PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 32, Fasc. 2 (2012), pp. 281–300

SECOND-ORDER THEORY FOR ITERATION STABLE TESSELLATIONS

BY

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Abstract. This paper deals with iteration stable (STIT) tessellations, and, more generally, with a certain class of tessellations that are infinitely divisible with respect to iteration. They form a new, rich and flexible family of space-time models considered in stochastic geometry. The previously developed martingale tools are used to study second-order properties of STIT tessellations. A general formula for the variance of the total surface area of cell boundaries inside an observation window is shown. This general expression is combined with tools from integral geometry to derive exact and asymptotic second-order formulas in the stationary and isotropic regime. Also a general formula for the pair-correlation function of the surface measure is found.

2000 AMS Mathematics Subject Classification: Primary: 60D05; Secondary: 52A22, 60G55.

Key words and phrases: Chord-power integral, integral geometry, iteration/nesting, martingale theory, pair-correlation function, random geometry, random tessellation, stochastic stability, stochastic geometry.

1. INTRODUCTION

Iteration stable random tessellations (or mosaics), called STIT tessellations for short, form a new model for random tessellations of the d-dimensional Euclidean space and were formally introduced in [8]–[11]. They have quickly attracted considerable interest in stochastic geometry, because of their flexibility and analytical tractability. They clearly show the potential to become a new mathematical reference model besides the hyperplane and Voronoi tessellations studied in classical stochastic geometry. Whereas much research in the last decades was devoted to mean values and mean value relations, modern stochastic geometry focuses on second-order theory, distributional results, and limit theorems; see [1], [3], [4], [18] or [20], to mention just a few. To introduce the non-specialized reader to the subject, we briefly recall the basic construction of STIT tessellations within compact convex windows $W \subset \mathbb{R}^d$ having interior points. To this end, let us fix a (in

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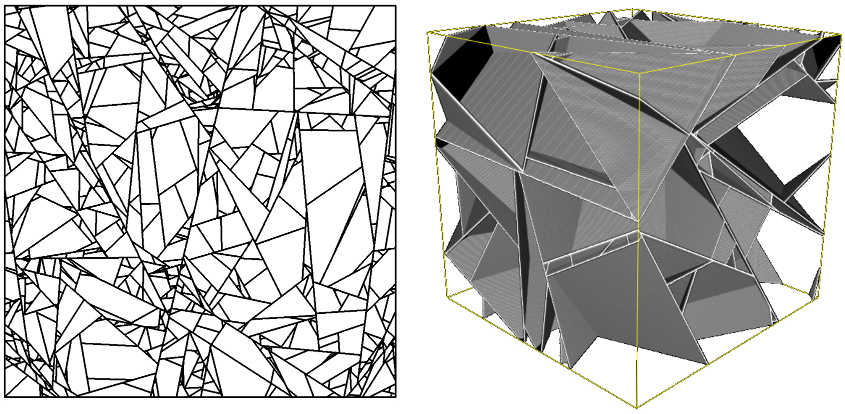

FIGURE 1. Realizations of a planar and a spatial stationary and isotropic STIT tessellation within a square or a cube, respectively

some sense non-degenerate) translation-invariant measure Λ on the space of hyperplanes. Further, let $t > 0$ be fixed and assign to the window W a random lifetime. Upon expiry of its lifetime, W dies and splits into two sub-cells separated by a hyperplane hitting W, which is chosen according to a suitable normalization of Λ. The resulting new cells are again assigned independent random lifetimes and the entire construction continues recursively until the deterministic time threshold t is reached; see Figure 1 for an illustration. In order to ensure the Markov property of the above construction in the continuous-time parameter t, we assume from now on that the lifetimes are exponentially distributed. Moreover, we assume that the parameter of the exponentially distributed lifetimes of individual cells c equals $\Lambda([c])$, where $[c]$ stands for the collection of hyperplanes hitting c. In this special situation, the random tessellation constructed by the described dynamics fulfills a stochastic **st**ability property under the operation of **it**eration of tessellations, and whence is indeed a *STIT tessellations*. We refer to Section 2 below for more details.

In [16] we have introduced a new technique relying on martingale theory for studying these tessellations. One feature of this new approach is that it allows us to investigate second-order parameters (i.e., variances) of the tessellation, which were out of reach so far and are in the focus of the present work. Based on a specialization of our martingale technique, we calculate in Section 4 the variance of a general face-functional and as a special case we find the variance of the total surface area of cell boundaries in a bounded convex window. The resulting integral expression can be explicitly evaluated in the stationary and isotropic regime by applying an integral-geometric transformation formula of Blaschke–Petkantschin type which is also developed in this paper; see Section 5. For the particular case of space dimension 3, an exact formula without further integrals is found. Another important task in our context is to determine for fixed terminal times t the large scale asymptotics of the afore calculated exact variance for a family of growing convex windows. Relying again on techniques from integral geometry, we will be able to determine asymptotic variance expressions, leading – most interestingly – in dimension $d = 2$ to a result of very different qualitative nature compared with space dimensions $d \geqslant 3$, where certain chord-power integrals known from convex and integral geometry will reflect the influence of the geometry of the observation window. In dimension $d = 2$ we will see that, in contrast to the described situation for higher space dimensions, the shape of the window does not play any rôle and only its area enters our formulas; see Section 6. We also derive an explicit expression for the so-called pair-correlation function of the random surface measure for arbitrary space dimensions (see Section 7), generalizing thereby findings from [21], which are based on completely different methods.

We would like to point out that the second-order theory developed in this paper is fundamental for our further work on STIT tessellations (see [13], [14], and [17]) and that its extended version [15] is available online.

In this paper we will make use of the following general notation: $B_R = B^d_R(o)$ is the d-dimensional ball around the origin with radius $R > 0$; $\kappa_j := \mathrm{Vol}_j(B^j_1)$ is the volume of the j-dimensional unit ball, and $j\kappa_j$ its surface area. The uniform distribution on the unit sphere $\mathcal{S}_{d-1}$ in $\mathbb{R}^d$ (normalized spherical surface measure) is denoted by ν_{d-1}.

2. CONSTRUCTION AND PROPERTIES OF THE TESSELLATIONS

Let Λ be a non-atomic and locally finite measure on the space $\mathcal{H}$ of hyperplanes in the d-dimensional Euclidean space $\mathbb{R}^d$. Further, let $t > 0$ and $W \subset \mathbb{R}^d$ be a compact convex window with interior points in which our construction of a random tessellation $Y(t\Lambda, W)$ is carried out. In a first step, we assign to the window W an exponentially distributed random lifetime with parameter $\Lambda([W])$, where $[W] := \{H \in \mathcal{H} :\ H \cap W \neq \emptyset\}$ means the collection of hyperplanes hitting W. Upon expiry of its lifetime, the cell W dies and splits into two polyhedral subcells W^+ and W^- separated by a hyperplane in $[W]$, which is chosen according to the conditional law $\Lambda(\,\cdot\,|[W])$. The resulting new cells W^+ and W^- are again assigned independent exponential lifetimes with respective parameters $\Lambda([W^+])$ and $\Lambda([W^-])$ (whence smaller cells live stochastically longer) and the entire construction continues recursively until the deterministic time threshold t is reached; for an illustration see Figure 1. The cell-separating $(d-1)$-dimensional facets (the term *facet* stands for a $(d-1)$-dimensional face here and throughout) arising in subsequent splits are referred to as $(d-1)$-dimensional maximal polytopes (or I-segments for $d = 2$ as assuming shapes similar to the letter *I*).

