PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 23, Fasc. 2 (2003), pp. 369–387

ON THE LOCAL TIME OF MULTIPARAMETER SYMMETRIC STABLE PROCESSES. REGULARITY AND LIMIT THEOREM IN BESOV SPACES

BY

B. BOUFOUSSI* (MARRAKESH) AND M. DOZZI* (NANCY)

Abstract. Let $X = (X_z, z \in T_N = [0, 1]^N)$ be a symmetric α-stable process, $1 < \alpha \leqslant 2$. Based on a Kolmogorov type continuity theorem we show Hölder conditions in L^p-norms for the local time of X with respect to the space and time variables, by distinguishing the cases where the time variables do or do not meet the axes. Weak convergence of the occupation integral is proved.

AMS Subject Classification: 60G17, 60G52, 60J55.

Key words and phrases: Local time, symmetric stable process, multiparameter process, anisotropic Besov space, tightness, limit theorem.

1. INTRODUCTION

Let $N \geqslant 1$, $T_N = [0, 1]^N$ and $X = (X_z, z \in T_N)$ be an N-parameter real-valued stochastic process defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. For $R \in \mathscr{B}(T_N)$, the Borel sets in T_N, the occupation measure of the process X on R is defined as follows: For a Borel set $B \subset \boldsymbol{R}$,

$$\nu_R(B) = \int_R I_B(X_\eta)\, d\eta.$$

If the random measure $\nu_{T_N}(\cdot)$ is absolutely continuous with respect to the Lebesgue measure, the same holds for $\nu_R(\cdot)$, for all $R \in \mathscr{B}(T_N)$, and we say that X has a *local time* on T_N, which is the Radon–Nikodym density $L(x, T_N) = d\nu_{T_N}(x)/dx$. In this case $L(x, R)$ exists x-a.e. for any $R \in \mathscr{B}(T_N)$ and the exceptional set depends on R. For $z = (s_1, \ldots, s_N) \in T_N$ and the rectangle

$$R_z = (0, z] = \{\eta = (u_1, \ldots, u_N) \in T_N\colon\ 0 < u_i \leqslant s_i,\ i = 1, \ldots, N\},$$

* Supported by the France-Morocco CNRS/CNRPRST convention, Project SPM/9661, and partially by the "Action Intégrée" programm MA/01/02.

let us define

$$L(x, z) := L(x, R_z).$$

$L(x, z)$ defines a real-valued function of two variables z, x. We know from Geman and Horowitz [18] that if X has a local time, then a version of $L(x, z)$ may be chosen, which defines a kernel on $\boldsymbol{R} \times \mathscr{B}(T_N)$, and such a version is called *regular*.

In the one-parameter case $N = 1$, the theory of local times of stochastic processes was initiated by the work of Paul Lévy [22] for the linear Brownian motion. Trotter [24] proved the first major result on the existence and the Hölder regularity (with respect to the uniform norm) of the local time of the Brownian motion; and much has been discovered since then by many other authors. The Hölder continuity condition in L^p-norm of the Brownian local time was proved by Boufoussi and Roynette [9], and generalized by Boufoussi and Ouknine [8] to the symmetric α-stable process $(1 < \alpha \leqslant 2)$. For N-parameter processes, we cite the powerful Fourier analytic method developed by Berman [2], who observed, in particular, that uniform smoothness of the local time implies uniform irregularity of the associated process. This observation explains an important connection between the behaviour of the process, such as the rate of its growth, and the behaviour of the local time.

The local time of the N-parameter Wiener process W can be defined by the integral of the (one-parameter) local time of the restrictions of W to lines which are parallel to some coordinate axis (cf. Walsh [26] for $N = 2$ and Davydov [14] for general N). A slightly different local time of W can be defined in terms of the N-parameter stochastic calculus (cf. Cairoli and Walsh [11] for $N = 2$ and Imkeller [19] for general N). Berman [3] (cf. also Berman [4], [5]) considered Gaussian processes with stationary increments, which are locally nondeterministic.

In his remarkable work Ehm [17] considered local times of multiparameter stable processes. In particular, he established a law of the iterated logarithm for the supremum of the local time and solved the Hausdorff measure problem for the graph and the range of these processes. For a survey of the theory and for references to other articles we refer to Geman and Horowitz [18], to the cited paper of Ehm [17] and to Dozzi [16].

Our aim in this paper is to prove that local times of real N-parameter symmetric α-stable Lévy processes $(N\alpha > 1)$ satisfy certain Hölder conditions in L^p-norms, which are more precise than the classical Hölder conditions in the uniform norm. This will be done by recalling notions on N-parameter Besov spaces; we use results of Kamont [20], [21] who has proved the characterization of these spaces in terms of the coefficients of the expansion of a continuous function on T_N with respect to a basis which consists of tensor products of Schauder functions. For an introduction to the one-parameter Besov spaces we refer to Bergh and Löfström [1]. The first characterization of these spaces in the L^p-norm with $1 \leqslant p \leqslant \infty$ by coefficients of the expansion in the Faber–Schauder basis has been given by Ciesielski [12] for the Hölder space, and by Ciesielski et al. [13] for the general Besov spaces. These characterizations make the Besov topology easy to handle, and many applications have been given in stochastic calculus such as the regularity of some classical sample paths (of fractional Brownian motion, of symmetric stable processes or of the local times of these processes).

The paper is organized as follows: In Section 2 we recall basic facts about Besov spaces and give the characterization theorem of these spaces which will be used throughout. A Kolmogorov criterion and a tightness one are given in these spaces. In Section 3 we establish inequalities on the moments of the increments of local times of symmetric α-stable processes; our approach consists in integrating the local time of a one-parameter restriction of these processes. As a consequence of Kolmogorov's criterion we prove a regularity result of the local time with respect to its space and its time parameter. Finally we consider the asymptotic behaviour of the occupation integral of these processes generalizing a result of Dozzi [15].

Most of the estimates in this paper contain constants whose value may change from line to line even if the notation of these constants does not change.

2. FUNCTION SPACES

In this section we begin by recalling basic notions on the multidimensional Besov spaces. We will be concerned with the anisotropic case only and adopt the notation of Kamont [21] to give a presentation and a characterization of these spaces.

2.1. Definitions. For an integer $N \geqslant 1$, let $T_N = [0, 1]^N$ and let $D = \{1, 2, \ldots, N\}$. Given a vector $\bar{a} = (a_1, a_2, \ldots, a_N) \in \boldsymbol{R}^N$, we write $|a|$ for $|a_1| + \ldots + |a_N|$. Moreover, T_N will be endowed with the partial order: for $\bar{a}, \bar{b} \in \boldsymbol{R}^N$, we write $\bar{a} \leqslant \bar{b}$ iff $a_i \leqslant b_i$ for all $i \in D$. We also use the notation $\bar{\lambda} = (\lambda, \lambda, \ldots, \lambda) \in \boldsymbol{R}^N$ for a real λ, and the sum $\sum_{\bar{j} \geqslant \bar{0}}$ will mean the sum $\sum_{j_1 \geqslant 0} \sum_{j_2 \geqslant 0} \ldots \sum_{j_N \geqslant 0}$.

$L^p(T_N)$, $1 \leqslant p < \infty$, is the class of functions whose p-th power is integrable on T_N, and $\mathscr{C}(T_N)$ is the space of continuous functions on T_N.

For f: $T_N \to \boldsymbol{R}$, $i \in D$ and $h \in \boldsymbol{R}$, the *progressive difference* of f of order one and in the direction e_i (where e_i means the i-th coordinate vector in $\boldsymbol{R}^N$) is defined by

$$(2.1) \qquad \Delta_{h,i} f(\bar{x}) = \begin{cases} f(\bar{x} + he_i) - f(\bar{x}) & \text{if } \bar{x},\ \bar{x} + he_i \in T_N, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $\bar{h} = (h_1, \ldots, h_N) \in \mathbf{R}^N$, $1 \leq k \leq N$ and $A = \{i_1, \ldots, i_k\} \subset D$, we define the progressive difference in direction A by

$$\Delta_{\bar{h},A} f = \Delta_{h_{i_1},i_1} \circ \ldots \circ \Delta_{h_{i_k},i_k} f, \tag{2.2}$$

where $\Delta_{\bar{h},A} f = f$ if $A = \emptyset$.

