}$ is a radius of convergence for f then f_a has property $(*)$ for all $a \in D^n_{\mathbb{R}}(R)$.

$S5_*$ If $f \in \mathcal{IN}_n$ has property $(*)$ then $\mathrm{Int}_j(f)$ has property $(*)$ for all $1 \leq j \leq n$.

Proposition

$f \in \mathcal{IN}_n \Longrightarrow \partial f / \partial X_j \in \mathcal{IN}_n$ for all j

Reminder:

Complexification:

$$\begin{aligned} f(X) &= \sum a_\alpha X^\alpha \in \mathcal{O}_n \\ &\Longrightarrow f(Z) = \sum a_\alpha Z^\alpha \in \mathcal{O}_n^{\mathbb{C}} \\ &\Longrightarrow \mathrm{Re}\, f, \mathrm{Im}\, f \in \mathcal{O}_{2n} \end{aligned}$$

Theorem

$\mathcal{IN}_n$ is closed under complexification:

$f(X) \in \mathcal{IN}_n \Longrightarrow \mathrm{Re}\, f, \mathrm{Im}\, f \in \mathcal{IN}_{2n}$

Corollary (Complex integration along piecewise polynomial curves)

Let $f \in \mathcal{IN}$ and let γ be a piecewise polynomial curve. Let

$$g(x') := \int_\gamma f(x', \zeta) d\zeta.$$

Then $g \in \mathcal{IN}_{n-1} \oplus i\mathcal{IN}_{n-1}$.

Proof: Axioms of $\mathcal{IN}$ and complexification

Using this we can prove

Weierstraß preparation theorem

Let $f \in \mathcal{IN}$ be regular in X_n of order d (i.e. $f(0, X_n) = aX_n^d + \ldots$). Then there are unique $P \in \mathcal{IN}_{n-1}[X_n]$ with degree d and $P(0, X_n) = aX_n^d$ and $u \in \mathcal{IN}$ with $u(0) \neq 0$ (i.e. a unit) such that

$$f = P \cdot u.$$

Also: **Weierstraß division theorem**

Corollary 1

$\mathcal{IN}_n$ is a regular (in particular noetherian) local ring. Its maximal ideal is generated by $X_1, \ldots, X_n$.

Corollary 2

$f \in \mathcal{IN}_n, h = (h_1, \ldots, h_n) \in (\mathcal{IN}_k)^n$ with $h(0) = 0$

$$\Longrightarrow f \circ h \in \mathcal{IN}_k$$

4. Main result

Restricted integrated Nash functions

For $n \in \mathbb{N}$ let $\mathcal{RIN}_n$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \begin{cases} \tilde{f}(x) & x \in [-1,1]^n \\ & \text{if} \\ 0 & x \notin [-1,1]^n \end{cases}$$

for some $\tilde{f} \in \mathcal{IN}_n$ that converges on a neighbourhood of $[-1,1]^n$.

Let

- $\mathcal{RIN} := \bigcup_{n \in \mathbb{N}} \mathcal{RIN}_n$
- $\mathbb{R}_{\mathcal{IN}} := \mathbb{R}((f)_{f \in \mathcal{RIN}})$ the structure generated by $\mathcal{RIN}$ over $\mathbb{R}$
- $\mathcal{L}^{\mathbb{Q}}_{\mathcal{IN}} := \{<, +, -, 0, 1, (r)_{r \in \mathbb{R}}, (f)_{f \in \mathcal{RIN}}, (x^q)_{q \in \mathbb{Q}}\}$

Proposition

The structure $\mathbb{R}_{\mathcal{IN}}$ is o-minimal, has quantifier elimination in $\mathcal{L}^{\mathbb{Q}}_{\mathcal{IN}}$, and definable functions are piecewise given by $\mathcal{L}^{\mathbb{Q}}_{\mathcal{IN}}$-terms.

Proof: $\mathcal{IN}$ is a Weierstraß system in the sense of

> D. Miller: A preparation theorem for Weierstrass systems. *Trans. Amer. Math. Soc.* **358**, no. 10 (2006), 4395-4439.

We can apply the results of the paper.

