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ON STRONG CONVERGENCE OF PRAMARTS IN BANACH SPACES

BY

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Abstract. We present new versions of strong convergence results for Banach-space-valued pramarts based on various tightness conditions combined with uniform integrability conditions of Mazur type. They extend a result of Egghe [7] stating that every Banach-space-valued pramart with a Cesàro-mean convergent subsequence converges strongly almost surely. Similar results are obtained for pramarts with values in the dual of a separable Banach space.

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1. INTRODUCTION

In [7] Egghe proved the following result:

THEOREM 1.1 ([7], Lemma 2.3). *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ satisfying the following condition*:

(C) *There exists a subsequence (f'_n) of (f_n) the Cesàro mean norm of which converges to a function $f_\infty \in L^1_X(\mathcal{F})$, that is*

$$\int_\Omega \left| \frac{1}{n} \sum_{i=1}^{i=n} f'_i - f_\infty \right| dP \to 0.$$

Then (f_n) converges strongly a.s. to f_∞.

In the above, $(X, |\cdot|)$ is a Banach space and $L^1_X(\mathcal{F})$ the space of (equivalence classes of) Bochner integrable X-valued functions over a probability space $(\Omega, \mathcal{F}, P)$.

Our goal in this paper is to extend Egghe's result to Banach-space-valued pramarts satisfying various tightness conditions combined with uniform integrability conditions of Mazur type. The paper is organized as follows. In Section 2 we give our notation and definitions. In Section 3 we deal with the strong convergence of pramarts in the space $L^1_X(\mathcal{F})$. The main result of this section (Theorem 3.1) ensures almost sure strong convergence of pramarts (f_n) under the following three conditions.

(1) There exist a sequence (g_n) with $g_n \in \text{co}\{f_i : i \geqslant n\}$ (i.e. g_n has the form $g_n = \sum_{i\geqslant n} \lambda_i^n f_i$, where $\lambda_i^n \geqslant 0$ for all $i \geqslant n$ and $\sum_{i\geqslant n} \lambda_i^n = 1$, but $\lambda_i^n > 0$ only for a finite number of indices) and an $\mathcal{R}(X_w)$-valued multifunction $\Gamma : \Omega \rightrightarrows X$ such that for almost all $\omega \in \Omega$ we have

$$g_n(\omega) \in \Gamma(\omega) \quad \text{for infinitely many indices } n.$$

(2) For each $y \in Y$, there exists a sequence (h_n) with $h_n \in \text{co}\{f_i : i \geqslant n\}$ such that

$$(\langle y, h_n \rangle) \text{ is uniformly integrable.}$$

(3) There exist a sequence (h'_n) with $h'_n \in \text{co}\{f_i : i \geqslant n\}$ and an integrable function φ from Ω into $\mathbb{R}^+$ such that

$$\liminf_{n\to\infty} |\langle y, h'_n \rangle| \leqslant \varphi \text{ a.s.} \quad \text{for each } y \in \overline{B}_Y.$$

In the above, $\mathcal{R}(X_w)$ denotes the collection of all nonempty weakly closed and weakly ball-compact subsets of X, Y is a norming subspace of X^*, and $\overline{B}_Y := Y \cap \overline{B}_{X^*}$ (X^* being the topological dual of X, and $\overline{B}_{X^*}$ its closed unit ball) (see Section 2).

We obtain this result by showing first that, under the above conditions (2) and (3), every pramart is pointwise bounded almost surely (Lemma 3.1).

We then proceed with two significant variants of Theorem 3.1 (Theorems 3.2 and 3.3) where the uniform integrability of the sequence (g_n) in condition (2) is relaxed to equi-integrability; but the tightness condition given in (1) is reinforced. The proofs depend on a new version of the well-known cluster point approximation theorem (Lemma 3.3).

Finally, in Section 4, we present results similar to those given in Section 3 for pramarts in the space $L^1_{X^*}[X](\Omega, F, P)$ of X-scalarly measurable functions f such that $\omega \to \|f(\omega)\|$ is P-integrable.

2. NOTATION AND PRELIMINARIES

In this paper, X stands for a Banach space, whose norm is denoted by $|\cdot|$, X^* for the topological dual of X, and $\overline{B}_{X^*}$ for the closed unit ball of X^*. We recall that a subspace Y of X^* is called *norming* if for every $x \in X$ we have

$$|x| = \sup\{\langle x^*, x\rangle : x^* \in \overline{B}_Y\},$$

where $\overline{B}_Y := Y \cap \overline{B}_{X^*}$. By w we denote the weak topology of X. The space X endowed with the topology w will be denoted by X_w. The collection of all subsets of X is denoted by 2^X. Further, recall that a subset C of X is said to be *w-ball* *compact* if the intersection of C with every closed ball is weakly compact. By $cwk(X_w)$ (resp. $\mathcal{R}(X_w)$) we denote the space of all nonempty w-compact convex subsets of X (resp. closed convex and w-ball compact subsets of X).

Next, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_n)_{n\geqslant 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is generated by $\bigcup_n \mathcal{F}_n$. We denote by $L^0_X(\mathcal{F})$ (resp. $L^1_X(\mathcal{F})$) the space of (equivalence classes of) $\mathcal{F}$-measurable (resp. Bochner integrable) X-valued functions. Recall that a sequence $(f_n)_{n\geqslant 1}$ in $L^1_X(\mathcal{F})$ is *uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \int_{\{\omega\in\Omega : |f_n(\omega)|\geqslant a\}} |f_n|\, dP = 0.$$

It is well known that (f_n) is uniformly integrable if it is bounded in $L^1_X(\mathcal{F})$ and equicontinuous, i.e.

$$\lim_{P(A)\to 0} \sup_{n\geqslant 1} \int_A |f_n|\, dP = 0.$$

A function $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ is called a *stopping time* with respect to $(\mathcal{F}_n)$ if for each $n \geqslant 1$, $\{\tau = n\} \in \mathcal{F}_n$. The set of all bounded stopping times with respect to $(\mathcal{F}_n)$ is denoted by T. For $\tau \in T$ and (f_n) an adapted sequence with respect to $(\mathcal{F}_n)$, recall that

$$f_\tau := \sum_{k=\min(\tau)}^{\max(\tau)} f_k 1_{\{\tau=k\}} \quad \text{and} \quad \mathcal{F}_\tau = \big\{A \in \mathcal{F} : A \cap \{\tau = k\} \in \mathcal{F}_k, \forall k \geqslant 1\big\}.$$

An adapted sequence $(f_n)_{n\geqslant 1}$ is a *pramart* in $L^1_X(\mathcal{F})$ if for every $\epsilon > 0$ there is $\sigma_0 \in T$ such that

$$\forall \sigma, \tau \in T,\ \big(\tau \geqslant \sigma \geqslant \sigma_0 \Rightarrow P(\{|f_\sigma - E^{\mathcal{F}_\sigma} f_\tau| > \epsilon\}) \leqslant \epsilon\big),$$

where $E^{\mathcal{F}_\sigma}$ denotes the conditional expectation with respect to $\mathcal{F}_\sigma$.

It is obvious that if $(f_n)_{n\geqslant 1}$ is a pramart in $L^1_X(\mathcal{F})$, then for every x^* in $\overline{B}_{X^*}$ the sequence $(\langle x^*, f_n\rangle)_{n\geqslant 1}$ is a pramart in $L^1_{\mathbb{R}}(\mathcal{F})$.

We end this section by introducing several concepts of tightness which will play a crucial role in this work. For this purpose, it will be convenient to say that a multifunction $\Gamma : \Omega \to 2^X$ is $\mathcal{F}$*-measurable* if its graph $\mathrm{Gr}(X)$ defined by

$$\mathrm{Gr}(X) := \{(\omega, x) \in \Omega \times X : x \in \Gamma(\omega)\}$$

belongs to $\mathcal{F} \otimes \mathcal{B}(X)$.