For a Borel function $f \in L^p(T_N)$ if $1 \leq p < \infty$ or $f \in \mathscr{C}(T_N)$ if $p = \infty$, the *smoothness* can be measured by its modulus of continuity computed in $L^p(T_N)$-norm and in the direction A. It is given by

$$\omega_{p,A}(f, \bar{t}) = \sup_{\bar{0} \leq \bar{h} \leq \bar{t}} \|\Delta_{\bar{h},A} f\|_p \quad \text{for } \bar{t} \in \mathbf{R}_+^N.$$

Let $b \in \mathbf{R}$ and $\bar{a} = (a_1, \ldots, a_N)$, where $a_i > 0$, $i \in D$. We will consider the real-valued function defined on T_N by

$$\omega_{\bar{a},b}(\bar{t}) = \prod_{i=1}^{N} t_i^{a_i} \left(1 + \sum_{i=1}^{N} \log \frac{1}{t_i}\right)^b.$$

The function $\omega_{\bar{a},b}$ is introduced here to measure the smoothness of a function f with respect to any direction A. More precisely, let

$$\omega_{\bar{a},b}(\bar{t}, A) = \prod_{i \in A} t_i^{a_i} \left(1 + \sum_{i \in A} \log \frac{1}{t_i}\right)^b \quad \text{for } \emptyset \neq A \subset D$$

and $\omega_{\bar{a},b}(\bar{t}, \emptyset) = 1$.

Definition 2.1. The *anisotropic Besov space associated with the modulus* $\omega_{\bar{a},b}$ is defined by

$$\mathrm{Lip}_p(\bar{a}, b)(T_N) = \left\{ f \in L^p(T_N)\colon \sum_{A \subset D} \sup_{\bar{t} > \bar{0}} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{\bar{a},b}(\bar{t}, A)} < +\infty \right\}.$$

$\mathrm{Lip}_p(\bar{a}, b)(T_N)$ endowed with the norm

$$\|f\|_p^{\bar{a},b} = \sum_{A \subset D} \sup_{\bar{t} > \bar{0}} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{\bar{a},b}(\bar{t}, A)}$$

is a non-separable Banach space containing a separable Banach subspace:

$$\mathrm{lip}_p^*(\bar{a}, b)(T_N) = \left\{ f \in L^p(T_N)\colon \forall (\emptyset \neq A \subset D) \lim_{\delta_A(\bar{t}) \to 0} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{\bar{a},b}(\bar{t}, A)} = 0 \right\},$$

where we have used the notation $\delta_A(\bar{t}) = \min\{t_i, i \in A\}$. As for the case $N = 1$ (see Ciesielski et al. [13]), these spaces are linearly isomorphic to some sequence spaces. In the next section we recall the characterization of the anisotropic Hölder class in L^p-norms.

Remark 2.1. Note that for $b = 0$ and $p = \infty$ the Besov space $\mathrm{Lip}_\infty(\bar{a}, 0)(T_N)$ is exactly the space of functions satisfying a classical Hölder condition (in each direction) with respect to the modulus $\omega_{\bar{a},0}(\bar{t}) = \prod_{i=1}^{N} t_i^{a_i}$.

2.2. The characterization of function spaces. The family of Schauder functions on [0, 1] is defined by

$$\varphi_0 = 1_{[0,1]}, \quad \varphi_1(s) = s1_{[0,1]}(s),$$

$$\varphi_n(s) = \varphi(2^{j+1}s-2k+1) \quad \text{for } n = 2^j+k,\ j \in N \text{ and } k = 1, \ldots, 2^j,$$

where $\varphi(u) := \max(0, 1-|u|)$. It is well known that $\{\varphi_n, n \geqslant 1\}$ is a Schauder basis of the Banach space of continuous functions on [0, 1], satisfying nice support properties which are fundamental in the characterization of Besov spaces in terms of sequence spaces. In the multidimensional case it was shown (cf. Kamont [20], [21]) that the tensor product of Schauder functions plays a similar role. For $\bar{n} = (n_1, n_2, \ldots, n_N)$ we will consider

$$\varphi_{\bar{n}} = \varphi_{n_1} \otimes \varphi_{n_2} \otimes \ldots \otimes \varphi_{n_N}.$$

We also need to consider the following dyadic decomposition of T_N. For $j \in M = N \cup \{-2, -1, 0\}$,

$$K_j = \begin{cases} \{j+2\} & \text{for } j = -2 \text{ or } j = -1, \\ \{2^j+k,\ k = 1, 2, \ldots, 2^j\} & \text{for } j \geqslant 0. \end{cases}$$

Now, for a vector $\bar{j} = (j_1, \ldots, j_N) \in M^N$, we put

$$\mathcal{K}_{\bar{j}} = K_{j_1} \times \ldots \times K_{j_N}.$$

One can show that for a function $f \in \mathscr{C}(T_N)$ we have the following decomposition:

$$f = \sum_{\bar{j} \in M^N} \sum_{\bar{n} \in \mathcal{K}_{\bar{j}}} C_{\bar{n}}(f)\, \varphi_{\bar{n}},$$

where for $\bar{n} = (n_1, n_2, \ldots, n_N)$

$$C_{\bar{n}}(f) = C_{1,n_1} \circ \ldots \circ C_{N,n_N}(f)$$

and for $\bar{x} \doteq (x_1, x_2, \ldots, x_N)$

$$C_{i,n}(f)(\bar{x}) = \begin{cases} f(\bar{x} - x_i e_i) & \text{if } n = 0, \\ f(\bar{x} + (1-x_i) e_i) - f(\bar{x} - x_i e_i) & \text{if } n = 1, \end{cases}$$

and for $n = 2^j+k$, with $j \geqslant 0$ and $k = 1, \ldots, 2^j$,

$$C_{i,n}(f)(\bar{x}) = f\left(\bar{x} + \left(\frac{2k-1}{2^{j+1}} - x_i\right) e_i\right) - \frac{f\left(\bar{x} + ((2k-2)/2^{j+1} - x_i) e_i\right) + f\left(\bar{x} + (2k/2^{j+1} - x_i) e_i\right)}{2}.$$

Now, for $\bar{j} \in M^N$ and $0 < p < \infty$, we will define

$$\Pi_{\bar{j},p}(f) = 2^{-|\bar{j}|/p} \Big(\sum_{\bar{n} \in \mathcal{X}_{\bar{j}}} |C_{\bar{n}}(f)|^p \Big)^{1/p}.$$

We will use the following result:

THEOREM 2.1. *Let $\bar{a} = (a_1, \ldots, a_N)$ be such that $\bar{0} < \bar{a} < \bar{1}$ and $b \in \mathbf{R}$. For $\bar{j} \in M^N$, we put*

$$t_{\bar{j}} = (2^{-j_1 \vee 0}, \ldots, 2^{-j_N \vee 0}).$$

Let $1 \leqslant p \leqslant \infty$ be such that $\min_{1 \leqslant i \leqslant N} a_i > 1/p$. Then $\mathrm{Lip}_p(\bar{a}, b)(T_N)$ is a space of continuous functions linearly isomorphic to a sequence space and we have the following equivalence:

$$f \in \mathrm{Lip}_p(\bar{a}, b)(T_N) \text{ iff } \Pi_{\bar{j},p}(f) = \mathcal{O}\big(\omega_{(\bar{a},b)}(t_{\bar{j}})\big) \quad \text{as } |\bar{j}| \to \infty,$$

$$f \in \mathrm{lip}_p^*(\bar{a}, b)(T_N) \text{ iff } \Pi_{\bar{j},p}(f) = o\big(\omega_{(\bar{a},b)}(t_{\bar{j}})\big) \quad \text{as } |\bar{j}| \to \infty.$$

The proof of the particular case $b = 0$ has been given by Kamont [20], and the general case follows the same ideas (cf. Kamont [21]).