Main theorem

Let $m, n \in \mathbb{N}$ and let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be definable in $\mathbb{R}_{\mathcal{IN}}$. Then there are functions $\varphi_1, \ldots, \varphi_r : \mathbb{R}^m \to \mathbb{R}$ definable in $\mathbb{R}_{\mathcal{IN}}$ and there is a a polynomial $P(X_1, \ldots, X_r, Y_1, \ldots, Y_r) \in \mathbb{R}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ such that

$$\mathrm{Int}(f) = P(\varphi_1, \ldots, \varphi_r, \log \varphi_1, \ldots, \log \varphi_r).$$

Proof: The methods of Lion, Rolin and Comte on integration of subanalytic functions can be adapted. By Dan Miller, the Lion-Rolin preparation theorem holds for Weierstraß systems:

> Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto f(x, y)$, be definable in $\mathbb{R}_{\mathcal{IN}}$. Then piecewise f can be written as
>
> $$f(x, y) = a(x)|y - \theta(x)|^r u(x, y)$$
>
> where $r \in \mathbb{Q}$ and $a(x), \theta(x), u(x, y)$ are definable in $\mathbb{R}_{\mathcal{IN}}$ with u being a unit and additional properties.

In our system the Lion-Rolin splitting holds:

> Let $f \in \mathcal{IN}_{n+2}$. Then there are $f_+, f_- \in \mathcal{IN}_{n+2}$ such that for all sufficiently small $x, y/z$ and $z \neq 0$
>
> $$f(x, y/z, z) = f_+(x, y, z) + (y/z) f_-(x, y, y/z).$$

5. Definability results

Let $\mathbb{R}_{\text{Int}} := \mathbb{R}((\text{Int(f)})_{f\,\text{semialg.}})$.

Remark

$\mathbb{R}_{\text{Int}}$ is o-minimal and a reduct of $\mathbb{R}_{\mathcal{IN},\text{exp}}$.

Goal:
To understand the structure $\mathbb{R}_{\mathcal{IN}}$ and its relation to $\mathbb{R}$ resp. $\mathbb{R}_{\text{Int}}$.

Let $\mathcal{M}$ be a structure on $\mathbb{R}$. We denote by $C^{\omega}_{\mathcal{M},n}$ the set of germs at $0 \in \mathbb{R}^n$ of analytic functions definable in $\mathcal{M}$.

Examples

a) $C^{\omega}_{\mathbb{R},n} = \mathcal{N}_n$

b) $C^{\omega}_{\mathbb{R}_{\text{an}},n} = \mathcal{O}_n$

Theorem

$C^{\omega}_{\mathbb{R}_{\mathcal{IN}},n} = \mathcal{IN}_n$ for all $n \in \mathbb{N}$

Definition

Let $\mathcal{M}$ be a structure on $\mathbb{R}$. We say that $\mathcal{M}$ is *analytically exhausting* if the following holds for all $n \in \mathbb{N}$:

$$f \in C^{\omega}_{\mathcal{M},n}, R \in \mathbb{R}^n_{>0} \text{ radius of convergence for f}$$
$$\Longrightarrow f_a \in C^{\omega}_{\mathcal{M},n} \ \forall a \in D^n_{\mathbb{R}}(R)$$

Examples

The structures $\mathbb{R}, \mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\mathcal{IN}}$ are analytically exhausting.

Definition

Let $\mathcal{M}, \mathcal{N}$ be structures on $\mathbb{R}$. We say that $\mathcal{N}$ is a *local analytic antiderivative closure* of $\mathcal{M}$ if the following holds.

I. $\mathcal{N}$ is an expansion of $\mathcal{M}$.

II. $f \in C^{\omega}_{\mathcal{N},n} \Rightarrow \text{Int}_j(f) \in C^{\omega}_{\mathcal{N},n} \ \ \forall 1 \leq j \leq n \ \ \forall n \in \mathbb{N}$.

III. $\mathcal{N}$ is analytically exhausting.

IV. If $\mathcal{N}'$ satisfies I. - III. then $\mathcal{N}'$ is an expansion of $\mathcal{N}$.

Theorem

The local analytic antiderivative closure of a structure exists and is unique.

Example

$\mathbb{R}_{\text{an}}$ is the local analytic antiderivative closure of itself.

Theorem

$\mathbb{R}_{\mathcal{IN}}$ is the local analytic antiderivative closure of $\mathbb{R}$.