The following notion of tightness is given in [3]:

DEFINITION 2.1. Let $\mathcal{C}$ be a subcollection of 2^X. A sequence (f_n) in $L^0_X(\mathcal{F})$ is $\mathcal{S}(\mathcal{C})$*-tight* if there exists an $\mathcal{F}$-measurable $\mathcal{C}$-valued multifunction $\Gamma : \Omega \rightrightarrows X$ such that for almost all $\omega \in \Omega$ we have

$$(*) \qquad f_n(\omega) \in \Gamma(\omega) \text{ infinitely often (i.o.)}$$

(that is, $f_n(\omega) \in \Gamma(\omega)$ for infinitely many indices n).

REMARK 2.1. *If the values of* Γ *are bounded*, *then it is easy to check that* $(*)$ *is equivalent to*

$$\liminf_{n\to+\infty} d\big(0, f_n(\omega)\cap\Gamma(\omega)\big) < +\infty.$$

REMARK 2.2. *By the Eberlein–Šmulian theorem*, *the following implication*

$$(f_n)\ \mathcal{S}\big(cwk(X_w)\big)\text{-}tight \Rightarrow w\text{-ls}\, f_n \neq \emptyset \ a.s.$$

holds true. Conversely, *if* w-ls $f_n \neq \emptyset$ *a.s.*, *then the condition* $(*)$ *in Definition* 2.1 *is satisfied*, *but the multifunction* Γ *can fail to be* $\mathcal{F}$*-measurable.*

It is also necessary to introduce stronger notions of the above tightness condition, namely $\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tightness and $\mathcal{A}d$-(B)-$\mathcal{S}(\mathcal{C})$-tightness. We need an extra definition:

DEFINITION 2.2. A multifunction $\Gamma : \Omega \Rightarrow X$ is called *adaptedly measurable* (*$\mathcal{A}d$-measurable* for short) if there exists an increasing sequence $(A_n)_{n\geqslant 1}$ in $\mathcal{F}$ adapted to $(\mathcal{F}_n)$ with $\lim_{n\to\infty} P(A_n) = 1$ such that for each $n \geqslant 1$ the multifunction $1_{A_n}\Gamma$ is $\mathcal{F}_n$-measurable.

DEFINITION 2.3. A sequence (f_n) in $L^0_X(\mathcal{F})$ is *$\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tight* if there exists an $\mathcal{A}d$-measurable $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X$ such that for almost all $\omega \in \Omega$ we have

$$f_n(\omega) \in \Gamma(\omega) \text{ i.o.}$$

DEFINITION 2.4. A sequence (f_n) in $L^0_X(\mathcal{F})$ is called *$\mathcal{A}d$-(B)-$\mathcal{S}(\mathcal{C})$-tight* if there exists an $\mathcal{A}d$-measurable $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X$ such that

$$f_n(\omega) \in \Gamma(\omega) \text{ i.o.}$$

and

$$(**) \qquad \sup_{\tau\in T} \int_{\{\omega\in\Omega: f_\tau(\omega)\in\Gamma(\omega)\}} |f_\tau|\, dP < \infty.$$

Let Y be a subset of X^*. If the condition $(**)$ is replaced with

$$(**)' \qquad \sup_{\tau\in T} \int_{\{\omega\in\Omega: f_\tau(\omega)\in\Gamma(\omega)\}} |\langle y, f_\tau\rangle|\, dP < \infty \quad \text{for all } y \in Y,$$

then (f_n) is called *Y-scalarly $\mathcal{A}d$-(B)-$\mathcal{S}(\mathcal{C})$-tight*.

Notice that, for each $\tau \in T$, the set $\{\omega \in \Omega : f_\tau(\omega) \in \Gamma(\omega)\}$ is $\mathcal{F}$-measurable. Indeed, since $(f_n, n \geqslant 1)$ are strongly measurable, they are almost surely separably valued, so we may assume that they all take values in a separable subspace F of X.

If we set $\Gamma' := \Gamma \cap F$, then we have

$$(\dagger)\quad \{\omega \in \Omega : f_\tau(\omega) \in \Gamma'(\omega)\} = \{\omega \in \Omega : f_\tau(\omega) \in \Gamma(\omega)\} \text{ a.s.} \quad \text{for all } \tau \in T$$

(because $f_\tau(\omega) \in F$ for all $\omega \in \Omega$). Further, the multifunction Γ' has its graph in $\mathcal{F} \otimes \mathcal{B}(F)$, since $\mathrm{Gr}(\Gamma') = \mathrm{Gr}(\Gamma) \cap (\Omega \times F)$ and $\mathrm{Gr}(\Gamma) \in \mathcal{F} \otimes \mathcal{B}(X)$ (here recall that $\mathcal{F} \otimes \mathcal{B}(F) = \big(\mathcal{F} \otimes \mathcal{B}(X)\big) \cap (\Omega \times F)$). As $\mathrm{Gr}(f_\tau) \in \mathcal{F}_\tau \otimes \mathcal{B}(F)$, it is clear that $\mathrm{Gr}(f_\tau) \cap \mathrm{Gr}(\Gamma') \in \mathcal{F} \otimes \mathcal{B}(F)$. Observing that $\{\omega \in \Omega : f_\tau(\omega) \in \Gamma'(\omega)\}$ is the projection of $\mathrm{Gr}(f_\tau) \cap \mathrm{Gr}(\Gamma')$ onto Ω, we conclude that the set $\{\omega \in \Omega : f_\tau(\omega) \in \Gamma'(\omega)\}$ is a member of $\mathcal{F}$, and so is the set $\{\omega \in \Omega : f_\tau(\omega) \in \Gamma(\omega)\}$ (by ($\dagger$)) in view of the completeness hypothesis on the probability space $(\Omega, \mathcal{F}, P)$, the separability of F, and the classical projection theorem ([5], Theorem III.23).

3. STRONG CONVERGENCE OF PRAMARTS IN $L^1_X(\mathcal{F})$

The following proposition shows that condition (C) of Theorem 1.1 can be replaced with a (weaker) condition of Mazur type. This is the starting point of the present paper.

PROPOSITION 3.1. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ satisfying the following condition*:

(M) *There exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ and a function $f_\infty \in L^1_X(\mathcal{F})$ such that*

$$\int_\Omega |g_n - f_\infty|\, dP \to 0.$$

Then (f_n) converges strongly a.s. to f_∞.

The proof of this proposition is based essentially on Theorem 4.1 in [9] and the following lemma which will also be used on some other occasions.

LEMMA 3.1. *Let $(f_n)_{n\geqslant 1}$ be an adapted sequence in $L^1_X(\mathcal{F})$ satisfying the above condition* (M). *Then, for every increasing sequence (τ_m) in T,*

$$|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| \leqslant \sup_{k\geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| \;\; \textit{a.s. for all } m \geqslant 1$$

and

$$\limsup_m |f_{\tau_m}(\omega) - f_\infty(\omega)| \leqslant \limsup_m \sup_{k\geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| \;\; \textit{a.s.}$$

P r o o f. Let (τ_m) be an increasing sequence in T. Then writing

$$g_n := \sum_{i=n}^{p_n} \lambda_i^n f_i \quad \text{with} \quad \sum_{i=n}^{p_n} \lambda_i^n = 1 \text{ and } \lambda_i^n \geqslant 0,$$

we get, for every m and for every n such that $n \geqslant \tau_m$,

$$\begin{aligned}|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}} g_n(\omega)| &\leqslant \sum_{i=n}^{p_n} \lambda_i^n |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_i)(\omega)| \\ &\leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)|\end{aligned}$$

for almost all $\omega \in \Omega$. Consequently, we have the following estimation:

$$\begin{aligned}&|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| \\ &\quad \leqslant |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(g_n)(\omega)| + |E^{\mathcal{F}_{\tau_m}} g_n(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| \\ &\quad \leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + |E^{\mathcal{F}_{\tau_m}} g_n(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)|\end{aligned}$$

a.s. for every m and for every n such that $n \geqslant \tau_m$. Now, since

$$\int_\Omega |E^{\mathcal{F}_{\tau_m}}(g_n) - E^{\mathcal{F}_{\tau_m}}(f_\infty)| \, dP \leqslant \int_\Omega \|g_n - f_\infty\| \, dP \to 0 \qquad \text{for all } m \geqslant 1,$$

using the diagonal method we find a subsequence of (g_n), denoted similarly, such that

$$\lim_n |E^{\mathcal{F}_{\tau_m}}(g_n)(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| = 0 \text{ a.s.} \quad \text{for all } m \geqslant 1.$$