Remark 2.2. Let $b \in \mathbf{R}$ and $\bar{0} < \bar{a} < \bar{1}$. Using Theorem 2.1, we can easily check that for any $\bar{0} < \bar{a} < \bar{a}_0$, $b \in \mathbf{R}$ and p large enough we have the following continuous injections:

$$\mathrm{Lip}_\infty(\bar{a}_0, b)(T_N) \subsetneq \mathrm{Lip}_p(\bar{a}_0, b)(T_N) \subsetneq \mathrm{lip}_\infty^*(\bar{a}, 0)(T_N). \tag{2.3}$$

We can also find in Kamont [21] an important application of Theorem 2.1, which consists in studying a local property of the fractional anisotropic Wiener field. More precisely, let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$, $\bar{0} < \bar{\alpha} < \bar{2}$, $W^{\bar{\alpha}} = (W_z^{\bar{\alpha}}, z \in T_N)$ be a centered Gaussian field with the covariance kernel

$$EW^{\bar{\alpha}}(z)\, W^{\bar{\alpha}}(z') = K^{\bar{\alpha}}(z, z'),$$

where

$$K^{\bar{\alpha}} = K^{\alpha_1} \otimes \ldots \otimes K^{\alpha_N}$$

and

$$K^{\alpha_i}(s, t) = \tfrac{1}{2}\{|s|^{\alpha_i} + |t|^{\alpha_i} - |s-t|^{\alpha_i}\}, \qquad s, t \in [0, 1].$$

Kamont has proved the following regularity theorem:

THEOREM 2.2. *For any $2 < p < \infty$,*

$$\mathbf{P}[W^{\bar{\alpha}} \in \mathrm{Lip}_p(\bar{\alpha}/2, 0)] = 1, \qquad \mathbf{P}[W^{\bar{\alpha}} \in \mathrm{lip}_p^*(\bar{\alpha}/2, 0)] = 0,$$

$$\mathbf{P}[W^{\bar{\alpha}} \in \mathrm{Lip}_\infty(\bar{\alpha}/2, \tfrac{1}{2})] = 1, \qquad \mathbf{P}[W^{\bar{\alpha}} \in \mathrm{lip}_\infty^*(\bar{\alpha}/2, \tfrac{1}{2})] = 0,$$

where $\bar{\alpha}/2 = (\alpha_1/2, \ldots, \alpha_N/2)$.

Note that $N = 2$ and $\bar{\alpha} = (1, 1)$ corresponds to the case of the Brownian sheet. The one-parameter case $N = 1$ is discussed in Ciesielski et al. [13].

2.3. Kolmogorov criterion. Let us denote by $\mathscr{R}$ the class of all N-parameter rectangles $R \subset T_N$ of the type

$$R = (\bar{s}, \bar{t}] = \mathop{\times}_{i=1}^{N} (s_i, t_i] \quad (0 \leqslant s_i < t_i,\ 1 \leqslant i \leqslant N).$$

Given a function $f\colon T_N \to \boldsymbol{R}$, the "increment" $f(R)$ of f over $R \in \mathscr{R}$ is defined in the same way as if f were a distribution, i.e. $f(R) = \int_R df$. More precisely, if $R = (\bar{s}, \bar{s}+\bar{h}]$, $\bar{h} = (h_1, \ldots, h_N) \in \boldsymbol{R}^N$, then

$$f(R) = \Delta_{h_1,1} \circ \ldots \circ \Delta_{h_N,N} f(\bar{s}),$$

where $\Delta_{h_i,i}$ is defined as in (2.1). Let us consider a direction which is a subset $A = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, N\}$. We put

$$R^A = \mathop{\times}_{i \in A} (s_i, s_i + h_i].$$

Let $A^c = \{j \in D\colon j \notin A\}$ and let us fix a family $x_A = \{x_j \in \{0, 1\}\colon j \in A^c\}$. For any function $f\colon [0, 1]^N \to \boldsymbol{R}$, we consider a function $f^{x_A}\colon [0, 1]^k \to \boldsymbol{R}$ defined by $f^{x_A}(z) = f(z^{x_A})$, where if $z = (s_{i_1}, \ldots, s_{i_k}) \in [0, 1]^k$, then $z^{x_A} = (\tilde{s}_1, \ldots, \tilde{s}_N) \in [0, 1]^N$ is such that $\tilde{s}_j = s_j$ for $j \in A$ and $\tilde{s}_j = x_j$ otherwise. We have the following Kolmogorov criterion in the space $\mathrm{Lip}_p(\bar{a}, b)(T_N)$.

LEMMA 2.1. *Let $(Y_z, z \in T)$ be an adapted continuous real-valued process satisfying the following assumption:*

There exists $\bar{a} = (a_1, \ldots, a_N)$, $a_i > 0$, $i \in D$, and for all $p \geqslant 1$ there exists a constant $C_p > 0$ such that for any rectangle $R = \times_{i=1}^{N} (s_i, s_i + h_i] \subset T_N$ we have for any direction $A = \{i_1, \ldots, i_k\} \subset D$ and for any choice of a family x_A of $N-k$ elements taken of the set $\{0, 1\}$:

$$E\,|Y^{x_A}(R^A)|^p \leqslant C_p \prod_{i \in A} |h_i|^{a_i p}.$$

Then for every $b > 0$ and $p > (N/b) \vee \max_i a_i^{-1}$ there exists a constant $C(p, b) > 0$ such that

$$\forall \lambda > 0,\ \boldsymbol{P}\{\|Y\|_p^{\bar{a},b} > \lambda\} \leqslant C(p, b)\, \lambda^{-p}.$$

In particular:

$$\boldsymbol{P}\{Y \in \mathrm{lip}_p^*(\bar{a}, b)(T_N)\} = 1.$$

Proof. Let $\lambda > 0$. By Theorem 2.1 one can use the following equivalence of norms:

$$\|Y\|_p^{\bar{a},b} \sim \sup_{\bar{j} \in M^N} \frac{\Pi_{\bar{j},p}(Y)}{\omega_{(\bar{a},b)}(t_{\bar{j}})},$$

and then we get

$$P\{\|Y\|_p^{\bar{a},b} > \lambda\} = P\left\{\sup_{\bar{j}\in M^N} \frac{\Pi_{\bar{j},p}(Y)}{\omega_{(\bar{a},b)}(t_{\bar{j}})} > \lambda\right\}$$
$$\leqslant \sum_{\bar{j}\in M^N} P\left\{\frac{2^{-|\bar{j}|/p}}{\omega_{(\bar{a},b)}(t_{\bar{j}})}\Big(\sum_{\bar{n}\in\mathcal{X}_{\bar{j}}} |C_{\bar{n}}(Y)|^p\Big)^{1/p} > \lambda\right\}$$
$$\leqslant \sum_{\bar{j}\in M^N} \frac{2^{-|\bar{j}|}}{(\omega_{(\bar{a},b)}(t_{\bar{j}}))^p} \sum_{\bar{n}\in\mathcal{X}_{\bar{j}}} E\,|C_{\bar{n}}(Y)|^p\,\lambda^{-p}.$$

Notice that the coefficients $C_{\bar{n}}(Y)$ are just the increments of Y on a dyadic rectangle of area proportional to $2^{-|\bar{j}|}$. Then, by the assumption of the lemma above, we get

$$P\{\|Y\|_p^{\bar{a},b} > \lambda\} \leqslant C_p\lambda^{-p} \sum_{\bar{j}\in M^N} \left(\frac{2^{-\sum_i a_i j_i}}{\omega_{(\bar{a},b)}(t_{\bar{j}})}\right)^p \tag{2.4}$$
$$\leqslant C_p\lambda^{-p} \sum_{\bar{j}\geqslant\bar{0}} \frac{1}{(1+|\bar{j}|\log 2)^{bp}},$$

and the series converges since $bp > N$. This implies in particular that

$$P\{Y\in \mathrm{Lip}_p(\bar{a}, b)(T_N)\} = 1.$$

Now, let $0 < b < b_0$. Using Theorem 2.1 one can show that the following continuous injection holds:

$$\forall p > \max_i a_i^{-1},\ \mathrm{Lip}_p(\bar{a}, b)(T_N) \hookrightarrow \mathrm{lip}_p^*(\bar{a}, b_0)(T_N). \tag{2.5}$$