The described process of recursive cell divisions is called the *MNW-construction* in honor of its inventors in the sequel and the resulting random tessellation created inside W is denoted by $Y(t\Lambda, W)$ as mentioned above. (We adopt the convention that a random tessellation is a random closed set, which is formed by the union of its cell boundaries. In particular, this induces a measurable structure on the space of tessellations.) The random tessellation $Y(t\Lambda, W)$ has the following properties (see [11] for the proofs):

• $Y(t\Lambda, W)$ is consistent in that $Y(t\Lambda, W) \cap V \overset{D}{=} Y(t\Lambda, V)$ for convex $V \subset W$, and thus $Y(t\Lambda, W)$ can be extended to a random tessellation $Y(t\Lambda)$ on the whole space $\mathbb{R}^d$. Hence, for arbitrary Borel sets $A \subset \mathbb{R}^d$ we shall write $Y(t\Lambda, A)$ for $Y(t\Lambda) \cap A$.

• If Λ is translation-invariant, $Y(t\Lambda)$ is stationary, i.e., stochastically translation invariant. If, moreover, Λ is the isometry-invariant hyperplane measure Λ_{iso} (with a normalization as in [12]), then $Y(t\Lambda_{\text{iso}})$ is even isotropic, i.e., stochastically invariant under rotations with respect to the origin.

• $Y(t\Lambda, W)$ is iteration infinitely divisible with respect to the operation $\boxplus$ of **it**eration of tessellations for any compact convex $W \subset \mathbb{R}^d$. This is to say

$$Y(t\Lambda, W) \overset{D}{=} m\Big(Y\big((t/m)\Lambda, W\big) \boxplus \ldots \boxplus Y\big((t/m)\Lambda, W\big)\Big), \qquad m = 2, 3, \ldots;$$

see [16] for more details. Because of this property we call $Y(t\Lambda, W)$ an iteration infinitely divisible *MNW-tessellation*. In addition, if Λ is translation-invariant, $Y(t\Lambda)$ is **st**able under the operation $\boxplus$, which is to say

$$Y(t\Lambda) \overset{D}{=} m\Big(Y\big((t/m)\Lambda\big) \boxplus \ldots \boxplus Y\big((t/m)\Lambda\big)\Big), \qquad m = 2, 3, \ldots$$

For this reason, $Y(t\Lambda)$ is called a random *STIT tessellation* in this case.

• In the stationary regime, the surface density, i.e., the mean surface area of cell boundaries of $Y(t\Lambda)$ per unit volume, equals t.

• STIT tessellations have the following scaling property:

$$tY(t\Lambda) \overset{D}{=} Y(\Lambda), \qquad t > 0,$$

i.e., the tessellation $Y(t\Lambda)$ of surface intensity t upon rescaling by factor t has the same distribution as $Y(\Lambda)$, the STIT tessellation with surface intensity one.

3. BACKGROUND MATERIAL

In this section we recall a few facts from [16], which will turn out to be crucial for our arguments below. Firstly, it follows directly from the MNW-construction that in the continuous-time parameter t, $\big(Y(t\Lambda, W)\big)_{t>0}$ is a pure jump Markov process on the space of tessellations of W, whose generator $\mathbb{L} := \mathbb{L}_{\Lambda;W}$ is given by the formula

$$\mathbb{L}F(Y) = \int_{[W]} \sum_{f \in \text{Cells}(Y \cap H)} [F(Y \cup \{f\}) - F(Y)]\, \Lambda(\mathrm{d}H) \tag{3.1}$$

for all F bounded and measurable on the space of tessellations of W. (Here, $Y \cap H$ stands for the tessellation induced by Y in the intersection plane H.) Similar to the approach taken in [16], the general theory of Markov processes can now be used to construct a class of martingales associated with iteration infinitely divisible MNW-tessellations or, more specifically, with STIT tessellations. Indeed, for bounded measurable $G = G(Y, t)$, considering the time-augmented Markov process $\big(Y(t\Lambda, W), t\big)_{t \geqslant 0}$ and applying standard theory (see Lemma 5.1 in Appendix 1, Section 5, in [6], or simply by performing a direct check) we obtain

PROPOSITION 3.1. *Assume that $G(Y,t)$ is twice continuously differentiable in t and that*

$$\sup_{Y,t} \left| \frac{\partial}{\partial t} G(Y,t) \right| + \left| \frac{\partial^2}{\partial t^2} G(Y,t) \right| < +\infty.$$

Then the stochastic process

$$G\big(Y(t\Lambda, W), t\big) - \int_0^t \left([\mathbb{L}G(\cdot, s)]\big(Y(s\Lambda, W)\big) + \frac{\partial}{\partial s} G\big(Y(s\Lambda, W), s\big) \right) \mathrm{d}s$$

is a martingale with respect to $\Im_t$, the filtration induced by $\big(Y(s\Lambda, W)\big)_{0 \leqslant s \leqslant t}$.

For Y abbreviating $Y(t\Lambda, W)$ for some t, Λ, and W, we define

$$\Sigma_\phi(Y) := \sum_{f \in \mathrm{MaxPolytopes}_{d-1}(Y)} \phi(f), \tag{3.2}$$

where $\mathrm{MaxPolytopes}_{d-1}(Y)$ are the $(d-1)$-dimensional maximal polytopes of Y (the I-segments in the two-dimensional case), whereas $\phi(\cdot)$ is a generic bounded and measurable functional on $(d-1)$-dimensional facets in W, that is to say, a bounded and measurable function on the space of closed $(d-1)$-dimensional polytopes in W, possibly chopped off by the boundary of W, with the standard measurable structure inherited from the space of closed sets in W. Whereas the so-defined Σ_ϕ is not bounded, we cannot directly apply Dynkin's formula (see Appendix 1, Section 5, in [6] for example) to conclude that the stochastic process $\Sigma_\phi\big(Y(t\Lambda, W)\big) - \int_0^t \mathbb{L}\Sigma_\phi\big(Y(s\Lambda, W)\big)\mathrm{d}s$ is an $\Im_t$-martingale. However, a suitable localization argument can be applied (see [16] for the details) to show

PROPOSITION 3.2. *The stochastic process*

$$\Sigma_\phi\big(Y(t\Lambda, W)\big) - \int_0^t \int_{[W]} \sum_{f \in \mathrm{Cells}(Y(s\Lambda, W) \cap H)} \phi(f)\, \Lambda(\mathrm{d}H)\mathrm{d}s$$

is a martingale with respect to $\Im_t$.

REMARK 3.1. We assume here and until the final section that the observation window W is convex. This is natural regarding the integral-geometric interpretation of variance formulas below. Moreover, the assumption is helpful when our results shall be compared with the formulas for the Poisson hyperplane tessellations in [3] for example. However, parts of our theory hold for arbitrary bounded Borel sets $W \subset \mathbb{R}^d$. This generality will be needed to obtain an explicit expression for the pair-correlation function of the surface measure at the end of this paper.