Therefore, passing to the limit as $n \to \infty$ in the estimation above, we get

$$|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| \leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)|$$

a.s. for every $m \geqslant 1$, and hence, by the triangle inequality and the classical Lévy theorem, we obtain

$$\begin{aligned}\limsup_m |f_{\tau_m}(\omega) - f_\infty(\omega)| &\leqslant \limsup_m \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}} f_k(\omega)| \\ &\quad + \lim_m |E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega) - f_\infty(\omega)| \\ &= \limsup_m \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| \text{ a.s.}\end{aligned}$$

This completes the proof of Lemma 3.1. ∎

Proof of Proposition 3.1. By the cluster point approximation theorem (Theorem 1 in [1]), we can choose a sequence (τ_m) in T with $\tau_m \geqslant m$ such that

(3.1) $$\limsup_n |f_n(\omega) - f_\infty(\omega)| = \lim_m |f_{\tau_m}(\omega) - f_\infty(\omega)| \text{ a.s.}$$

Then, by Lemma 3.1, we have

$$\limsup_m |f_{\tau_m}(\omega) - f_\infty(\omega)| \leqslant \limsup_m \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| \text{ a.s.}$$

On the other hand, as (f_n) is a pramart, by Theorem 4.1 in [9] (see also Theorem I.3.5.5 in [8] or Lemma 2.1 in [7]) we have

$$\lim_m \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| = 0 \text{ a.s.} \tag{3.2}$$

This equation together with (3.1) give

$$\lim_n |f_n(\omega) - f_\infty(\omega)| = 0 \text{ a.s.} \quad \blacksquare$$

Lemma 3.1 and the proof of Proposition 3.1 permit us also to formulate a more general result in which the condition (M) can be replaced with the following weaker one:

(SM) There exist a function $f_\infty \in L^1_X(\mathcal{F})$ and a norming subspace Y of X^* such that

$$\forall y \in Y,\ \lim_n \inf_{g \in \mathrm{co}\{f_i : i \geqslant n\}} \int_\Omega |\langle y, g\rangle - \langle y, f_\infty\rangle| \, dP = 0. \tag{‡}$$

Obviously, condition (‡) has the following equivalent formulation:

For each $y \in Y$, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ such that

$$\lim_n \int_\Omega |\langle y, g_n\rangle - \langle y, f_\infty\rangle| \, dP = 0.$$

PROPOSITION 3.2. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ satisfying the above condition* (SM). *Then (f_n) converges strongly a.s. to f_∞.*

P r o o f. Since f_n $(n \geqslant 1)$ and f_∞ are strongly measurable, they are a.s. separably valued, we can and do assume that X is separable. Consequently, $\overline{B}_{X^*}$ is weak* metrizable, and hence weak* separable, since it is weak* compact. Now, let f_∞ be as mentioned in (SM) and let y be an arbitrary fixed element in $\overline{B}_Y$. Then there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ (which may depend on y) such that

$$\int_\Omega |\langle y, g_n\rangle - \langle y, f_\infty\rangle| \, dP \to 0.$$

Noting that each $\langle y, g_n\rangle$ is a member of the set $\mathrm{co}\{\langle y, f_i\rangle : i \geqslant n\}$, it is possible to apply Lemma 3.1 to the real-valued adapted sequences $(\langle y, f_n\rangle)$, which gives

$$\begin{aligned} |\langle y, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)\rangle| &\leqslant \sup_{k \geqslant \tau_m} |\langle y, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)\rangle| \\ &\leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| \end{aligned}$$

a.s. for every $m \geqslant 1$, where (τ_m) is any increasing sequence in T satisfying (3.1). Since Y is a norming space, $\overline{B}_Y$ is weak* dense in $\overline{B}_{X^*}$. Consequently,

$$|\langle x^*, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)\rangle| \leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)|$$

a.s. for each $x^* \in \overline{B}_{X^*}$. Using the weak* separability of $\overline{B}_{X^*}$ we get, by a routine argument,

$$|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)| \leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)|$$

a.s. for every $m \geqslant 1$. Hence

$$|f_{\tau_m}(\omega) - f_\infty(\omega)| \leqslant \sup_{k \geqslant \tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + |E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega) - f_\infty(\omega)|$$

a.s. for every $m \geqslant 1$. Now, it remains only to use (3.1), (3.2) and the classical Lévy theorem. ■

As a consequence of Propositions 3.1 and 3.2 we have:

PROPOSITION 3.3. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ and let Y be a norming subspace of X^* such that the following conditions hold*:

(a) *There exists an $\mathcal{S}(cwk(X_w))$-tight sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$.*

(b) *For each $y \in Y$, there exists a sequence (h_n) with $h_n \in \mathrm{co}\{f_i : i \geqslant n\}$, such that $(\langle y, h_n\rangle)$ is uniformly integrable.*

(c) *There exist a sequence (h'_n) with $h'_n \in \mathrm{co}\{f_i : i \geqslant n\}$ and an integrable function φ from Ω into $\mathbb{R}^+$ such that*

$$\liminf_{n\to\infty} |\langle y, h'_n\rangle| \leqslant \varphi \text{ a.s.} \quad \text{for each } y \in \overline{B}_Y.$$

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that

$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

P r o o f. Since the functions f_n $(n \geqslant 1)$ are almost surely separably valued, we can assume without loss of generality that X is separable. Let $y \in Y$ be fixed and let (h_n) be a sequence associated with y according to (b). As the sequence $(\langle y, h_n\rangle)$ is uniformly integrable, there exist a subsequence (h_{n_k}) of (h_n) and a function $\varphi_y \in L^1_{\mathbb{R}}(\mathcal{F})$ such that

$$\lim_{k\to\infty} \int_\Omega |\langle y, h_{n_k}\rangle - \varphi_y| \, dP. \tag{3.3}$$

Consequently, since $h_{n_k} \in \mathrm{co}\{f_i : i \geqslant n_k\} \subset \mathrm{co}\{f_i : i \geqslant k\}$, the condition (SM) is satisfied for the $L^1_{\mathbb{R}}(\mathcal{F})$-pramart $(\langle y, f_n\rangle)$. So, by Proposition 3.1, we have

$$\lim_{n\to+\infty} \langle y, f_n\rangle = \varphi_y \text{ a.s.} \tag{3.4}$$

On the other hand, by (a), there exist a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ and an $\mathcal{F}$-measurable $cwk(X_w)$-valued multifunction $\Gamma : \Omega \rightrightarrows X$ such that for almost all $\omega \in \Omega$ we have

$$g_n(\omega) \in \Gamma(\omega) \quad \text{for infinitely many indices } n.$$

Then, for each ω outside a negligible set N, $\big(g_n(\omega)\big)$ admits a subsequence $\big(g'_n(\omega)\big)$ whose members belong to the w-compact set $\Gamma(\omega)$. Therefore, by the Eberlein–Šmulian theorem, one can find a subsequence of $\big(g'_n(\omega)\big)$, still denoted in the same manner, and an element $x_\omega \in X$ such that

$$\big(g'_n(\omega)\big) \ w\text{-converges to } x_\omega.$$

Define $f_\infty(\omega) := x_\omega$ for $\omega \in \Omega \setminus N$ and $f_\infty(\omega) := 0$ for $\omega \in N$. Then, taking into account (3.4), we get

$$\lim_{n\to+\infty} \langle y, f_n(\omega)\rangle = \lim_{n\to+\infty} \langle y, g'_n(\omega)\rangle = \langle y, f_\infty(\omega)\rangle = \varphi_y(\omega) \text{ a.s.} \tag{3.5}$$

This implies the $\mathcal{F}$-measurability of $\langle y, f_\infty\rangle$ for all $y \in Y$. Recalling that the ball $\overline{B}_Y$ is weak* dense in $\overline{B}_{X^*}$ and X is separable, we conclude that f_∞ is $\mathcal{F}$-measurable. Furthermore, (3.5) also shows that

$$\langle y, f_\infty\rangle = \lim_{n\to+\infty} \langle y, h'_n\rangle \text{ a.s.} \quad \text{for all } y \in Y,$$

where (h'_n) is a sequence as given in condition (c). This yields

$$|\langle y, f_\infty\rangle| \leqslant \varphi \text{ a.s.} \quad \text{for all } y \in \overline{B}_Y.$$

Equivalently,

$$|\langle x^*, f_\infty\rangle| \leqslant \varphi \text{ a.s.} \quad \text{for all } x^* \in \overline{B}_{X^*}.$$

Using the weak* separability of $\overline{B}_{X^*}$ we get

$$|f_\infty| \leqslant \varphi \text{ a.s.},$$

which, in view of the integrability of φ, shows that $|f_\infty|$ is integrable. Thus, we have $f_\infty \in L^1_X(\mathcal{F})$. Finally, from (3.3) and (3.5) it follows that

$$\lim_{k\to\infty} \int_\Omega |\langle y, h_{n_k}\rangle - \langle y, f_\infty\rangle|\, dP \quad \text{for all } y \in Y.$$

Therefore the condition (SM) is satisfied. The conclusion then follows from Proposition 3.2. ■

REMARK 3.1. *The above conditions* (b) *and* (c) *are implied by* (SM).