Then for any $b > 0$ and $p > \max_i a_i^{-1} \vee (N/b)$ we have

$$P\{Y\in \mathrm{lip}_p^*(\bar{a}, b)(T_N)\} = 1. \ \blacksquare$$

2.4. Tightness criterion in Besov spaces. Let $\bar{t} = (t_1, \ldots, t_N)\in T_N$, $\delta > 0$ and $A = \{i_1, \ldots, i_k\} \subset D$. Let $\delta_A(\bar{t}) = \inf\{t_{i_1}, \ldots, t_{i_k}\}$ and

$$K_{\bar{a},b}(\delta, p, f) = \sum_{\substack{A\subset D\\ A\neq\emptyset}} \sup_{\bar{t}:\delta_A(\bar{t})\leqslant\delta} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)}.$$

LEMMA 2.2. *Let $1 \leqslant p < \infty$ and $b\in\mathbf{R}$ be such that $p > \max_i a_i^{-1} \vee (N/b)$. A bounded set K of $\mathrm{Lip}_p(\bar{a}, b)(T_N)$ is relatively compact if*

$$\limsup_{\delta\to 0}\, \sup_{f\in K} K_{\bar{a},b}(\delta, p, f) = 0.$$

The proof of the two-parameter case of Lemma 2.2 is given in Boufoussi and Lakhel [7]. The proof of the general case follows the same techniques, but we present it here for the sake of completeness.

Proof. Note that K is clearly bounded in $L^p(T_N)$. Since $\mathrm{Lip}_p(\bar{a}, b)(T_N)$, $p > \max_i a_i^{-1}$ is a space of continuous functions on T_N, it follows from the Fréchet–Kolmogorov theorem (see Brézis [10], p. 72) that K is relatively compact in $L^p(T_N)$. This enables us to extract from each sequence $(f_n)_{n\geqslant 1}$ of K a subsequence $(f_{n_k})_{k\geqslant 1}$ which converges to f in $L^p(T_N)$. Now it is enough to see that:

(i) f is in $\mathrm{Lip}_p(\bar{a}, b)(T_N)$;

(ii) $(f_{n_k})_{k\geqslant 1}$ is a Cauchy sequence in $\mathrm{Lip}_p(\bar{a}, b)(T_N)$.

Since $f_{n_k} \to f$ in L^p-norm as $k \to \infty$, we can take a subsequence, also denoted by f_{n_k}, which converges a.e. to f. Fatou's lemma implies that for any set $A \subset D$ we have

$$\omega_{p,A}(f, \bar{t}) \leqslant \sup_{n_k} \omega_{p,A}(f_{n_k}, \bar{t}). \tag{2.6}$$

Then

$$\|f\|_p^{\bar{a},b} \leqslant \sup_{g\in K} \|g\|_p^{\bar{a},b} < \infty.$$

On the other hand, by (2.6) and the second assumption in Lemma 2.2, we get for all $A \subset D$

$$\lim_{\delta_A(\bar{t})\to 0} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} = 0,$$

which proves that f is in the separable space $\mathrm{lip}_p^*(\bar{a}, b)(T_N)$.

Now we turn to the proof of (ii). Let $n, n' \geqslant 0$; we have

$$\|f_n - f_{n'}\|_p^{\bar{a},b} = \sum_{A\subset D} \sup_{\bar{0}<\bar{t}\leqslant\bar{1}} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)}.$$

If $A = \emptyset$, then

$$\frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} = \|f_n - f_{n'}\|_p \to 0 \quad \text{as } n, n' \to \infty.$$

For $\delta > 0$, we have

$$\sup_{\bar{t}\leqslant\bar{1}} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} \leqslant \sup_{\delta_A(\bar{t})\leqslant\delta} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} + \sup_{\delta_A(\bar{t})>\delta} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)}.$$

Then

$$\begin{aligned}
\|f_n - f_{n'}\|_p^{\bar{a},b} &\leqslant \|f_n - f_{n'}\|_p + \sum_{\substack{A\subset D\\ A\neq\emptyset}} \sup_{\delta_A(\bar{t})\leqslant\delta} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} + \sum_{\substack{A\subset D\\ A\neq\emptyset}} \sup_{\delta_A(\bar{t})>\delta} \frac{\omega_{p,A}(f_n - f_{n'}, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} \\
&\leqslant \|f_n - f_{n'}\|_p + K_{\bar{a},b}(\delta, p, f_n - f_{n'}) + \|f_n - f_{n'}\|_p \sum_{\substack{A\subset D\\ A\neq\emptyset}} \frac{1}{\omega_{(\bar{a},b)}(\bar{\delta}, A)} \\
&\leqslant \|f_n - f_{n'}\|_p \Big(1 + \sum_{\substack{A\subset D\\ A\neq\emptyset}} \delta^{-\sum_{i\in A} a_i} \Big(1 + |A| \log\frac{1}{\delta}\Big)^{-|A|b}\Big) + K_{\bar{a},b}(\delta, p, f_n - f_{n'}).
\end{aligned}$$

Then $(f_{n_k})_{k\geqslant 1}$ is clearly a Cauchy sequence in $\mathrm{Lip}_p(\bar{a}, b)(T_N)$, which completes the proof of Lemma 2.2. ∎

As a consequence we will prove the following compact injection.

LEMMA 2.3. *Let $\bar{a} > \bar{0}$ and $b < b'$. Then, for any $p > \max_i a_i^{-1} \vee (N/b)$, the following injection is compact:*

$$\mathrm{Lip}_p(\bar{a}, b)(T_N) \hookrightarrow \mathrm{Lip}_p(\bar{a}, b')(T_N).$$

Proof. By the continuous injection (2.5), it is clear that any bounded subset K of $\mathrm{Lip}_p(\bar{a}, b)(T_N)$ is bounded in $\mathrm{Lip}_p(\bar{a}, b')(T_N)$. It is enough to show that

$$\lim_{\delta\to 0} \sup_{f\in K} K_{\bar{a},b'}(\delta, p, f) = 0.$$

We have

$$K_{\bar{a},b'}(\delta, p, f) = \sum_{\substack{A\subset D\\ A\neq\emptyset}} \sup_{\delta_A(\bar{t})\leqslant\delta} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{(\bar{a},b')}(\bar{t}, A)} \tag{2.7}$$

$$= \sum_{\substack{A\subset D\\ A\neq\emptyset}} \sup_{\delta_A(\bar{t})\leqslant\delta} \frac{\omega_{p,A}(f, \bar{t})}{\omega_{(\bar{a},b)}(\bar{t}, A)} \frac{\omega_{(\bar{a},b)}(\bar{t}, A)}{\omega_{(\bar{a},b')}(\bar{t}, A)}.$$

Remark that

$$\frac{\omega_{(\bar{a},b)}(\bar{t}, A)}{\omega_{(\bar{a},b')}(\bar{t}, A)} = \left(1+\sum_{i\in A}\log\frac{1}{t_i}\right)^{b-b'}.$$

Recall that $\bar{0} < \bar{t} \leqslant \bar{1}$, and since $b-b' < 0$, we get

$$\frac{\omega_{(\bar{a},b)}(\bar{t}, A)}{\omega_{(\bar{a},b')}(\bar{t}, A)} \leqslant \left(1+c(A)\log\frac{1}{\delta}\right)^{b-b'}, \tag{2.8}$$

where $c(A)$ is the number of $i\in A$ such that $t_i \leqslant \delta$. Now, from (2.7) and (2.8) we have

$$\sup_{f\in K} K_{\bar{a},b'}(\delta, p, f) \leqslant \left(1+\log\frac{1}{\delta}\right)^{b-b'} \sup_{f\in K} \|f\|_p^{\bar{a},b}.$$

Since $b-b' < 0$, the term on the right-hand side goes to 0 as $\delta \to 0$, and this completes the proof of the lemma. ∎

Remark 2.3. Note that the continuous injection in (2.5) is also compact.