4. A GENERAL VARIANCE FORMULA

The general martingale statements from the previous section admit a convenient specialization to deal with second-order characteristics of iteration infinitely divisible MNW-tessellations or stationary STIT tessellations. Let us fix through this section a compact convex window $W \subset \mathbb{R}^d$ with interior points. From now on we will focus our attention on translation-invariant face functionals ϕ of $(d-1)$-dimensional facets, regarded as usual as closed subsets of W, of the form

$$\phi(f) := \mathrm{Vol}_{d-1}(f)\zeta\big(\vec{\mathbf{n}}(f)\big) \tag{4.1}$$

with $\vec{\mathbf{n}}(f)$ standing for the unit normal to f and ζ for a bounded measurable even function on the $(d-1)$-dimensional unit sphere $\mathcal{S}_{d-1}$. Recall now the definition (3.2) of $\Sigma_\phi\big(Y(t\Lambda, W)\big)$, introduce the bar notation

$$\bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big) := \Sigma_\phi\big(Y(t\Lambda, W)\big) - \mathbb{E}\Sigma_\phi\big(Y(t\Lambda, W)\big),$$

write $\bar{\Sigma}^2_\phi\big(Y(t\Lambda, W)\big)$ for $\big(\bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big)\big)^2$, and put

$$A_\phi\big(Y(t\Lambda, W)\big) := \int\limits_{[W]} \sum_{f \in \mathrm{Cells}(Y(t\Lambda, W) \cap H)} \phi(f)\, \Lambda(\mathrm{d}H). \tag{4.2}$$

PROPOSITION 4.1. *The stochastic process*

$$\bar{\Sigma}^2_\phi\big(Y(t\Lambda, W)\big) - \int\limits_0^t A_{\phi^2}\big(Y(s\Lambda, W)\big)\mathrm{d}s \tag{4.3}$$

is an $\Im_t$-martingale.

P r o o f. Let $Y = Y(t\Lambda, W)$ for some t and W and define

$$G(Y, t) := \big(\Sigma_\phi(Y) - \mathbb{E}\Sigma_\phi(Y)\big)^2,$$

so that $G\big(Y(t\Lambda, W), t\big) = \bar{\Sigma}^2_\phi\big(Y(t\Lambda, W)\big)$. For so defined G we have

$$\begin{aligned} \frac{\partial}{\partial t} G(Y, t) &= 2\bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big)\, \frac{\partial}{\partial t} \bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big) \\ &= -2\bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big)\, \frac{\partial}{\partial t} \mathbb{E}\Sigma_\phi\big(Y(t\Lambda, W)\big) \\ &= -2\bar{\Sigma}_\phi\big(Y(t\Lambda, W)\big)\, \mathbb{E}A_\phi\big(Y(t\Lambda, W)\big) \end{aligned} \tag{4.4}$$

according to Proposition 3.2. In order to apply Proposition 3.1 it remains to find an expression for $[\mathbb{L}G(\cdot,t)](Y)$. Using (3.1) we obtain

$$[\mathbb{L}G(\cdot,t)](Y) = \int_{[W]} \sum_{f\in \mathrm{Cells}(Y\cap H)} \left[\left(\Sigma_\phi(Y\cup\{f\}) - \mathbb{E}\Sigma_\phi(Y)\right)^2 - \bar{\Sigma}^2_\phi(Y)\right] \Lambda(\mathrm{d}H).$$

By rearranging terms we find that

$$[\mathbb{L}G(\cdot,t)](Y) = \int_{[W]} \sum_{f\in \mathrm{Cells}(Y\cap H)} [\phi^2(f) + 2\phi(f)\bar{\Sigma}_\phi(Y)]\, \Lambda(dH). \tag{4.5}$$

We would now like to apply Proposition 3.1 and use (4.4) and (4.5) to conclude that

$$\begin{aligned} G\big(Y(t\Lambda,W),t\big) &- \int_0^t \left([\mathbb{L}G(\cdot,s)]\big(Y(s\Lambda,W)\big) + \frac{\partial}{\partial s}G\big(Y(s\Lambda,W),s\big)\right)\mathrm{d}s \quad (4.6)\\ &= \bar{\Sigma}^2_\phi\big(Y(t\Lambda,W)\big) - \int_0^t \int_{[W]} \sum_{f\in\mathrm{Cells}(Y(s\Lambda,W)\cap H)} \phi^2(f)\,\Lambda(\mathrm{d}H)\mathrm{d}s\\ &\quad - 2\int_0^t \Big[\int_{[W]} \sum_{f\in\mathrm{Cells}(Y(s\Lambda,W)\cap H)} \phi(f)\,\bar{\Sigma}_\phi\big(Y(s\Lambda,W)\big)\,\Lambda(\mathrm{d}H)\\ &\quad - \bar{\Sigma}_\phi\big(Y(s\Lambda,W)\big)\,\mathbb{E}A_\phi\big(Y(s\Lambda,W)\big)\Big]\mathrm{d}s\\ &= \bar{\Sigma}^2_\phi\big(Y(t\Lambda,W)\big) - \int_0^t A_{\phi^2}\big(Y(s\Lambda,W)\big)\mathrm{d}s \quad (4.7)\\ &\quad - 2\int_0^t \bar{A}_\phi\big(Y(s\Lambda,W)\big)\bar{\Sigma}_\phi\big(Y(s\Lambda,W)\big)\mathrm{d}s \end{aligned}$$

is an $\Im_t$-martingale with $\bar{A}_\phi\big(Y(s\Lambda,W)\big) := A_\phi\big(Y(s\Lambda,W)\big) - \mathbb{E}A_\phi\big(Y(s\Lambda,W)\big)$. However, this is not possible directly, because $G(Y,t)$ does not necessarily fulfill the conditions of Proposition 3.1. However, we can apply a suitable localization and truncation argument similar to the one leading to Proposition 2 in [16] to get the result. In our case we replace G by G_N chosen so that $\big(G_N(\cdot,\cdot)\wedge N\big)\vee -N \equiv \big(G(Y,t)\wedge N\big)\vee -N$, that $|G_N(\cdot,\cdot)| \leqslant N+1$, and that $G_N(\cdot,t)$ is twice continuously differentiable in t. The localizing stopping times are defined by

$$T_N = \inf\left\{t>0 : \left(|G(Y,t)| \vee \left|\frac{\partial}{\partial t}G(Y,t)\right| \vee \left|\frac{\partial^2}{\partial t^2}G(Y,t)\right|\right) \geqslant N\right\}.$$

Then Proposition 3.1 can be applied to infer that the stochastic process (4.6) with G replaced by G_N is a local martingale for the localizing sequence $(T_N)_{N\geqslant 1}$ as defined above. Letting $N\to\infty$ we see that even (4.6) with the original function G

is a local martingale. Moreover, it is of class DL, which finally shows that (4.6) with $G(Y,t) = \bar{\Sigma}_\phi^2(Y)$ and consequently (4.7) is a martingale as claimed; see Definition 4.8 and Problem 5.19 (i) in [5].