Indeed, let f_∞ and Y be given as in the condition (SM). Then to each $y \in Y$ there corresponds a sequence (g_n) with $g_n \in \operatorname{co}\{f_i : i \geqslant n\}$ such that

$$\lim_n \int_\Omega |\langle y, g_n\rangle - \langle y, f_\infty\rangle| \, dP = 0,$$

which yields the uniform integrability of the sequence $(\langle y, g_n\rangle)$, and so the condition (b) is satisfied. Furthermore, by the classical Fatou lemma we have

$$\liminf_n |\langle y, g_n\rangle - \langle y, f_\infty\rangle| = 0 \text{ a.s.},$$

which implies

$$\liminf_n |\langle y, g_n\rangle| \leqslant |\langle y, f_\infty\rangle| \leqslant |f_\infty| \text{ a.s.} \quad \text{for all } y \in \overline{B}_Y.$$

Hence the condition (c) follows by taking $\varphi := |f_\infty|$. ■

Surprisingly, the following theorem, which is our first main result, shows that in the above condition (a) the collection $cwk(X_w)$ can be replaced with $\mathcal{R}(X_w)$.

THEOREM 3.1. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ and Y be a norming subspace of X^* such that the following conditions hold*:

(a$'$) *There exists an $\mathcal{S}\big(\mathcal{R}(X_w)\big)$-tight sequence (g_n) with $g_n \in \operatorname{co}\{f_i : i \geqslant n\}$.*

(b) *For each $y \in Y$, there exists a sequence (h_n) with $h_n \in \operatorname{co}\{f_i : i \geqslant n\}$, such that $(\langle y, h_n\rangle)$ is uniformly integrable.*

(c) *There exist a sequence (h'_n) with $h'_n \in \operatorname{co}\{f_i : i \geqslant n\}$ and an integrable function φ from Ω into $\mathbb{R}^+$ such that*

$$\liminf_{n\to\infty} |\langle y, h'_n\rangle| \leqslant \varphi \textit{ a.s.} \quad \textit{for each } y \in \overline{B}_Y.$$

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that

$$(f_n) \textit{ converges strongly a.s. to } f_\infty.$$

Before proceeding with the proof of Theorem 3.1, we note an immediate corollary.

COROLLARY 3.1. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ satisfying the following conditions*:

(a$'$) *There exists an $\mathcal{S}\big(\mathcal{R}(X_w)\big)$-tight sequence (g_n) with $g_n \in \operatorname{co}\{f_i : i \geqslant n\}$.*

(b$'$) *There exists a uniformly integrable sequence (h_n) with $h_n \in \operatorname{co}\{f_i : i \geqslant n\}$.*

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that

$$(f_n) \textit{ converges strongly a.s. to } f_\infty.$$

P r o o f. This follows from the fact that the conditions (b) and (c) are implied by (b$'$). ■

The proof of Theorem 3.1 uses the following important lemma (compare with Lemma VIII 2.4.1 in [8]).

LEMMA 3.2. *Let* $(f_n)_{n\geqslant 1}$ *be a pramart in* $L^1_X(\mathcal{F})$ *and* Y *be a norming subspace of* X^* *such that the conditions* (b) *and* (c) *hold. Then* $\limsup_n |f_n| \in L^1_{\mathbb{R}}(\mathcal{F})$. *Consequently*, (f_n) *is pointwise bounded almost surely.*

Proof. As in the preceding proofs, we may suppose that X is separable. According to the cluster point approximation theorem (Theorem 1 in [1]), we can choose an increasing sequence (τ_m) in T with $\tau_m \geqslant m$ such that

$$\limsup_n |f_n(\omega)| = \lim_m |f_{\tau_m}(\omega)| \text{ a.s.} \tag{3.6}$$

Now, from the condition (b) and the proof of Proposition 3.3 we know that for each $y \in \overline{B}_Y$ the $L^1_{\mathbb{R}}(\mathcal{F})$-pramart $(\langle y, f_n\rangle)$ satisfies (3.3) and (3.4). Therefore, by (3.3) and Lemma 3.1, we have

$$\begin{aligned}
&|\langle y, f_{\tau_m}(\omega)\rangle - \varphi_y(\omega)| \\
&\qquad \leqslant \sup_{k\geqslant\tau_m} |\langle y, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)\rangle| + |E^{\mathcal{F}_{\tau_m}}(\varphi_y)(\omega) - \varphi_y(\omega)| \\
&\qquad \leqslant \sup_{k\geqslant\tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + E^{\mathcal{F}_{\tau_m}}(|\varphi_y|)(\omega) + |\varphi_y(\omega)|
\end{aligned}$$

a.s. for every $m \geqslant 1$. This implies

$$|\langle y, f_{\tau_m}(\omega)\rangle| \leqslant \sup_{k\geqslant\tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + E^{\mathcal{F}_{\tau_m}}(|\varphi_y|)(\omega) + 2|\varphi_y(\omega)| \text{ a.s.}$$

Further, let (h'_n) be a sequence as given in the condition (c). Since, by (3.4) we have

$$|\varphi_y(\omega)| = \lim_{n\to+\infty} |\langle y, h'_n(\omega)\rangle| \leqslant \varphi(\omega) \text{ a.s.}$$

and $\varphi \in L^1_{\mathbb{R}}(\mathcal{F})$ (by (c)), we get

$$|\langle y, f_{\tau_m}(\omega)\rangle| \leqslant \sup_{k\geqslant\tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + E^{\mathcal{F}_{\tau_m}}(\varphi)(\omega) + 2\varphi(\omega) \text{ a.s.}$$

Hence, by the weak* denseness of $\overline{B}_Y$ in $\overline{B}_{X^*}$ and the weak* separability of $\overline{B}_{X^*}$,

$$|f_{\tau_m}(\omega)| \leqslant \sup_{k\geqslant\tau_m} |f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)| + E^{\mathcal{F}_{\tau_m}}(\varphi)(\omega) + 2\varphi(\omega)$$

a.s. for every $m \geqslant 1$. Using (3.2), (3.6), and the classical Lévy theorem we get

$$\limsup_n |f_n(\omega)| \leqslant 3\varphi(\omega) \text{ a.s.}$$

Thus $\limsup_n |f_n| \in L^1_{\mathbb{R}}(\mathcal{F})$. ■

P r o o f o f T h e o r e m 3.1. By conditions (b), (c), and Lemma 3.2, (f_n) is pointwise bounded. Noting that a sequence whose members are convex combinations of a pointwise bounded sequence is still pointwise bounded, we conclude that (f_n) satisfies condition (a), since it satisfies (a′). Thus we return to the situation of Proposition 3.3. ■

REMARK 3.2. For Proposition 3.3 as well as for Theorem 3.1 and its Corollary 3.1, the measurability of the multifunction Γ that appears in Definition 2.1 is not essential.