Now we are able to give a tightness criterion in Besov spaces. With the same notation as in Lemma 2.1 we have

LEMMA 2.4. *Let $(X_n(z), z\in T_N)_{n\geqslant 1}$ be a sequence of continuous adapted processes satisfying the following condition:*

There exists $\bar{a} = (a_1, \ldots, a_N)$, $a_i > 0$ for all $i \in \{1, \ldots, N\}$, and for all $p \geqslant 1$ there exists a constant $C_p > 0$ such that for any rectangle $R = \times_{i=1}^{N} (s_i, s_i + h_i] \subset T_N$ we have for any direction $A = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$ and for any choice of the family x_A of $N-k$ elements taken in $\{0, 1\}$:

$$E\,|X_n^{x_A}(R^A)|^p \leqslant C_p \prod_{i \in A} |h_i|^{a_i p} \quad \text{for all } n \in \mathbf{N}.$$

Then X^n is tight in $\mathrm{lip}_p^*(\bar{a}, b)(T_N)$ for any $p > \max_i a_i^{-1} \vee (N/b)$.

Proof. As in (2.4) we have

$$P\{\|X_n\|_p^{\bar{a},b} > \lambda\} \leqslant C_p \lambda^{-p} \sum_{\bar{j} \geqslant \bar{0}} \frac{1}{(1 + |\bar{j}| \log 2)^{bp}}.$$

By virtue of Lemma 2.3 and Remark 2.3 we can choose λ large enough to get the desired result. ∎

As an application of the above lemma we have the following approximation result obtained by Nualart [23] in the space of continuous functions (see also Yor [27]):

THEOREM 2.3. *Let $(B_s^n, s \in [0, 1])_{n \in \mathbf{N}}$ and $(C_t^n, t \in [0, 1])_{n \in \mathbf{N}}$ be two families of standard linear independent Brownian motions. Then for any $b > 0$ and $p > 2/b$ the sequence of processes*

$$W_{s,t}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} B_s^j C_t^j$$

converges weakly to the Brownian sheet in the separable Banach spaces $\mathrm{lip}_p^*((\frac{1}{2}, \frac{1}{2}), b)$.

Proof. We apply the tightness criterion (Lemma 2.4) with $N = 2$. Since $W_{\cdot,0}^n = W_{0,\cdot}^n = 0$, it is enough to estimate the moments of the increments of W^n on rectangles. So let $R = [s, s+h] \times [t, t+k] \subset T$, $h, k \in \mathbf{R}$, and let $1 < p < \infty$. Then

$$E\,|W^n(R)|^p = |hk|^{p/2} E\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \eta_i\right]^p,$$

where $\{\xi_i, \eta_i, i \geqslant 1\}$ is a double sequence of independent identically distributed random variables with common law $\mathcal{N}(0, 1)$. There exists a constant $C_p > 0$ such that

$$\begin{aligned} E\,|W^n(R)|^p &\leqslant C_p |hk|^{p/2} E\,[n^{-1} \sum_{i=1}^{n} \xi_i^2]^{p/2} \\ &\leqslant C_p |hk|^{p/2} n^{-1} \sum_{i=1}^{n} E\,|\xi_i|^p \leqslant C_p' |hk|^{p/2}. \end{aligned}$$

This gives the tightness of distributions of W^n in $\mathrm{lip}_p^*(\bar{a}, \beta)$ for all $\beta > 0$,

$p > (2/\beta) \vee 2$ and $\bar{a} = (\frac{1}{2}, \frac{1}{2})$. Finally, recall that $\mathrm{lip}_p^*((\frac{1}{2}, \frac{1}{2}), \beta)$ is also a space of continuous functions. The convergence of finite-dimensional distributions can be deduced from the result of Yor [27] or Nualart [23] who has proved the weak convergence $W^n \stackrel{\mathscr{L}}{\Rightarrow} W$ as $n \to \infty$ in the space of continuous functions $C([0, 1]^2)$. ∎

3. LOCAL TIME OF N-PARAMETER SYMMETRIC STABLE PROCESSES

3.1. Regularity. Let $1 < \alpha \leqslant 2$ and let $(X_z, z \in T = [0, 1]^N)$ be a symmetric α-stable process, that is, for any rectangle $R \subset \boldsymbol{R}_+^N$ the random variable $X(R)$ is symmetric α-stable with

$$E \exp i\gamma X(R) = \exp\{-c\lambda(R)|\gamma|^\alpha\},$$

where $\gamma \in \boldsymbol{R}$, c is a real constant, and λ is the Lebesgue measure on $\boldsymbol{R}^N$.

In the work of Ehm [17], regularity properties of the local time of X have been studied, and relations with be behaviour of the sample paths of X have been pointed out (such as the dimension of the level sets $N_x = \{z\colon X_z = x\}$). In particular, he proved the existence of a jointly continuous version of the local time and Hölder conditions for its trajectories by means of the Fourier analytic approach. Our aim in this section is to study the regularity properties of the trajectories of the local time from the point of view of Besov spaces.

3.1.1. *Local time of the Brownian sheet.* Let us consider $(W_z, z \in [0, 1]^2)$, the two-parameter Brownian sheet (with values in $\boldsymbol{R}$). The local time of W has been constructed as the Radon–Nikodym derivative of the occupation measure, and via integration of local times on lines (cf. Walsh [26]), or directly by using the Fourier analytic approach due to Berman [2]–[5]. For fixed $t > 0$, we consider the rescaled one-parameter Brownian motion $(W_{s,t}, s \in [0, 1])$. Let us denote its local time by L_1. For $x \in \boldsymbol{R}$ and $t > 0$, we clearly have

$$(L_1(x, s, t), s \geqslant 0) \sim \left(\frac{1}{\sqrt{t}} l\left(\frac{x}{\sqrt{t}}, s\right), s \geqslant 0\right),$$

where $\sim$ means the equivalence in distribution, and l is the local time of the one-parameter Brownian motion. It is clear that $L_1(x, s, t) \to \infty$ in probability as $t \to 0$. The behaviour in Besov topology of L_1 for fixed $t > 0$ has been studied in detail, since it is a Brownian local time. It has been proved in Boufoussi and Roynette [9] that the spatial trajectory of L_1 satisfies a Hölder condition in $L^p([0, 1])$ of order $\frac{1}{2}$ for all $p > 2$. A similar regularity result has been given in Boufoussi and Kamont [6] jointly in the space and the time parameter. Then it is interesting to investigate the behaviour of L_1 in t for fixed x and s.

Walsh [26] has proved that for fixed x and s the trajectories of L_1 satisfy a Hölder condition of order $\gamma < \frac{1}{4}$. He uses the following Tanaka formulae:

$$L_1(x, s, t) = \frac{2}{t}\Big((W_{s,t}-x)^+ - x^- - \int_0^s I_{(W_{u,t}>x)}\,\partial_1 W_{u,t}\Big),$$

where the integral is the Itô integral with respect to the rescaled Brownian motion $W_{u,t}$. For fixed $\varepsilon > 0$ Walsh [26] has proved by means of the Itô calculus for $W_{\cdot,t}$ the following regularity result:

For each $p > 1$, there is a constant $C(p, s, \varepsilon)$ depending only on s, p and ε such that

$$\forall t, t' > \varepsilon: \ |t-t'| < 1, \ E|L_1(x, s, t)-L_1(x, s, t')|^p \leqslant C(p, s, \varepsilon)|t-t'|^{p/4}.$$

Then, using the Kolmogorov criterion in Besov topology (Lemma 2.1 with $N = 1$), we get

THEOREM 3.1. *Let $s > 0$, $0 < \varepsilon < 1$ and $x \in \boldsymbol{R}$. For all $b > 0$ and $p > 4 \vee b^{-1}$*

$$\boldsymbol{P}\{L_1(x, s, \cdot)|_{[\varepsilon,1]} \in \mathrm{lip}_p^*(\tfrac{1}{4}, b)\} = 1,$$

where $L_1(x, s, \cdot)|_{[\varepsilon,1]}$ denotes the restriction of $L_1(x, s, \cdot)$ to the interval $[\varepsilon, 1]$.