We now take advantage of the special form (4.1) of the face functional ϕ to conclude that

$$A_\phi \equiv \int_{[W]} \mathrm{Vol}_{d-1}(H \cap W)\zeta\big(\vec{\mathbf{n}}(H)\big)\,\Lambda(\mathrm{d}H) = \mathrm{const}. \tag{4.8}$$

This implies that $\bar{A}_\phi \equiv 0$, and thus, using (4.7), we can complete the proof. ∎

The so-far established theory is now used to calculate the variance of face functionals as given by the formula (4.1) of iteration infinitely divisible random MNW-tessellations $Y(t\Lambda, W)$ restricted to a compact convex window $W \subset \mathbb{R}^d$ with $\mathrm{Vol}_d(W) > 0$.

THEOREM 4.1. *For arbitrary diffuse and locally finite measures Λ on $\mathcal{H}$ and ϕ as in* (4.1) *we have*

$$\mathrm{Var}\Big(\Sigma_\phi\big(Y(t\Lambda, W)\big)\Big) = \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \int_{W\cap H}\int_{W\cap H} \frac{1-\exp\big(-t\Lambda([xy])\big)}{\Lambda([xy])}\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H).$$

P r o o f. Recall first (4.2) and note that it implies

$$A_{\phi^2}\big(Y(t\Lambda, W)\big) = \int_{[W]} \sum_{f\in\mathrm{Cells}(Y(t\lambda,W)\cap H)} \phi^2(f)\,\Lambda(\mathrm{d}H)$$

$$= \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \int_{W\cap H}\int_{W\cap H} \mathbf{1}[x,y \text{ are in the same cell of } Y\cap H]\,\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H).$$

Taking the expectation and using Fubini's theorem we see that

$$\mathbb{E}A_{\phi^2}\big(Y(t\Lambda, W)\big) = \int_{[W]} \sum_{f\in\mathrm{Cells}(Y(t\Lambda,W)\cap H)} \phi^2(f)\,\Lambda(\mathrm{d}H)$$

$$= \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \int_{W\cap H}\int_{W\cap H} \mathbb{P}(x,y \text{ are in the same cell of } Y\cap H)\,\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H).$$

Moreover, the martingale property of the stochastic process in (4.3) implies that

$$\mathrm{Var}\,\Sigma_\phi\big(Y(t\Lambda, W)\big) = \mathbb{E}\bar{\Sigma}_\phi^2\big(Y(t\Lambda, W)\big) = \int_0^t \mathbb{E}A_{\phi^2}\big(Y(s\Lambda, W)\big)\mathrm{d}s.$$

Thus,

$$\mathrm{Var}\,\Sigma_\phi\big(Y(t\Lambda, W)\big) = \int_0^t \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \tag{4.9}$$

$$\times \int_{H\cap W}\int_{H\cap W} \mathbb{P}\big(x,y \text{ are in the same cell of } Y(s\Lambda, W)\cap H\big)\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H)\mathrm{d}s.$$

We now determine the probability that two points $x, y \in W \cap H$ are in the same cell of $Y(s\Lambda, W)$. We note that this is equivalent to the event that the line segment xy connecting x and y does not intersect $Y(s\Lambda, W)$, i.e.,

$$\mathbb{P}\big(x, y \text{ are in the same cell of } Y(s\Lambda, W) \cap H\big) = \mathbb{P}\big(xy \cap Y(s\Lambda, W) = \emptyset\big);$$

recall that we regard $Y(s\Lambda, W)$ as a random closed subset of W. The latter probability is the so-called capacity functional of $Y(s\Lambda, W)$ evaluated for the line segment xy. Thus, using Lemma 3 in [11], we obtain

$$\mathbb{P}\big(x, y \text{ are in the same cell of } Y(s\Lambda, W) \cap H\big) = \exp\big(-s\Lambda([xy])\big).$$

Combining this with (4.9), we end up with

$$\begin{aligned}
&\operatorname{Var}\Big(\Sigma_\phi\big(Y(t\Lambda, W)\big)\Big) \\
&\quad = \int_0^t \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \int_{W\cap H} \int_{W\cap H} \exp\big(-s\Lambda([xy])\big)\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H)\mathrm{d}s \\
&\quad = \int_{[W]} \zeta^2\big(\vec{\mathbf{n}}(H)\big) \int_{W\cap H} \int_{W\cap H} \frac{1-\exp\big(-t\Lambda([xy])\big)}{\Lambda([xy])}\mathrm{d}x\mathrm{d}y\Lambda(\mathrm{d}H).
\end{aligned}$$

This completes our argument. ∎

For general hyperplane measures Λ this cannot be simplified further, even not in the stationary case. However, in the special case where Λ is the isometry-invariant measure Λ_{iso}, tools from integral geometry become available to evaluate the integral further.

5. EXACT VARIANCE EXPRESSION FOR THE ISOTROPIC STIT TESSELLATION

For the stationary and isotropic case $\Lambda = \Lambda_{\text{iso}}$ we want to evaluate the variance expression from Theorem 4.1 in the special case $\phi = \mathrm{Vol}_{d-1}$, i.e., when $\zeta \equiv 1$. To simplify the notation we will write from now on $Y(t)$ instead of $Y(t\Lambda_{\text{iso}})$.

THEOREM 5.1. *Let W be a compact and convex subset of $\mathbb{R}^d$ having interior points and let $\overline{\gamma}_W(r) = \int_{\mathcal{S}_{d-1}} \mathrm{Vol}_d\big(W \cap (W + ru)\big)\nu_{d-1}(\mathrm{d}u)$ be the isotropized set-covariance function of W. For the stationary and isotropic STIT tessellation $Y(t)$ with surface intensity $t > 0$ we have*

$$\begin{aligned}
(5.1)\quad &\operatorname{Var}\Big(\mathrm{Vol}_{d-1}\big(Y(t, W)\big)\Big) \\
&\quad = \frac{d-1}{2} \int_W \int_W \left[1 - \exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d} t\,\|x-y\|\right)\right] \|x-y\|^{-2}\mathrm{d}x\mathrm{d}y
\end{aligned}$$

and, *consequently*,

$$\operatorname{Var}\Big(\operatorname{Vol}_{d-1}\big(Y(t,W)\big)\Big) = \frac{d(d-1)\kappa_d}{2}\int_0^\infty \overline{\gamma}_W(r)r^{d-3}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right]\mathrm{d}r. \tag{5.2}$$

The key to Theorem 5.1 is an integral-geometric transformation formula of Blaschke–Petkantschin type, which is interesting in its own right. We develop this formula in a slightly more general setting than presently needed. This generality will be useful in the discussion of the pair-correlation function of the random surface measure at the end of the paper.