Before passing to the next result, we provide an interesting corollary. For the convenience of the reader, recall first the following classical notion of tightness:

A sequence (f_n) in $L^0_X(\mathcal{F})$ is *$\mathcal{C}$-tight* if, for every $\epsilon > 0$, there is a $\mathcal{C}$-valued $\mathcal{F}$-measurable multifunction $\Gamma_\epsilon : \Omega \rightrightarrows X$ such that

$$\inf_n P\big(\{\omega \in \Omega : f_n(\omega) \in \Gamma_\epsilon(\omega)\}\big) \geqslant 1 - \epsilon.$$

REMARK 3.3. *If a sequence (f_n) in $L^0_X(\mathcal{F})$ is $\mathcal{C}$-tight, then it is $\mathcal{S}(\mathcal{C})$-tight.*

Indeed, consider $\epsilon_q := 1/q$, $q \geqslant 1$. Then there is an $\mathcal{F}$-measurable $\mathcal{C}$-valued multifunction $\Gamma_{\epsilon_q} : \Omega \rightrightarrows X$, denoted simply by Γ_q, such that $\inf_n P(A_{n,q}) \geqslant 1 - \epsilon_q$, where we put $A_{n,q} := \{\omega \in \Omega : f_n(\omega) \in \Gamma_q(\omega)\}$. Now, we define the sequence $(\Omega_q)_{q\geqslant 1}$ by $\Omega_q = \limsup_{n\to+\infty} A_{n,q}$ and the multifunction Γ on Ω by $\Gamma = 1_{\Omega'_1}\Gamma_1 + \sum_{q\geqslant 2} 1_{\Omega'_q}\Gamma_q$, where $\Omega'_1 = \Omega_1$ and $\Omega'_q = \Omega_q \setminus \bigcup_{i<q} \Omega_i$ for all $q > 1$. Then the inequality $\inf_n P(A_{n,q}) \geqslant 1 - \epsilon_q$ implies

$$P(\Omega_q) = \lim_{n\to\infty} P\Big(\bigcup_{m\geqslant n} A_{m,q}\Big) \geqslant 1 - \epsilon_q \to 1.$$

Further, for each $\omega \in \Omega_q$ we have $\omega \in A_{n,q}$ i.o. This means $f_n(\omega) \in \Gamma_q(\omega)$ i.o., in view of the definition of $A_{n,q}$. Since $\Omega'_q \subset \Omega_q$, $\bigcup_q \Omega_q = \bigcup_q \Omega'_q = \Omega$ a.s., and $\Gamma(\omega) = \Gamma_q(\omega)$ on Ω'_q, it follows that $f_n(\omega) \in \Gamma(\omega)$ i.o. for almost all $\omega \in \Omega$. ■

COROLLARY 3.2. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ such that the conditions* (a″), (b), *and* (c) *hold*, *where*

(a″) *There exists an $\mathcal{R}(X_w)$-tight sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$.*

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that

$$(f_n) \textit{ converges strongly a.s. to } f_\infty.$$

Conditions (b) *and* (c) *can be replaced with* (b′).

P r o o f. It is a consequence of Remark 3.3 and Theorem 3.1. ■

REMARK 3.4. An inspection of the preceding proofs reveals that in conditions (a), (a′), (a″), (b), (b′) and (c), we may change $\mathrm{co}\{f_i\colon i \geqslant n\}$ into $\mathrm{co}\{f_{\tau_i}\colon i \geqslant n\}$, where (τ_n) is an arbitrary cofinal increasing sequence in T corresponding to each condition. Without loss of generality, these new (weaker) conditions will be denoted in a similar way.

We now deal with the case of $\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tight pramarts. An approximation lemma that may be of independent interest is obtained first. (Compare with Theorem 1 in [1].)

LEMMA 3.3. *Let $(f_n)_{n\geqslant 1}$ be an $\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tight adapted sequence in $L^1_X(\mathcal{F})$. Then there exists an increasing sequence (τ_n) in T with $\tau_n \geqslant n$ such that for each ω outside a negligible set there exists a positive integer n_ω such that*

$$\forall n \geqslant n_\omega,\ \ f_{\tau_n}(\omega) \in \Gamma(\omega),$$

where Γ is the multifunction which appears in the σ-$\mathcal{S}(\mathcal{C})$-tightness condition.

Proof. $\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tightness entails the existence of a σ-measurable $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X$ such that

$$f_n(\omega) \in \Gamma(\omega) \text{ i.o.} \tag{3.7}$$

Now, by the $\mathcal{A}d$-measurability of Γ, one can choose an increasing sequence $(A_n)_{n\geqslant 1}$ in $\mathcal{F}$ adapted to $(\mathcal{F}_n)$ with $\lim_{n\to\infty} P(A_n) = 1$ such that the multifunction $1_{A_n}\Gamma$ is $\mathcal{F}_n$-measurable for each $n \geqslant 1$. For each $m, n \geqslant 1$ let us define the set

$$F_n^m := \begin{cases} \{\omega \in A_m : f_n(\omega) \in \Gamma(\omega)\} & \text{if } n \geqslant m, \\ \emptyset & \text{otherwise} \end{cases}$$

and the function

$$\theta_n^m := 1_{F_n^m}.$$

Then $(\theta_n^m)_n$ is adapted with respect to $(\mathcal{F}_n)$ and, by (3.7), we have

$$\omega \in F_n^m \text{ i.o.}$$

for every $m \geqslant 1$ and for every $\omega \in A_m$ outside a negligible set N. This means that there exists an increasing sequence (n_k) of positive integers such that $\omega \in F_{n_k}^m$ for all $k \geqslant 1$. Equivalently, for each $m \geqslant 1$ and for each $\omega \in A_m \setminus N$, $1_{A_m}(\omega)$ is a cluster point of the sequence $\big(\theta_n^m(\omega)\big)$. Consequently, using the cluster point approximation theorem one can find an increasing sequence (τ_n^m) in T with $\tau_n^m \geqslant n$ such that

$$1_{A_m} = \lim_{n\to+\infty} \theta_{\tau_n^m}^m \text{ a.s.},$$

and hence in probability. Thus it is possible to find two strictly increasing sequences $\big(\alpha(n)\big)$ and $\big(\beta(n)\big)$ of positive integers with $\alpha(n) \geqslant \beta(n)$ for all $n \geqslant 1$

such that if we set $\tau_n := \tau_{\alpha(n)}^{\beta(n)}$, then

$$\lim_{n\to+\infty} \theta_{\tau_n}^{\beta(n)} - 1_{A_{\beta(n)}} = 0 \text{ a.s.},$$

and so

$$\lim_{n\to+\infty} \theta_{\tau_n}^{\beta(n)} = 1 \text{ a.s.},$$

since $A_{\beta(n)} \uparrow \Omega$ a.s. Note that $\tau_n \in T$ and $\tau_n = \tau_{\alpha(n)}^{\beta(n)} \geqslant \alpha(n) \geqslant \beta(n) \geqslant n$ for all $n \geqslant 1$. So, by the definition of the F_n^m's, we have

$$\theta_{\tau_n}^{\beta(n)}(\omega) = 1_{\{\omega' \in A_{\beta(n)} : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega)$$

for all $\omega \in \Omega$ and for all $n \geqslant 1$. Further, passing to a subsequence, if necessary, we may suppose $\tau_{n+1} \geqslant \tau_n$ for all $n \geqslant 1$. Finally, from the two previous equalities, the fact that $A_{\beta(n)} \uparrow \Omega$ a.s. and the following decomposition

$$\begin{aligned} &1_{\{\omega' \in \Omega : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega) \\ &\quad = 1_{\{\omega' \in \Omega \setminus A_{\beta(n)} : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega) + 1_{\{\omega' \in A_{\beta(n)} : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega) \\ &\quad = 1_{\{\omega' \in \Omega \setminus A_{\beta(n)} : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega) + \theta_{\tau_n}^{\beta(n)}(\omega) \qquad (\omega \in \Omega) \end{aligned}$$

it follows that

$$\lim_{n\to+\infty} 1_{\{\omega' \in \Omega : f_{\tau_n(\omega)}(\omega') \in \Gamma(\omega')\}}(\omega) = 1 \text{ a.s.}$$

Obviously, this means that for each ω outside a negligible set, there exists a positive integer n_ω such that $f_{\tau_n}(\omega) \in \Gamma(\omega)$ for all $n \geqslant n_\omega$, which is the desired conclusion. ■

We can now prove the following significant variant of Theorem 3.1.