Remark 3.1. According to the definition of the Besov space in the case $N = 1$, it follows from Theorem 3.1 that for any $b > 0$ and for any $p > 4 \vee b^{-1}$

$$\omega_p\big(L_1(x, s, \cdot)|_{[\varepsilon,1]}, \delta\big) = o\big(\delta^{1/4}(1+\log\delta^{-1})^b\big) \quad \text{as } \delta \to 0$$

P-a.s. By virtue of (2.3), this establishes in particular that for any fixed x, s, the trajectory $L_1(x, s, \cdot)|_{[\varepsilon,1]}$ satisfies a Hölder condition of order $\alpha < \frac{1}{4}$.

3.1.2. *Local time of symmetric stable processes.* Let $z = (s_1, \ldots, s_N) \in T = [0, 1]^N$, $\bar{h} = (h_1, \ldots, h_N) \in \boldsymbol{R}^N$, $x \in \boldsymbol{R}$ and consider the rectangle $R = (z, z+\bar{h}]$. In order to get bounds for the moments of the increment $L(x, R)$, fix $\tilde{z} = (s_2, \ldots, s_N)$ and consider the rescaled one-parameter α-stable symmetric process $s_1 \to X_{(s_1,\tilde{z})}$. Let $L_1(x, s_1, \tilde{z})$ be a continuous version of its local time. A local time of X may be obtained by integrating L_1:

$$L(x, (0, z]) = L(x, z) = \int_{(0,\tilde{z}]} L_1(x, s_1, \tilde{s})\,d\tilde{s}. \tag{3.1}$$

The importance of this formula lies in the fact that it is comparatively simple to obtain upper bounds for the moments of L_1, which may be used to get bounds on the moments of L by (3.1). By the scaling property, for each $\tilde{z} \in [0, 1]^{N-1}$ and $x \in \boldsymbol{R}$ we get

(3.2) $\big(L_1(x, s_1, \tilde{z}), s_1 \in [0, 1]\big)$ is distributed as $\big(|\tilde{z}|^{-1/\alpha} l(|\tilde{z}|^{-1/\alpha} x, s_1), s_1 \in [0, 1]\big)$, where l is the local time of the one-parameter process

$$s_1 \to X_{(s_1,\tilde{1})} \quad \text{with } \tilde{1} = (\underbrace{1, \ldots, 1}_{N-1}),$$

and $|\tilde{z}| = \lambda([0, \tilde{z}])$.

LEMMA 3.1. *Let $m \in N^*$. There exists a constant $C(m, \alpha)$ depending only on m and α such that*

$$E|L(x, R)|^{2m} \leqslant C(m, \alpha)[\lambda(R)]^{2m(1-1/\alpha)}.$$

Proof. Let $n = 2m$. For $z = (s_1, \ldots, s_N) \in T = [0, 1]^N$ we will put $\tilde{z} = (s_2, \ldots, s_N)$. Let $\bar{h} = (h_1, \ldots, h_N) \in \boldsymbol{R}^N$, $x \in \boldsymbol{R}$ and consider the rectangle $R = (z, z + \bar{h}]$. We will also use the notation $\tilde{h} = (h_2, \ldots, h_N)$ and $\tilde{R} = (\tilde{z}, \tilde{z} + \tilde{h}]$. By (3.1), for any $x \in \boldsymbol{R}$ we have

$$\begin{aligned} E(L(x, R))^n &= E\Big(\int_{\tilde{R}} L_1(x, s_1 + h_1, \tilde{u}) - L_1(x, s_1, \tilde{u})\, d\tilde{u}\Big)^n \\ &= \int_{\tilde{R}^n} E \prod_{j=1}^{n} |L_1(x, s_1 + h_1, \tilde{u}^j) - L_1(x, s_1, \tilde{u}^j)|\, d\tilde{u}^j. \end{aligned}$$

By an application of Hölder's inequality and the scaling property (3.2) we get

$$\begin{aligned} (3.3)\qquad E(L(x, R))^n &\leqslant \int_{\tilde{R}^n} \prod_{j=1}^{n} [E|L_1(x, s_1 + h_1, \tilde{u}^j) - L_1(x, s_1, \tilde{u}^j)|^n]^{1/n} d\tilde{u}^j \\ &\leqslant \int_{\tilde{R}^n} \prod_{j=1}^{n} |\tilde{u}^j|^{-1/\alpha} \big[E\,\big|l(|\tilde{u}^j|^{-1/\alpha} x, s_1 + h_1) - l(|\tilde{u}^j|^{-1/\alpha} x, s_1)\big|^n\big]^{1/n} d\tilde{u}^j \\ &\leqslant \Big[\int_{\tilde{R}} |\tilde{u}|^{-1/\alpha} \big[E\,\big|l(|\tilde{u}|^{-1/\alpha} x, s_1 + h_1) - l(|\tilde{u}|^{-1/\alpha} x, s_1)\big|^n\big]^{1/n} d\tilde{u}\Big]^n. \end{aligned}$$

Applying Berman's method to the one-parameter process (see for example Vares [25]), we get

$$E\,\big|l(|\tilde{u}^j|^{-1/\alpha} x, s_1 + h_1) - l(|\tilde{u}^j|^{-1/\alpha} x, s_1)\big|^n \leqslant C(n, \alpha)\,|h_1|^{n(1-1/\alpha)}.$$

This and (3.3) imply that

$$\begin{aligned} E(L(x, R))^n &\leqslant C(n, \alpha)\,|h_1|^{n(1-1/\alpha)} \Big[\int_{\tilde{R}} |\tilde{u}|^{-1/\alpha} d\tilde{u}\Big]^n \\ &\leqslant C(n, \alpha)\big(\lambda(R)\big)^{n(1-1/\alpha)}. \quad \blacksquare \end{aligned}$$

As a consequence of Lemma 3.1, we get from Lemma 2.1 the following regularity result of the local time in Besov spaces:

THEOREM 3.2. *Let $1 < \alpha \leqslant 2$. For any fixed $x \in \boldsymbol{R}$, we have*

$$P\{L(x, \cdot) \in \mathrm{lip}_p^*(1 - 1/\alpha, b)(T_N)\} = 1$$

for all $b > 0$ and $p > (N/b) \vee \alpha/(\alpha - 1)$.

Remark 3.2. 1. We know that $(L(x, z), z \in T_N)$ satisfies a Hölder condition of order $\eta < 1 - 1/\alpha$, by virtue of the continuous injection (2.3). The regularity result given in Theorem 3.2 for a large parameter p is more precise than the classical one.

2. We have not been able to prove limit regularity results which consist in answering the question: is it true that

$$P\{L(x,\cdot)\in \mathrm{lip}_p^*(1-\alpha^{-1},0)(T_N)\}=0, \quad P\{L(x,\cdot)\in \mathrm{Lip}_p(1-\alpha^{-1},0)(T_N)\}=1?$$

We now turn to the regularity of the local time with respect to the space variable. Two cases will be treated.

a. The first case. Let $z=(s_1,\ldots,s_N)\in(0,1]^N$. We look at the bound of the moments of $L(x,R_z)=L(x,z)$, where $R_z=(0,z]$. We have the following lemma:

LEMMA 3.2. *For all* $0<\eta<(\alpha-1)/2$, $n=2m\in \boldsymbol{N}^*$ *and* $x,h\in \boldsymbol{R}$, *we have*

$$E|L(x+h,z)-L(x,z)|^n \leqslant C(n,\eta,\alpha)\big(\lambda(R_z)\big)^{n(1-(1+\eta)/\alpha)}|h|^{n\eta},$$

where $C(n,\eta,\alpha)$ *is a finite constant.*

Proof. The notation is the same as in the proof of Lemma 3.1. As in (3.3) we have

$$E|L(x+h,z)-L(x,z)|^n \leqslant E\int_{(0,\tilde z]^n}\prod_{j=1}^{n}|L_1(x+h,s_1,\tilde s^j)-L_1(x,s_1,\tilde s^j)|\,d\tilde s^j$$

$$\leqslant \int_{(0,\tilde z]^n}\prod_{j=1}^{n}\big[E|L_1(x+h,s_1,\tilde s^j)-L_1(x,s_1,\tilde s^j)|^n\big]^{1/n}d\tilde s^j.$$