PROPOSITION 5.1. *Let $W_1, W_2 \subset \mathbb{R}^d$ be Borel sets and let $g\colon W_1\times W_2\to\mathbb{R}$ be a non-negative measurable function. Then*

$$\int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H} g(x,y)\,\mathrm{d}x\mathrm{d}y\Lambda_{\mathrm{iso}}(\mathrm{d}H) = \frac{(d-1)\kappa_{d-1}}{d\kappa_d}\int_{W_1}\int_{W_2}\frac{g(x,y)}{\|x-y\|}\mathrm{d}x\mathrm{d}y. \tag{5.3}$$

Proof. First, we use the affine Blaschke–Petkantschin formula (see [12], Theorem 7.2.7) with $q=1$ to deduce that for any non-negative measurable function $h:(\mathbb{R}^d)^2\to\mathbb{R}$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{d\kappa_d}{2}\int_{\mathcal{L}}\int_L\int_L h(x,y)\,\|x-y\|^{d-1}\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L,$$

where $\mathcal{L}$ is the space of lines in $\mathbb{R}^d$ with invariant measure $\mathrm{d}L$ normalized as in [12] and ℓ_L is the Lebesgue measure on L. Taking now

$$h(x,y) = \mathbf{1}[x\in W_1]\mathbf{1}[y\in W_2]\,\|x-y\|^k\,g(x,y)$$

for some $k>-d$ and another non-negative measurable function $g: W_1\times W_2\to\mathbb{R}$ we obtain

$$\int_{W_1}\int_{W_2}\|x-y\|^k\,g(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{d\kappa_d}{2}\int_{\mathcal{L}}\int_{W_1\cap L}\int_{W_2\cap L}\|x-y\|^{d-1+k}\,g(x,y)\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L. \tag{5.4}$$

For $k=-1$ this yields

$$\int_{W_1}\int_{W_2}\frac{g(x,y)}{\|x-y\|}\mathrm{d}x\mathrm{d}y = \frac{d\kappa_d}{2}\int_{\mathcal{L}}\int_{W_1\cap L}\int_{W_2\cap L}\|x-y\|^{d-2}\,g(x,y)\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L. \tag{5.5}$$

We replace now in (5.4), for $k = 0$, W_1 by $W_1 \cap H$ and W_2 by $W_2 \cap H$ for some fixed hyperplane H, and d by $d-1$. Then we get

$$\int_{W_1\cap H}\int_{W_2\cap H} g(x,y)\,\mathrm{d}x\mathrm{d}y$$
$$= \frac{(d-1)\kappa_{d-1}}{2}\int_{\mathcal{L}^H}\int_{W_1\cap H\cap L}\int_{W_2\cap H\cap L} \|x-y\|^{d-2}\, g(x,y)\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L^H,$$

where by $\mathcal{L}^H$ we mean the space of lines within the hyperplane H (again with a normalization as in [12]). Averaging the last expression over all hyperplanes H and using the fact that $\Lambda_{\mathrm{iso}}(\mathrm{d}H) \otimes \mathrm{d}L^H = \mathrm{d}L$ (see the remark at the end of Section 7.1 in [12]) yields

$$\text{(5.6)}\quad \int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H} g(x,y)\,\mathrm{d}x\mathrm{d}y\Lambda_{\mathrm{iso}}(\mathrm{d}H)$$
$$= \frac{(d-1)\kappa_{d-1}}{2}\int_{\mathcal{H}}\int_{\mathcal{L}^H}\int_{W_1\cap H\cap L}\int_{W_2\cap H\cap L} \|x-y\|^{d-2}$$
$$\times\; g(x,y)\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L^H\Lambda_{\mathrm{iso}}(\mathrm{d}H)$$
$$= \frac{(d-1)\kappa_{d-1}}{2}\int_{\mathcal{L}}\int_{W_1\cap L}\int_{W_2\cap L} \|x-y\|^{d-2}\, g(x,y)\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L.$$

By comparing (5.5) and (5.6) we finally conclude that

$$\int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H} g(x,y)\,\mathrm{d}x\mathrm{d}y\Lambda_{\mathrm{iso}}(\mathrm{d}H) = \frac{(d-1)\kappa_{d-1}}{d\kappa_d}\int_{W_1}\int_{W_2}\frac{g(x,y)}{\|x-y\|}\,\mathrm{d}x\mathrm{d}y,$$

completing thereby the proof of the proposition. ■

P r o o f o f T h e o r e m 5.1. First, in view of the general formula from Theorem 4.1, we put

$$g(x,y) = \frac{1-\exp\big(-t\Lambda_{\mathrm{iso}}([xy])\big)}{\Lambda_{\mathrm{iso}}([xy])}$$
$$= \left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\,\|x-y\|\right)\right]\left(\frac{2\kappa_{d-1}}{d\kappa_d}\,\|x-y\|\right)^{-1},$$

where the equality follows from the fact that

$$\Lambda_{\mathrm{iso}}([xy]) = \frac{2\kappa_{d-1}}{d\kappa_d}\|x-y\|;$$

cf. [12], Theorem 6.2.2 with $q = j = d-1$ there. Thus, upon applying the transformation formula (5.3) with $W_1 = W_2 = W$, we conclude the following identity for $\mathrm{Var}\left(\Sigma_{\mathrm{Vol}_{d-1}}(Y(t,W))\right) = \mathrm{Var}\left(\mathrm{Vol}_{d-1}(Y(t,W))\right)$:

$$\begin{aligned}
&\int_{[W]} \int_{H\cap W} \int_{H\cap W} g(x,y)\,\mathrm{d}x\mathrm{d}y\Lambda_{\mathrm{iso}}(\mathrm{d}H)\\
&\qquad = \frac{d-1}{2}\int_W\int_W \left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\,\|x-y\|\right)\right]\|x-y\|^{-2}\mathrm{d}x\mathrm{d}y\\
&\qquad = \frac{d(d-1)\kappa_d}{2}\int_0^\infty \overline{\gamma}_W(r)\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right]r^{-2}r^{d-1}\mathrm{d}r\\
&\qquad = \frac{d(d-1)\kappa_d}{2}\int_0^\infty \overline{\gamma}_W(r)r^{d-3}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right]\mathrm{d}r,
\end{aligned}$$

where we have passed to spherical coordinates in the second step. ■

In the special case $W = B_R^3$, the isotropized set-covariance function $\overline{\gamma}_{B_R^3}(r)$ is given by

$$\overline{\gamma}_{B_R^3}(r) = \begin{cases} \frac{4\pi}{3}R^3\left(1-\frac{3r}{4R}+\frac{r^3}{16R^3}\right) & \text{for } 0\leqslant r\leqslant 2R,\\ 0 & \text{for } r>2R.\end{cases}$$

In this practically relevant situation the variance integral can be evaluated in a closed form and we obtain

$$\begin{aligned}
(5.7)\quad &\mathrm{Var}\left(\mathrm{Vol}_2\left(Y(t,B_R^3)\right)\right)\\
&\qquad = \frac{4\pi^2}{3t^4}\left(t^2R^2(12-8tR+3t^2R^2)+24(1+tR)e^{-tR}-24\right).
\end{aligned}$$

The same closed form cannot be achieved in other space dimensions, since for example $\overline{\gamma}_{B_R^2}(r)$ has a more complicated structure. More precisely,

$$\overline{\gamma}_{B_R^2}(r) = 2R^2\arccos\left(\frac{r}{2R}\right) - \frac{r}{2}\sqrt{4R^2-r^2}$$

for r between 0 and $2R$ and $\overline{\gamma}_{B_R^2}(r) = 0$ for $r > 2R$. For general space dimensions (including the separately discussed cases $d=2$ and $d=3$), $\overline{\gamma}_{B_R^d}(r)$ is given by

$$\overline{\gamma}_{B_R^d}(r) = 2\kappa_{d-1}R^d\int_{r/(2R)}^1 (1-u^2)^{(d-1)/2}\,\mathrm{d}u$$

if $0\leqslant r\leqslant 2R$ and $\overline{\gamma}_{B_R^d}(r) = 0$ otherwise. Unfortunately, the resulting integrals in this case cannot be further simplified.