THEOREM 3.2. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L_X^1(\mathcal{F})$ and Y be a norming subspace of X^* such that the following conditions hold*:

(a)$^+$ *(f_n) is $\mathcal{A}d$-$\mathcal{S}\big(cwk(X_w)\big)$-tight.*

(b)$^-$ *For every $y \in Y$ and for every increasing sequence (τ_n) in T there exists a sequence (h_n) with $h_n \in \operatorname{co}\{f_{\tau_i} : i \geqslant n\}$ such that $(\langle y, h_n\rangle)$ is equi-integrable.*

(c) *There exist an increasing sequence (τ_n) in T, a sequence (h'_n) with $h'_n \in \operatorname{co}\{f_{\tau_i} : i \geqslant n\}$, and an integrable function φ from Ω into $\mathbb{R}^+$ such that*

$$\liminf_{n\to\infty} |\langle y, h'_n\rangle| \leqslant \varphi \ \text{ a.s.} \quad \text{for each } y \in \overline{B}_Y.$$

Then there exists a function $f_\infty \in L_X^1(\mathcal{F})$ such that

$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

P r o o f. By the $\mathcal{A}d$-$\mathcal{S}(\mathcal{C})$-tightness assumption and Lemma 3.3, there exist an $\mathcal{A}d$-measurable $cwk(X_w)$-valued multifunction $\Gamma : \Omega \Rightarrow X$ and an increasing sequence (τ_n) in T, with $\tau_n \geqslant n$, such that for each ω outside a negligible set N one can find an integer $n_\omega \geqslant 1$ satisfying:

$$\forall n \geqslant n_\omega, \;\; f_{\tau_n}(\omega) \in \Gamma(\omega). \tag{3.8}$$

Now, let $y \in Y$ be arbitrarily fixed and let (h_n) be a sequence associated with y according to (b)$^-$. We claim that the sequence $(\langle y, h_n\rangle)$ is bounded in $L^1_{\mathbb{R}}(\mathcal{F})$. To prove this let

$$B_m := \{\omega \in \Omega : \sup_n |f_{\tau_n}(\omega)| \leqslant m\}.$$

From (3.8) and the fact that the sets $\Gamma(\omega)$ $(\omega \in \Omega)$ are bounded it follows that $\lim_{m\to\infty} P(B_m) = 1$. As the sequence $(\langle y, h_n\rangle)$ is equi-integrable, there exists an integer $m_0 \geqslant 1$ such that

$$\sup_n \int_{\Omega\setminus B_{m_0}} |\langle y, h_n\rangle| \, dP \leqslant 1. \tag{3.9}$$

Further, again by (3.8), it is not difficult to check that for each ω in $\Omega \setminus N$

$$h_n(\omega) \in \Gamma(\omega) \quad \text{ for all } n \geqslant n_\omega$$

since Γ is convex-valued. Equivalently,

$$\lim_{n\to\infty} 1_{\{\omega\in\Omega: h_n(\omega)\in\Gamma(\omega)\}} = 1 \text{ a.s.},$$

which implies

$$\lim_{n\to\infty} P\big(\{\omega \in \Omega : h_n(\omega) \in \Gamma(\omega)\}\big) = 1,$$

and so, again using the equi-integrability of $(\langle y, h_n\rangle)$, we obtain

$$\lim_{n\to\infty} \int_{\{\omega\in\Omega: h_n(\omega)\notin\Gamma(\omega)\}} |\langle y, h_n\rangle| \, dP = 0.$$

This equation together with the decomposition

$$\int_{B_{m_0}} |\langle y, h_n\rangle| \, dP = \int_{B_{m_0}\cap\{\omega\in\Omega: h_n(\omega)\in\Gamma(\omega)\}} |\langle y, h_n\rangle| \, dP + \int_{B_{m_0}\cap\{\omega\in\Omega: h_n(\omega)\notin\Gamma(\omega)\}} |\langle y, h_n\rangle| \, dP$$

imply

$$\limsup_{n\to\infty} \int_{B_{m_0}} |\langle y, h_n\rangle| \, dP \leqslant \int_{B_{m_0}} \sup_n |f_{\tau_n}(\omega)| \, dP \leqslant m_0 P(B_{m_0}).$$

Hence
$$\sup_n \int_{B_{m_0}} |\langle y, h_n\rangle|\, dP < \infty.$$
Putting this formula together with (3.9), we get
$$\sup_n \int_{\Omega} |\langle y, h_n\rangle|\, dP < \infty,$$
as claimed. Consequently, the sequence $(\langle y, h_n\rangle)$ is uniformly integrable, so that condition (b) is satisfied. Conditions (a) and (c) are also satisfied. Therefore, taking into account Remark 3.4, we see that Theorem 3.2 is a consequence of Theorem 3.1. ■

COROLLARY 3.3. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ satisfying the following two conditions*:

(a)$^+$ *(f_n) is $\mathcal{A}d$-$\mathcal{S}\big(cwk(X_w)\big)$-tight.*

(b$'$)$^-$ *For every increasing sequence (τ_n) in T, there exists a sequence (h_n) with $h_n \in \mathrm{co}\{f_{\tau_i} : i \geqslant n\}$ such that $(|h_n|)$ is equi-integrable.*

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that
$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

P r o o f. Following *mutatis mutandis* the arguments of the proof above by replacing $|\langle y, f_{\tau_n}\rangle|$ with $|f_{\tau_n}|$, and by using (b$'$)$^-$ instead of (b)$^-$, we show that the sequence $(|f_{\tau_n}|)$ is uniformly integrable. So, by Remark 3.4, condition (b$'$) is satisfied. Condition (a$'$) is also satisfied, since it is implied by (a)$^+$. Thus, we return to the situation of Corollary 3.1. ■

We conclude this section by providing a version of Theorem 3.2 for $\mathcal{C} = \mathcal{R}(X_w)$, but this time under the $\mathcal{A}d$-(B)-$\mathcal{S}(\mathcal{C})$-tightness condition.

THEOREM 3.3. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ and Y be a norming subspace of X^* such that the conditions* (a$'$)$^+$, (b)$^-$, *and* (c) *hold*, *where*

(a$'$)$^+$ *(f_n) is $\mathcal{A}d$-(B)-$\mathcal{S}\big(\mathcal{R}(X_w)\big)$-tight.*

Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that
$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

P r o o f. We will show that condition (b) is satisfied; once this is done we can use Theorem 3.1 and Remark 3.4 to get the desired conclusion. By the $\mathcal{A}d$-(B)-$\mathcal{S}\big(\mathcal{R}(X_w)\big)$-tightness assumption and Lemma 3.3, there exist a σ-measurable $\mathcal{R}(X_w)$-valued multifunction $\Gamma : \Omega \rightrightarrows X$ and an increasing sequence (τ_n) in T with $\tau_n \geqslant n$ such that for each ω outside a negligible set N one can find an integer $n_\omega \geqslant 1$ satisfying (3.8) and
$$\forall y \in Y,\ \sup_n \int_{\{\omega\in\Omega : f_{\tau_n}\in\Gamma(\omega)\}} |\langle y, f_{\tau_n}\rangle|\, dP < \infty. \tag{3.10}$$

Next, let $y \in Y$ be arbitrarily fixed and let (h_n) be a sequence associated with y according to (b)$^-$. We want to show that the sequence $(\langle y, h_n\rangle)$ is bounded in $L^1_{\mathbb{R}}(\mathcal{F})$. First, observe that (3.8) can be expressed as follows:

$$\lim_{n\to\infty} 1_{\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} = 1 \text{ a.s.,}$$

which implies

$$\lim_{n\to\infty} P\big(\big\{\omega \in \Omega : \bigcup_{p\geqslant n} \{f_{\tau_p}(\omega)\} \subset \Gamma(\omega)\big\}\big) = 1.$$

As the sequence $(\langle y, h_n\rangle)$ is equi-integrable (by (b)$^-$), we deduce that

$$\lim_{n\to\infty} \int_{\Omega\setminus\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_n}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, h_n\rangle|\, dP = 0.$$