For any fixed $\tilde s\in(0,\tilde z]$ and $x,h\in\boldsymbol{R}$, by the scaling property, the process

$$\big((L_1(x+h,u,\tilde s),L_1(x,u,\tilde s)),\ u\in[0,1]\big)$$

is distributed as

$$\big((|\tilde s|^{-1/\alpha}l(|\tilde s|^{-1/\alpha}(x+h),u),\ |\tilde s|^{-1/\alpha}l(|\tilde s|^{-1/\alpha}x,u)),\ u\in[0,1]\big).$$

Then we get

$$E|L(x+h,z)-L(x,z)|^n$$

$$\leqslant \int_{(0,\tilde z]^n}\prod_{j=1}^{n}|\tilde s^j|^{-1/\alpha}\big[E|l(|\tilde s^j|^{-1/\alpha}(x+h),s_1)-l(|\tilde s^j|^{-1/\alpha}x,s_1)|^n\big]^{1/n}d\tilde s^j.$$

We apply Lemma 1.2 of Ehm [17], which remains true for $Q=[0,s_1]$, $s_1>0$ in case $N=1$ and when the dimension of the space variable is equal to 1: for any $0<\eta<(\alpha-1)/2$,

$$E|l(x,s_1)-l(y,s_1)|^n\leqslant C(n,\eta,\alpha)|x-y|^{n\eta}s_1^{n(1-(1+\eta)/\alpha)}, \tag{3.4}$$

where $C(n,\eta,\alpha)$ is a positive constant. Now (3.4) implies that

$$E|L(x+h,z)-L(x,z)|^n\leqslant C(n,\eta,\alpha)s_1^{n(1-(1+\eta)/\alpha)}|h|^{n\eta}\int_{(0,\tilde z]^n}\prod_{j=1}^{n}|\tilde s^j|^{-(1+\eta)/\alpha}d\tilde s^j.$$

Since $0 < \eta < (\alpha - 1)/2$, the integral on the right-hand side is finite, and we get

$$E|L(x+h, z) - L(x, z)|^n \leqslant C(n, \eta, \alpha)\,\lambda(R_z)^{n(1-(1+\eta)/\alpha)}|h|^{n\eta},$$

which completes the proof of the lemma. ∎

As a consequence of Lemma 3.2 and the result of Lemma 2.1 for the one-parameter case, we have

THEOREM 3.3. *Let $1 < \alpha \leqslant 2$, $0 < \eta < (\alpha-1)/2$ and fix $z \in T_N$. Then for any $b > 0$ and $p > b^{-1} \vee \eta^{-1}$ we have*

$$P\{L(\cdot, z) \in \mathrm{lip}_p^*(\eta, b)(T_1)\} = 1.$$

Remark 3.3. By virtue of (2.3), Theorem 3.3 implies in particular that the spatial trajectory of L satisfies a Hölder condition of order $\eta < (\alpha-1)/2$.

b. The second case. Now let the time parameters be away from the axes. That means, we consider the trajectory $(L(x, R), x \in [0, 1])$, where

$$R \in \mathscr{R}^* = \{R = (z, z']:\ \text{where } z = (s_1, \ldots, s_N),\ z' = (s'_1, \ldots, s'_N)\ \text{and } 0 < s_i < s'_i \leqslant 1\}.$$

In this case we can distinguish two situations:

• If $(N\alpha-1)/2 > 1$, then there is a version of the local time L such that $x \to L(x, R)$, $x \in [0, 1]$, is a.s. continuously differentiable. In this case we will look at the regularity in Besov spaces of the derivative

$$L^{(1)}(x, R) = \frac{\partial}{\partial x} L(x, R).$$

• The case $(N\alpha - 1)/2 \leqslant 1$ contains only the cases $N = 1$ and $N = 2$ since we restrict ourselves to $1 < \alpha \leqslant 2$.

LEMMA 3.3. *Let $1 < \alpha \leqslant 2$, $n = 2m \in \mathbf{N}^*$, and let $x, h \in \mathbf{R}$.*

(1) *Let $(N\alpha-1)/2 \leqslant 1$. For any $0 < \eta < (N\alpha-1)/2$ there exists a finite constant $C(m, N, \eta, \alpha) > 0$ such that, for any rectangle $R = (z, z'] \in \mathscr{R}^*$,*

$$E|L(x+h, R) - L(x, R)|^n \leqslant C(m, N, \eta, \alpha)\,\lambda([0, z])^{-n(1-1/N)(1+\eta)/\alpha}\,\lambda(R)^{n(1-(1+\eta)/N\alpha)}|h|^{n\eta}.$$

(2) *Let $1 < (N\alpha-1)/2 \leqslant 2$. For any $0 < \eta < (N\alpha-1)/2 - 1$ there exists a finite constant $C'(m, N, \eta, \alpha) > 0$ such that, for any rectangle $R = (z, z'] \in \mathscr{R}^*$,*

$$E|L^{(1)}(x+h, R) - L^{(1)}(x, R)|^n \leqslant C'(m, N, \eta, \alpha)\,\lambda([0, z])^{-n(1-1/N)(2+\eta)/\alpha}\,\lambda(R)^{n(1-(2+\eta)/N\alpha)}|h|^{n\eta}.$$

For the proof of this result we refer to Lemma 1.2 in Ehm [17]. As a consequence of Lemma 3.3 and the Kolmogorov criterion in Besov spaces (Lemma 2.1) we get

THEOREM 3.4. *Let* $1 < \alpha \leqslant 2$ *and* $R \in \mathscr{R}^*$.

(1) *If* $(N\alpha-1)/2 \leqslant 1$, *then for* $0 < \eta < (N\alpha-1)/2$ *we have*

$$P\{L(\cdot, R) \in \mathrm{lip}_p^*(\eta, b)(T_1)\} = 1$$

for all $b > 0$ *and* $p > b^{-1} \vee \eta^{-1}$.

(2) *If* $1 < (N\alpha-1)/2 \leqslant 2$, *then for* $0 < \eta < (N\alpha-1)/2-1$ *we have*

$$P\{L^{(1)}(\cdot, R) \in \mathrm{lip}_p^*(\eta, b)(T_1)\} = 1$$

for all $b > 0$ *and* $p > b^{-1} \vee \eta^{-1}$.

Remark 3.4. 1. Analogous results with higher order derivatives can be formulated for $k < (N\alpha-1)/2 \leqslant k+1$, $k \geqslant 2$.

2. In the case $N = 1$ and for $1 < \alpha \leqslant 2$ it has been proved in Boufoussi and Ouknine [8] that for any fixed $z \in [0, 1]$ and any $p \geqslant 1/\alpha$

$$P\{L(\cdot, z) \in \mathrm{Lip}_p((\alpha-1)/2, 0)(T_1)\} = 1. \tag{3.5}$$

Observe that (3.5) is stronger than the result given in Theorem 3.4. The reason is that the proof of (3.5) is not based on Kolmogorov's criteria, but it uses a classical Dynkin isomorphism theorem relating the sample path properties of the local time of X to those of an associated Gaussian process which is easy to handle. Note also that (3.5) generalizes the case of the Brownian motion ($N = 1$ and $\alpha = 2$) studied by Boufoussi and Roynette [9]. They use direct calculations and prove a more precise regularity result, which reads as follows: for any fixed $z \in [0, 1]$ and for any $p > 1$,

$$P\{L(\cdot, z) \in \mathrm{Lip}_p(\tfrac{1}{2}, 0)(T_1)\} = 1, \qquad P\{L(\cdot, z) \in \mathrm{lip}_p^*(\tfrac{1}{2}, 0)(T_1)\} = 0.$$

3. If $N > 2$ and the time parameters are taken away from the axes (in the sense that $R \in \mathscr{R}^*$), then Theorem 3.4 says that the trajectory $x \to L(x, R)$ is a.s. much more regular than that stated in Theorem 3.3. Notice that in this case the limiting situation $\eta = (\alpha-1)/2$ in Theorem 3.3 is reached.