6. THE VARIANCE IN THE ASYMPTOTIC REGIME

Another important task in our context is to determine for fixed t the large R asymptotics of the variance $\operatorname{Var}\left(\operatorname{Vol}_{d-1}\left(Y(t,W_R)\right)\right)$ for the family of growing windows $W_R = R\cdot W$ as $R\to\infty$ and with W as in the previous section. We concentrate once more on the isotropic case, where explicit computations are possible. Let us write $\sim$ for the asymptotic equivalence of functions, i.e., $f(R)\sim g(R)$ iff $f(R)/g(R)\to 1$ as $R\to\infty$.

THEOREM 6.1. *For $d=2$,*

$$\operatorname{Var}\Big(\operatorname{Vol}_1\left(Y(t,W_R)\right)\Big)\sim \pi\operatorname{Vol}_2(W)R^2\log R, \tag{6.1}$$

whereas for $d\geqslant 3$ we have

$$\operatorname{Var}\Big(\operatorname{Vol}_{d-1}\left(Y(t,W_R)\right)\Big)\sim \frac{d-1}{2}E_2(W)R^{2(d-1)} \tag{6.2}$$

with $E_2(W)$ being the two-energy of W given by

$$E_2(W)=\int\limits_W\int\limits_W \|x-y\|^{-2}\,\mathrm{d}x\mathrm{d}y. \tag{6.3}$$

In particular, this establishes (weak) long range dependencies present in stationary and isotropic STIT tessellations $Y(t)$. In the planar case, these dependencies are rather weak in that

$$\frac{\operatorname{Var}\Big(\operatorname{Vol}_1\left(Y(t,W_R)\right)\Big)}{\operatorname{Vol}_2(W_R)}\sim\pi\log R\to\infty\qquad\text{as } R\to\infty.$$

For $d\geqslant 3$ these dependencies are much stronger, as the variance of the total surface area grows asymptotically like $R^{2(d-1)}$, which should be compared with the volume-order R^d.

P r o o f. Formula (6.1) can be established by using (5.2), the relation

$$\overline{\gamma}_{W_R}\sim\operatorname{Vol}_2(W_R)=R^2\operatorname{Vol}_2(W),$$

valid uniformly for arguments $r=O(R/\log R)$, the observation that $\overline{\gamma}_{W_R}(r)\to 0$ for $r=\Omega(R\log R)$, together with the fact that

$$\int\limits_0^{L(R)}\left(1-e^{-cr}\right)\frac{\mathrm{d}r}{r}\sim\log R,\qquad c>0,$$

as soon as $\log L(R)\sim\log R$, and the scaling property of STIT tessellations:

$$\begin{aligned}\operatorname{Var}\Big(\operatorname{Vol}_1\left(Y(t,W_R)\right)\Big)&=t^{-2}\operatorname{Var}\Big(\operatorname{Vol}_1\left(Y(1,W_{tR})\right)\Big)\\&=\frac{\pi}{t^2}\int\limits_0^\infty\overline{\gamma}_{W_{tR}}(r)\left[1-\exp\left(-\frac{2}{\pi}r\right)\right]\frac{\mathrm{d}r}{r}\\&\sim\pi t^{-2}\operatorname{Vol}_2(W_{tR})\log(tR)\sim\pi R^2\operatorname{Vol}_2(W)\log R.\end{aligned}$$

To see (6.2), we use (5.1) and again the scaling property of STIT tessellations to obtain

$$\begin{aligned}&\mathrm{Var}\Big(\mathrm{Vol}_{d-1}\big(Y(t,W_R)\big)\Big) = R^{2(d-1)}\,\mathrm{Var}\Big(\mathrm{Vol}_{d-1}\big(Y(Rt,W)\big)\Big)\\ &= R^{2(d-1)}\frac{d-1}{2}\int\limits_W\int\limits_W\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}Rt\,\|x-y\|\right)\right]\|x-y\|^{-2}\mathrm{d}x\mathrm{d}y\\ &\sim R^{2(d-1)}\frac{d-1}{2}E_2(W)\qquad \text{as } R\to\infty.\end{aligned}$$

Observe that this does not extend to the above separately treated case $d=2$ because there the integral in (6.3) diverges. ■

It is easily seen that $E_2(\,\cdot\,)$ enjoys a *superadditivity* property in that

$$E_2(W_1\cup W_2)\geqslant E_2(W_1)+E_2(W_2),\qquad W_1\cap W_2=\emptyset,$$

which stands in contrast to (6.1), where the asymptotic expression is linear in $\mathrm{Vol}_2(W)$. We will now derive an integral-geometric interpretation of this energy functional. Taking $W_1=W_2=W$, $g(x,y)\equiv 1$, and $k=-2$ in (5.4) yields the identity

$$\begin{aligned}E_2(W) &= \int\limits_W\int\limits_W \|x-y\|^{-2}\,\mathrm{d}x\mathrm{d}y\\ &= \frac{d\kappa_d}{2}\int\limits_{\mathcal{L}}\int\limits_{W\cap L}\int\limits_{W\cap L}\|x-y\|^{d-3}\,\ell_L(\mathrm{d}x)\ell_L(\mathrm{d}y)\mathrm{d}L\\ &= \frac{d\kappa_d}{(d-1)(d-2)}\int\limits_{\mathcal{L}}\mathrm{Vol}_1(W\cap L)^{d-1}\,\mathrm{d}L\\ &= \frac{2}{(d-1)(d-2)}I_{d-1}(W)\end{aligned}\tag{6.4}$$

with $I_{d-1}(W)$ being the $(d-1)$-st *chord power integral* of W in the sense of [12], p. 363. More precisely,

$$I_{d-1}(W)=\frac{d\kappa_d}{2}\int\limits_{\mathcal{L}}\mathrm{Vol}_1^{d-1}(W\cap L)\,\mathrm{d}L.$$

Hence, combining (6.4) with (6.2) from above and using the fact that $I_{d-1}(\cdot)$ is homogeneous of degree $2(d-1)$, we arrive at the following result.

COROLLARY 6.1. *For space dimensions $d\geqslant 3$ the asymptotic variance of the total surface area induced by the cells of $Y(t,W_R)$ is given by*

$$\mathrm{Var}\Big(\mathrm{Vol}_{d-1}\big(Y(t,W_R)\big)\Big)\sim\frac{1}{d-2}I_{d-1}(W_R)=\frac{1}{d-2}R^{2(d-1)}I_{d-1}(W)$$

as $R\to\infty$.