This equation together with the decomposition

$$\begin{aligned}\int_\Omega |\langle y, h_n\rangle|\, dP = &\int_{\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, h_n\rangle|\, dP\\ &+ \int_{\Omega\setminus\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, h_n\rangle|\, dP\end{aligned}$$

imply

$$\limsup_{n\to\infty} \int_\Omega |\langle y, h_n\rangle|\, dP \leqslant \sup_n \int_{\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, h_n\rangle|\, dP.$$

Since each h_n is of the form

$$h_n := \sum_{i\in J_n} \mu_i^n f_{\tau_{i+n}} \quad \text{with } \sum_{i\in J_n} \mu_i^n = 1 \text{ and } \mu_i^n \geqslant 0,$$

we get

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega |\langle y, h_n\rangle|\, dP &\leqslant \sup_n \sum_{i\in J_n} \mu_i^n \int_{\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, f_{\tau_{i+n}}\rangle|\, dP\\ &\leqslant \sup_n \sup_{m\geqslant n} \int_{\{\omega\in\Omega:\cup_{p\geqslant n}\{f_{\tau_p}(\omega)\}\subset\Gamma(\omega)\}} |\langle y, f_{\tau_m}\rangle|\, dP\\ &\leqslant \sup_m \int_{\{\omega\in\Omega: f_{\tau_m}(\omega)\in\Gamma(\omega)\}} |\langle y, f_{\tau_m}\rangle|\, dP,\end{aligned}$$

where the last inequality follows from the fact that

$$\big\{\omega \in \Omega : \bigcup_{p\geqslant n} \{f_{\tau_p}(\omega)\} \subset \Gamma(\omega)\big\} \subset \{\omega \in \Omega : f_{\tau_m}(\omega) \in \Gamma(\omega)\} \quad \text{for all } m \geqslant n.$$

Thus, by (3.10), we obtain

$$\sup_n \int_\Omega |\langle y, h_n\rangle|\, dP < \infty.$$

Consequently, the sequence $(\langle y, h_n\rangle)$ is uniformly integrable, so that condition (b) is satisfied. ∎

Under the condition (b′)$^-$, Theorem 3.3 has the following formulation:

COROLLARY 3.4. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_X(\mathcal{F})$ such that the conditions* (a′)$^+$ *and* (b′)$^-$ *hold. Then there exists a function $f_\infty \in L^1_X(\mathcal{F})$ such that*

$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

Proof. If we replace in the proof of Theorem 3.3 the sequence $(|\langle y, f_{\tau_n}\rangle|)$ with $(|f_{\tau_n}|)$, we obtain the uniform integrability of (f_{τ_n}), and so the condition (b′) is satisfied. The conclusion then follows from Corollary 3.1 and Remark 3.4. ∎

4. THE CASE OF PRAMARTS WITH VALUES IN A DUAL SPACE

In this section $(X, |\cdot|)$ is a separable Banach space and $(x_\ell)_{\ell\geqslant 1}$ is a fixed dense sequence in the closed unit ball $\overline{B}_X$ of X. We denote by X^* the topological dual of X, and the dual norm by $\|\cdot\|$. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. If t is a topology on X^*, the space X^* endowed with t is denoted by X^*_t. Three topologies will be considered on X^*, namely the norm topology s^*, the weak topology $w = \sigma(X^*, X^{**})$, and the weak-star topology $w^* = \sigma(X^*, X)$. The collection of all subsets of X^* is denoted by 2^{X^*}.

Let $(C_n)_{n\geqslant 1}$ be a sequence of subsets of X^*. The *sequential weak upper limit* w-ls C_n of (C_n) is defined by

$$w\text{-ls}\, C_n = \{x^* \in X^* : x^* = w\text{-}\lim_{j\to+\infty} x^*_{n_j}\,,\ x^*_{n_j} \in C_{n_j}\}$$

and the *topological weak upper limit* w-LS C_n of (C_n) is denoted by w-LS C_n and is defined by

$$w\text{-LS}\, C_n = \bigcap_{n\geqslant 1} w\text{-cl} \bigcup_{k\geqslant n} C_n,$$

where w-cl denotes the closed hull operation in the weak topology. The following inclusion is easy to check:

$$w\text{-ls}\, C_n \subseteq w\text{-LS}\, C_n.$$

Conversely, if the C_n are contained in a fixed weakly compact subset, then both sides coincide.

As in the previous section, $(\Omega, \mathcal{F}, P)$ stands for a complete probability space and $(\mathcal{F}_n)_{n \geqslant 1}$ for an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\bigcup_n \mathcal{F}_n$. The set of all bounded stopping times with respect to $(\mathcal{F}_n)$ is denoted by T. A function $f : \Omega \to X^*$ is said to be *X-scalarly $\mathcal{F}$-measurable* (or, simply, *scalarly $\mathcal{F}$-measurable*) if the real-valued function $\omega \to \langle x, f(\omega)\rangle$ is measurable with respect to the σ-field $\mathcal{F}$ for all $x \in X$. We say also that f is *weak*-$\mathcal{F}$-measurable*. Recall that if $f : \Omega \to X^*$ is a scalarly $\mathcal{F}$-measurable function such that $\langle x, f\rangle \in L^1_{\mathbb{R}}(\mathcal{F})$ for all $x \in X$, then for each $A \in \mathcal{F}$ there is $x^* \in X^*$ such that

$$\forall x \in X,\ \langle x, x^*\rangle = \int_A \langle x, f\rangle \, dP.$$

The vector x^* is called the *weak* integral* (or *Gelfand integral*) of f over A and is denoted simply by $\int_A f \, dP$. We denote by $L^0_{X^*}[X](\mathcal{F})$ (resp. $L^1_{X^*}[X](\mathcal{F})$) the space of all (classes of) scalarly $\mathcal{F}$-measurable functions (resp. scalarly $\mathcal{F}$-measurable functions f such that $\omega \to \|f(\omega)\|$ is P-integrable).

Next, let $(f_n)_{n \geqslant 1}$ be a sequence in $L^1_{X^*}[X](\mathcal{F})$. If each f_n is $\mathcal{F}_n$-scalarly measurable, we say that (f_n) is *adapted* with respect to $(\mathcal{F}_n)$. For $\tau \in T$ and (f_n) an adapted sequence with respect to $(\mathcal{F}_n)$, recall that

$$f_\tau := \sum_{k=\min(\tau)}^{\max(\tau)} f_k 1_{\{\tau = k\}} \quad \text{and} \quad \mathcal{F}_\tau = \big\{A \in \mathcal{F} : A \cap \{\tau = k\} \in \mathcal{F}_k, \forall k \geqslant 1\big\}.$$

It is readily seen that f_τ is $\mathcal{F}_\tau$-scalarly measurable.

DEFINITION 4.1. An adapted sequence $(f_n)_{n \geqslant 1}$ in $L^1_{X^*}[X](\mathcal{F})$ is a *pramart* if for every $\epsilon > 0$ there is $\sigma_0 \in T$ such that for every σ and τ in T with $\tau \geqslant \sigma \geqslant \sigma_0$ we have

$$P(\{\|f_\sigma - E^{\mathcal{F}_\sigma} f_\tau\| > \epsilon\}) < \epsilon,$$

where $E^{\mathcal{F}_n}$ denotes the (Gelfand) conditional expectation with respect to $\mathcal{F}_n$. It should be noted that conditional expectation of a Gelfand function in $L^1_{X^*}[X](\mathcal{F})$ always exists (see [11], Proposition 7, p. 366, and [13], Theorem 3).

It is obvious that if $(f_n)_{n \geqslant 1}$ is a pramart in $L^1_{X^*}[X](\mathcal{F})$, then for every x in $\overline{B}_X$, the sequence $(\langle x, f_n\rangle)_{n \geqslant 1}$ is a pramart in $L^1_{\mathbb{R}}(\mathcal{F})$.