3.2. Limit theorem. Let f: $\boldsymbol{R} \to \boldsymbol{R}$ be a bounded measurable function with compact support, and let $(X_z, z \in T_N)$ be a real symmetric stable process with index $1 < \alpha \leqslant 2$. For $z = (s_1, \ldots, s_N)$ we will note $\lambda z = (\lambda s_1, \ldots, \lambda s_N)$. Our goal in this section is to give a functional limit of

$$\Big(\lambda^{N(1/\alpha-1)} \int_{[0,\lambda z]} f(X_u)\, du,\ \lambda > 0\Big) \quad \text{as } \lambda \to \infty,$$

with respect to the Besov topology. We have

THEOREM 3.5. *Let* $b > 0$. *For all* $p > (N/b) \vee \alpha/(\alpha-1)$ *we have*

$$\lambda^{N(1/\alpha-1)} \int_{[0,\lambda z]} f(X_u)\, du \overset{\mathscr{L}}{\Rightarrow} \int f(x)\, dx\, L(0, z) \quad \text{as } \lambda \to \infty$$

in the Besov space $\mathrm{lip}_p^*(1-1/\alpha, b)(T_N)$, *where* $\overset{\mathscr{L}}{\Rightarrow}$ *means the weak convergence.*

Proof. Let $\lambda \in \boldsymbol{R}_+$. Using the scaling property $X_{\lambda z} \sim \lambda^{N/\alpha} X_z$, by an obvious change of variables we get

$$\Big(\lambda^{N(1/\alpha-1)} \int_{[0,\lambda z]} f(X_u)\,du,\ z \in T_N\Big) \sim \Big(\lambda^{N/\alpha} \int_{[0,z]} f(\lambda^{N/\alpha} X_u)\,du,\ z \in T_N\Big).$$

Now, using the occupation formula and again changing the variables, we obtain

$$\lambda^{N/\alpha} \int_{[0,z]} f(\lambda^{N/\alpha} X_u)\,du = \lambda^{N/\alpha} \int f(\lambda^{N/\alpha} x)\, L(x, z)\,dx = \int f(x)\, L(\lambda^{-N/\alpha} x, z)\,dx.$$

Since f is bounded and of compact support, the process $\int f(x) L(\lambda^{-N/\alpha} x, z)\,dx$ a.s. converges uniformly (in z) to $\int f(x)\,dx\, L(0, z)$. Now, since, for $p > (N/b) \vee \alpha/(\alpha-1)$, $\mathrm{lip}_p^*(1-1/\alpha, b)(T_N)$ is a space of continuous functions, it is enough to show that the family of processes

$$X_z^\lambda = \int f(x)\, L(\lambda^{-N/\alpha} x, z)\,dx, \qquad \lambda > 0,$$

is tight in $\mathrm{lip}_p^*(1-1/\alpha, b)(T_N)$, and this can be done by using Lemma 2.4. To this end let $R = [z, z'] \subset T_N$. Since f is bounded and of compact support, one can easily show, with the estimate of Lemma 3.1, that

$$E\big(X^\lambda(R)\big)^{2m} \leqslant C(m)\, \lambda\, (R)^{2m(1-1/\alpha)} \quad \text{for all } m \in N^*,$$

where $C(m) > 0$ is a constant. The tightness criterion in Lemma 2.4 completes the proof. ∎

REFERENCES

[1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.

[2] S. M. Berman, *Local times and sample function properties of stationary Gaussian processes*, Trans. Amer. Math. Soc. 137 (1969), pp. 277–299.

[3] S. M. Berman, *Gaussian processes with stationary increments: Local times and sample function properties*, Ann. Math. Statist. 41 (1970), pp. 1260–1272.

[4] S. M. Berman, *Gaussian sample functions: uniform dimension and Hölder conditions nowhere*, Nagoya Math. J. 46 (1972), pp. 63–86.

[5] S. M. Berman, *Local nondeterminism and local times of Gaussian processes*, Indiana Univ. Math. J. 23 (1973), pp. 69–94.

[6] B. Boufoussi and A. Kamont, *Temps local brownien et spaces de Besov anisotropiques*, Stoch. Stoch. Reports 61 (1997), pp. 89–105.

[7] B. Boufoussi and E. Lakhel, *Un résultat d'approximation de la solution d'une EDPS en norme de Besov anisotropique*, C. R. Acad. Sci. Paris 330 (2000), pp. 883–888.

[8] B. Boufoussi and Y. Ouknine, *Régularité du temps local du processus symétrique stable en norme de Besov*, Stoch. Stoch. Reports 66 (3–4) (1999), pp. 167–175.

[9] B. Boufoussi and B. Roynette, *Le temps local brownien appartient p.s. à l'espace de Besov* $B^{1/2}_{p,\infty}$, C. R. Acad. Sci. Paris 316 (1993), pp. 843–848.

[10] H. Brezis, *Analyse fonctionnelle. Théorie et applications*, Masson, 1993.

[11] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. 134 (1975), pp. 111–183.

[12] Z. Ciesielski, *On the isomorphism of the space H_α and m*, Bull. Acad. Pol. Sci. 8 (1960), pp. 217–222.
[13] Z. Ciesielski, G. Kerkyacharian and B. Roynette, *Quelques espaces fonctionnels associés à des processus gaussiens*, Studia Math. 107 (1993), pp. 171–204.
[14] Y. Davydov, *Local times of multi-parameter random process*, Theory Probab. Appl. 23 (1978), pp. 573–583.
[15] M. Dozzi, *On the local time of the multi-parameter Wiener process and the asymptotic behaviour of an associated integral*, Stochastics 25 (1988), pp. 155–169.
[16] M. Dozzi, *Occupation density and sample path properties of N-parameter processes*, Lecture Notes in Math. 1802 (2003), pp. 127–166.
[17] W. Ehm, *Sample function properties of multi-parameter stable processes*, Z. Wahrsch. verw. Gebiete 56 (1981), pp. 195–228.
[18] D. Geman and J. Horowitz, *Occupation densities*, Ann. Probab. 8 (1980), pp. 1–67.
[19] P. Imkeller, *Stochastic analysis and local time for (N, d)-Wiener processes*, Ann. Inst. H. Poincaré 20 (1984), pp. 75–101.
[20] A. Kamont, *Isomorphism of some anisotropic Besov and sequence spaces*, Studia Math. 110 (2) (1994), pp. 169–189.
[21] A. Kamont, *On the fractional anisotropic Wiener field*, Probab. Math. Statist. 16 (1) (1996), pp. 85–98.
[22] P. Lévy, *Sur certains processus stochastiques homogènes*, Compositio Math. 7 (1939), pp. 283–339.
[23] D. Nualart, *Weak convergence to the law of two-parameter continuous processes*, Z. Wahrsch. verw. Gebiete 55 (1981), pp. 255–259.
[24] H. Trotter, *A property of Brownian motion paths*, Illinois J. Math. 2 (1958), pp. 425–433.
[25] M. E. Vares, *Local times for two-parameter Lévy processes*, Stochastic Process. Appl. 15 (1983), pp. 59–82.
[26] J. B. Walsh, *The local time of the Brownian sheet*, Astérisque 52–53 (1978), pp. 47–61.
[27] M. Yor, *Le drap brownien comme limite en loi de temps locaux linéaires*, in: *Séminaire de Probabilités XVII*, Lecture Notes in Math. 986 (1983), pp. 89–105.

B. Boufoussi
Cadi Ayyad University
FSSM, Department of Mathematics (LPS)
P.B.O. 2390, Marrakesh, Morocco
E-mail: boufoussi@ucam.ac.ma

M. Dozzi
Université Henri Poincaré Nancy I
Institut Elie Cartan
B.P. 239
54506 Vandoeuvre-Lès-Nancy, France
E-mail: dozzi@iecn.u-nancy.fr

Received on 6.8.2002;
revised version on 10.10.2003