In general, $I_{d-1}(W)$ is rather difficult to evaluate explicitly. But in the special case $W = B_1^d$ we can apply Theorem 8.6.6 in [12] (with a corrected constant), which yields

$$I_{d-1}(B_1^d) = d2^{d-2}\frac{\kappa_d \kappa_{2d-2}}{\kappa_{d-1}}$$

and, thus, the two-energy of the d-dimensional unit ball B_1^d equals

$$E_2(B_1^d) = \frac{d2^{d-1}}{(d-1)(d-2)}\frac{\kappa_d \kappa_{2d-2}}{\kappa_{d-1}} = \frac{2\pi^d}{(d-1)(d-2)}\Gamma\left(\frac{d}{2}\right)^{-2}$$

(here $\Gamma(\cdot)$ is the usual gamma function). In the particular case $d = 3$ we obtain the value $E_2(B_1^3) = 4\pi^2$, which agrees with our explicit variance formula (5.7).

7. PAIR-CORRELATION FUNCTION

It is our next goal to establish a closed formula for the so-called pair-correlation function $g_{d,t}(r)$ of the random surface measure of a STIT tessellation $Y(t)$. To introduce the pair-correlation function formally, we start by recalling that the *second-moment measure* $\mu_{d,t}^{(2)}$ of the random surface measure of $Y(t)$ is defined by

$$\mu_{d,t}^{(2)}(W_1 \times W_2) = \mathbb{E}\left[\operatorname{Vol}_{d-1}\left(Y(t, W_1)\right)\ \operatorname{Vol}_{d-1}\left(Y(t, W_2)\right)\right]$$

for measurable subsets $W_1, W_2 \subset \mathbb{R}^d$. Since $Y(t)$ is stationary, the *reduced second-moment measure* $\mathcal{K}_{d,t}$ on $\mathbb{R}^d$ can be introduced by

$$\mu_{d,t}^{(2)}(W_1 \times W_2) = t^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}[x \in W_1, x + h \in W_2]\, \mathrm{d}x \mathcal{K}_{d,t}(\mathrm{d}h) \tag{7.1}$$

and the *reduced second-moment function* by $K_{d,t}(r) := \mathcal{K}(B_r^d)$; see Section 2.5 in [2] or Sections 4.5 and 7.2 in [19]. Finally, the pair-correlation function $g_{d,t}(r)$ is related to $K_{d,t}(r)$ via

$$g_{d,t}(r) = \frac{1}{d\kappa_d r^{d-1}}\frac{\partial K_{d,t}(r)}{\partial r}.$$

This function is a commonly used tool in spatial statistics, stochastic geometry and physics to describe the second-order structure of a random set. It describes the expected surface density of $Y(t)$ at a given distance r from a typical point of $Y(t)$; cf. [19].

THEOREM 7.1. *The pair-correlation function $g_{d,t}(r)$ of the random surface measure of the stationary and isotropic random STIT tessellation $Y(t)$ is given by*

$$g_{d,t}(r) = 1 + \frac{d-1}{2t^2r^2}\left[1 - \exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right].$$

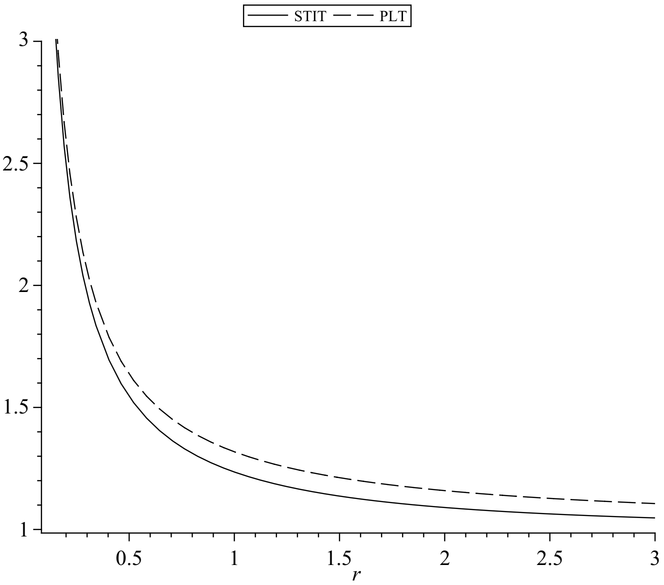

FIGURE 2. Pair-correlation functions of a stationary and isotropic STIT tessellation $Y(1)$ in the plane (solid curve) and a stationary and isotropic Poisson line tessellation (PLT) with edge length density one (dashed curve)

Proof. We start by noticing that the stochastic process in (4.7) is a martingale for any ϕ on the space of $(d-1)$-polytopes of the form

$$\phi(f) = \zeta\big(\vec{\mathbf{n}}(f)\big) \int\limits_f g(x)\mathrm{d}x,$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function with bounded support and ζ is a bounded measurable even function on $\mathcal{S}_{d-1}$; compare, for example, with the discussion in Section 3.2 of [13]. Let $\psi(f) = \xi\big(\vec{\mathbf{n}}(f)\big) \int_f h(x)\mathrm{d}x$ be another such functional and apply the martingale property of (4.7) to $\phi + \psi$ and $\phi - \psi$. Subtracting the resulting expressions we conclude that

$$\text{(7.2)} \quad \bar{\Sigma}_\phi\big(Y(t)\big)\bar{\Sigma}_\psi\big(Y(t)\big) - \int\limits_0^t A_{\phi\psi}\big(Y(s)\big)\mathrm{d}s \\ - \int\limits_0^t \bar{A}_\phi\big(Y(s)\big)\bar{\Sigma}_\psi\big(Y(s)\big) + \bar{A}_\psi\big(Y(s)\big)\bar{\Sigma}_\phi\big(Y(s)\big)\mathrm{d}s$$

is a martingale with respect to $\Im_t$; compare with Proposition 1 in [13] or Proposition 2 in [14]. Here,

$$\Sigma_\phi\big(Y(t)\big) = \sum_{f\in \mathrm{MaxPolytopes}_{d-1}(Y(t))} \phi(f),$$
$$A_\phi\big(Y(t)\big) = \int_{\mathcal{H}} \sum_{f\in \mathrm{Cells}(Y(t)\cap H)} \phi(f)\,\Lambda_{\mathrm{iso}}(\mathrm{d}H)$$

with similar expressions for ψ and with the standard bar-notation $\bar{\Sigma}_\phi\big(Y(t)\big) = \Sigma_\phi\big(Y(t)\big) - \mathbb{E}\Sigma_\phi\big(Y(t)\big)$, etc. It is important to note that even if the integrals in (7.2) are defined for the whole space tessellation $Y(t)$, we can safely replace $Y(t)$ by $Y(t,W)$ for some $W\subset\mathbb{R}^d$ containing the supports of ϕ and ψ.

Now, we let $W_1\subset\mathbb{R}^d$ and $W_2\subset\mathbb{R}^d$ be two bounded Borel sets and take $g(x) = \mathbf{1}[x\in W_1]$, $h(x) = \mathbf{1}[x\in W_2]$, and $\zeta = \xi \equiv 1$ in the definition of ϕ and ψ so that

$$\phi = \mathrm{Vol}_{d-1}(\,\cdot\,\cap W_1) \quad \text{and} \quad \psi = \mathrm{Vol}_{d-1}(\,\cdot\,\cap W_2).$$

Then (4.8) implies that $\bar{A}_\phi(\,\cdot\,) = \bar{A}_\psi(\,\cdot\,) = 0$. Taking now expectations in (7.2) yields

$$\mathbb{E}\big[\,\mathrm{Vol}_{d-1}\big(Y(t,W_1)\big)\,\mathrm{Vol}_{d-1}\big(Y(t,W_1)\big)\big]$$
$$- \mathbb{E}\,\mathrm{Vol}_{d-1}\big(Y(t,W_1)\big)\,\mathbb{E}\,\mathrm{Vol}_{d-1}\big(Y(t,W_2)\big) = \mathbb{E}\int_0^t A_{\phi\psi}\big(Y(s)\big)\mathrm{d}s$$

with $A_{\phi\psi}\big(Y(s)\big)$ given by

$$A_{\phi\psi}\big(Y(s)\big) = \int_{\mathcal{H}} \sum_{f\in\mathrm{Cells}(Y(s)\cap H)} \mathrm{Vol}_{d-1}(f\cap W_1)\,\mathrm{Vol}_{d-1}(f\cap W_2)\,\Lambda_{\mathrm{iso}}(\mathrm{d}H).$$

Proceeding now as in the proof of Theorem 4.1 and using Proposition 5.1 in its general form we obtain

$$\begin{aligned}
(7.3)\quad & \mathbb{E}\int_0^t A_{\phi\psi}\big(Y(s)\big)\,\mathrm{d}s \\
&= \int_0^t\int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H} \mathbb{P}(x,y \text{ are in the same cell of } Y(s)\cap H)\mathrm{d}x\mathrm{d}y\mathrm{d}H\mathrm{d}s \\
&= \int_0^t\int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H} \exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}s\|x-y\|\right)\mathrm{d}x\mathrm{d}y\mathrm{d}H\mathrm{d}s \\
&= \int_{\mathcal{H}}\int_{W_1\cap H}\int_{W_2\cap H}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\|x-y\|\right)\right]\|x-y\|^{-1}\mathrm{d}x\mathrm{d}y\mathrm{d}H \\
&= \frac{d-1}{2}\int_{W_1}\int_{W_2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\|x-y\|\right)\right]\|x-y\|^{-2}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\big[\operatorname{Vol}_{d-1}\big(Y(t,W_1)\big)\operatorname{Vol}_{d-1}\big(Y(t,W_1)\big)\big]\\ -\mathbb{E}\operatorname{Vol}_{d-1}\big(Y(t,W_1)\big)\,\mathbb{E}\operatorname{Vol}_{d-1}\big(Y(t,W_2)\big)\\ =\frac{d-1}{2}\int_{W_1}\int_{W_2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\|x-y\|\right)\right]\|x-y\|^{-2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Since $\mathbb{E}\operatorname{Vol}_{d-1}\big(Y(t,W_i)\big)=t\operatorname{Vol}_d(W_i)$ for $i=1,2$, we see that

$$\begin{aligned}\mu^{(2)}_{d,t}(W_1\times W_2)&=\mathbb{E}\big[\operatorname{Vol}_{d-1}\big(Y(t,W_1)\big)\operatorname{Vol}_{d-1}\big(Y(t,W_2)\big)\big]\\ &=t^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbf{1}[x\in W_1,x+h\in W_2]\\ &\quad\times\left(1+\frac{d-1}{2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\|h\|\right)\right]t^{-2}\|h\|^{-2}\right)\mathrm{d}x\mathrm{d}h.\end{aligned}$$

A glance at (7.1) then shows that the reduced second-moment measure $\mathcal{K}_{d,t}$ is given by

$$\mathcal{K}_{d,t}(\mathrm{d}h)=\left(1+\frac{d-1}{2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}t\|h\|\right)\right]t^{-2}\|h\|^{-2}\right)\mathrm{d}h.$$

The definition of the reduced second-moment function implies now that

$$\begin{aligned}(7.4)\quad K_{d,t}(r)&=\mathcal{K}(B^d_r)\\ &=d\kappa_d\int_0^r\left(1+\frac{d-1}{2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tu\right)\right]t^{-2}u^{-2}\right)u^{d-1}\,\mathrm{d}u.\end{aligned}$$

Finally, the relationship between $g_{d,t}(r)$ and $K_{d,t}(r)$ leads in view of (7.4) to

$$\begin{aligned}g_{d,t}(r)&=\frac{1}{d\kappa_d r^{d-1}}\,d\kappa_d\left(1+\frac{d-1}{2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right]t^{-2}r^{-2}\right)r^{d-1}\\ &=1+\frac{d-1}{2}\left[1-\exp\left(-\frac{2\kappa_{d-1}}{d\kappa_d}tr\right)\right]t^{-2}r^{-2}.\end{aligned}$$

This completes the proof. ■

It is interesting to note the joint scale invariance of $g_{d,t}(r)$, i.e.,

$$g_{d,t}(\lambda r)=g_{d,t/\lambda}(r),\qquad \lambda>0.$$

We further note that in the special case $d = 2$ the pair-correlation function $g_{d,t}(r)$ becomes

$$g_{2,t}(r) = 1 + \frac{1}{2t^2r^2}\left[1 - \exp\left(-\frac{2}{\pi}rt\right)\right],$$

which was independently discovered by Weiß, Ohser, and Nagel by entirely different methods and is presented in [21]. However, it should be emphasized though that our approach developed above yields information also on higher dimensional cases. For example, we have for the spatial case $d = 3$

$$g_{3,t}(r) = 1 + \frac{1}{t^2r^2}\left[1 - \exp\left(-\frac{1}{2}tr\right)\right].$$

The result in Theorem 7.1 should be compared with pair-correlation function of the surface measure of a stationary and isotropic Poisson hyperplane tessellation $\mathrm{PHT}(t)$ having the same surface intensity $t > 0$. The latter will be denoted by $g_d^{\mathrm{PHT}(t)}(r)$. Using Slivnyak's theorem for Poisson processes (see [12], Theorem 3.3.5) one can easily show that

$$g_d^{\mathrm{PHT}(t)}(r) = 1 + \frac{(d-1)\kappa_{d-1}}{d\kappa_d tr}.$$

In particular, for the planar case $d = 2$, i.e., for the Poisson line tessellation abbreviated by $\mathrm{PLT}(t)$, we have

$$g_2^{\mathrm{PLT}(t)}(r) = 1 + \frac{1}{\pi tr}.$$

A comparison of $g_{2,t}(r)$ and $g_2^{\mathrm{PLT}(t)}(r)$ is shown in Figure 2.

Acknowledgments. The authors would like to thank Joachim Ohser (Darmstadt) and Claudia Redenbach (Kaiserslautern) for providing the simulations shown in Figure 1. The second author also thanks Matthias Reitzner (Osnabrück) and an anonymous referee for valuable comments, which were very helpful to improve the paper.

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Received on 16.1.2012;
revised version on 20.11.2012