Now it is convenient to reformulate the tightness conditions introduced in Section 3 in the space $L^0_{X^*}[X](\mathcal{F})$. For this purpose, let $\mathcal{C} = cwk(X^*_w)$ or $\mathcal{R}(X^*_w)$, where $cwk(X^*_w)$ (resp. $\mathcal{R}(X^*_w)$) denotes the space of all nonempty $\sigma(X^*, X^{**})$-compact convex subsets of X^*_w (resp. closed convex subsets of X^*_w such that their intersections with any closed ball are weakly compact). A $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X^*$ is *$\mathcal{F}$-measurable* if its graph $\mathrm{Gr}(\Gamma)$, defined by

$$\mathrm{Gr}(\Gamma) := \{(\omega, x^*) \in \Omega \times X^* : x^* \in \Gamma(\omega)\},$$

belongs to $\mathcal{F} \otimes \mathcal{B}(X^*_{w^*})$.

Let (f_n) be a sequence in $L^0_{X^*}[X](\mathcal{F})$.

• (f_n) is said to be $\mathcal{S}(\mathcal{C})$-*tight* if there exists an $\mathcal{F}$-measurable $\mathcal{C}$-valued multifunction $\Gamma : \Omega \Rightarrow X^*$ such that for almost all $\omega \in \Omega$ we have

$$f_n(\omega) \in \Gamma(\omega) \text{ i.o.}$$

• (f_n) is said to be $\mathcal{C}$-*tight* if for every $\epsilon > 0$ there is a $\mathcal{C}$-valued $\mathcal{F}$-measurable multifunction $\Gamma_\epsilon : \Omega \Rightarrow X^*$ such that

$$\inf_n P\big(\{\omega \in \Omega : f_n(\omega) \in \Gamma_\epsilon(\omega)\}\big) \geqslant 1 - \epsilon.$$

Next, let us introduce the following notion of σ-measurability (see [10]).

DEFINITION 4.2. A function f in $L^0_{X^*}[X](\mathcal{F})$ is said to be σ-*measurable* if there exists an adapted sequence $(\Gamma_n)_{n\geqslant 1}$ (that is, for each integer $n \geqslant 1$, Γ_n is $\mathcal{F}_n$-measurable) of $\mathcal{R}(X^*_w)$-valued multifunctions such that

$$f(\omega) \in s^*\text{-cl co}\Big(\bigcup_n \Gamma_n\Big) \text{ a.s.}$$

The sequence (Γ_n) given in this definition can be assumed to be adapted with respect to a subsequence of $(\mathcal{F}_n)$.

PROPOSITION 4.1 ([10], Proposition 4.2). *Let $(f_n)_{n\geqslant 1}$ be an adapted sequence in $L^0_{X^*}[X](\mathcal{F})$ and f_∞ be a function in $L^0_{X^*}[X](\mathcal{F})$ such that*
(i) (f_n) *is* $\mathcal{S}\big(cwk(X^*_w)\big)$-*tight*;
(ii) $\lim_{n\to\infty}\langle x_\ell, f_n\rangle = \langle x_\ell, f_\infty\rangle$ *a.s. for all* ℓ.
Then f_∞ is σ-measurable.

The following version of Lévy's theorem in the framework of a dual space will be crucial in the sequel.

THEOREM 4.1 ([10], Proposition 4.5). *Let f be a function in $L^1_{X^*}[X](\mathcal{F})$. Then the following two statements are equivalent*:
(i) $\big(E^{\mathcal{F}_n}(f)\big)$ *converges strongly a.s. to f.*
(ii) f *is σ-measurable.*

REMARK 4.1. *Condition* (i) *of Theorem* 4.1 *can be replaced with the following*
(i)$'$ *For every increasing sequence (τ_n) in T*

$$\big(E^{\mathcal{F}_{\tau_n}}(f)\big) \textit{ converges strongly a.s. to } f.$$

The implication (i)$'$ $\Rightarrow$ (i) is obvious, whereas (i) $\Rightarrow$ (i)$'$ follows from the next lemma, which is an easy adaptation of Theorem VII.2.4 in [8].

LEMMA 4.1. *Let $f \in L^1_{X^*}[X](\mathcal{F})$. Then for every $\tau \in T$ and every $n \geqslant 1$*

$$E^{\mathcal{F}_n}(f) = E^{\mathcal{F}_\tau}(f) \textit{ a.s.} \quad \textit{on } \{\tau = n\}.$$

REMARK 4.2. To prove the existence of a scalarly measurable function that is not σ-measurable suppose that X is a separable Banach space whose dual X^* is not strongly separable (equivalently, X^* does not have the Radon–Nikodym property, by a theorem of Stegall; see [6], p. 195). Then Theorem II.2.2.1 in [8] allows us to show the existence of a probability space $(\Delta, \mathcal{G}, \mu)$ and a uniformly integrable martingale $(g_n, \mathcal{G}_n)$ in $L^1_{X^*}(\Delta, \mathcal{G}, \mu)$ which diverges in the $L^1_{X^*}(\Delta, \mathcal{G}, \mu)$-norm. Noting that the functions g_n are also members of $L^1_{X^*}[X](\Delta, \mathcal{G}, \mu)$ and invoking Theorem 6.1 (4) of [4], it is possible to find a subsequence (g_{n_k}) of (g_n) and a function $g_\infty \in L^1_{X^*}[X](\Delta, \mathcal{G}, \mu)$ such that

$$\lim_{k\to\infty} \int_A \langle x_\ell, g_{n_k}\rangle\, dP = \int_A \langle x_\ell, g_\infty\rangle\, dP$$

for all $\ell \geqslant 1$ and $A \in \mathcal{F}$. Now, since for each $\ell \geqslant 1$, $(\langle x_\ell, g_n\rangle)_n$ is a martingale in $L^1_{\mathbb{R}}(\mathcal{F})$, we have

$$\int_A \langle x_\ell, g_{n_k}\rangle\, dP = \int_A \langle x_\ell, g_m\rangle\, dP$$

for all $m \geqslant 1$, $k \geqslant m$, and $A \in \mathcal{F}_m$. Therefore,

$$\int_A \langle x_\ell, g_m\rangle\, dP = \int_A \langle x_\ell, g_\infty\rangle\, dP \quad \text{for all } A \in \mathcal{F}_m,$$

which is equivalent to

$$\langle x_\ell, g_m\rangle = E^{\mathcal{F}_m}(\langle x_\ell, g_\infty\rangle) = \langle x_\ell, E^{\mathcal{F}_m}(g_\infty)\rangle \text{ a.s} \quad \text{for all } m \geqslant 1.$$

This holds for all $\ell \geqslant 1$. Hence

$$\forall m \geqslant 1,\ g_m = E^{\mathcal{F}_m}(g_\infty) \text{ a.s.}$$

By Theorem 4.1, it follows that g_∞ is not σ-measurable. Otherwise, (g_m) s^*-converges almost surely to g_∞ and also strongly in $L^1_{X^*}(\Delta, \mathcal{G}, \mu)$. ■

Now we are ready to extend the results of the preceding section to the space $L^1_{X^*}[X](\mathcal{F})$. Before going further, we need the following $L^1_{X^*}[X](\mathcal{F})$-extension of Lemma 3.1.

LEMMA 4.2. *Let $(f_n)_{n\geqslant 1}$ be an adapted sequence in $L^1_{X^*}[X](\mathcal{F})$ satisfying the following condition*:

(M)* *There exist a sequence (g_n) with $g_n \in \operatorname{co}\{f_i : i \geqslant n\}$ and a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that*

$$\int_\Omega \|g_n - f_\infty\|\, dP \to 0.$$

Then, for every increasing sequence (τ_m) in T,

(i) $$\|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}} f_\infty(\omega)\| \leqslant \sup_{k\geqslant \tau_m} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}} f_k(\omega)\| \text{ *a.s.* } \quad \textit{for all } m \geqslant 1$$

and

(ii) $\limsup_m \|f_{\tau_m}(\omega) - f_\infty(\omega)\| \leqslant \limsup_m \sup_{k \geqslant \tau_m} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}} f_k(\omega)\|$ *a.s.*

P r o o f. Inequality (i) is an easy adaptation of the corresponding one in Lemma 3.1. It remains to verify (ii). Since $\int_\Omega \|g_n - f_\infty\| \, dP \to 0$, one can find a subsequence of (g_n) whose norm converges a.s. to f_∞. Therefore, noting that (g_n) is adapted with respect to a subsequence of $(\mathcal{F}_n)$, we conclude that f_∞ is σ_∞-measurable. Conversely, by Theorem 4.1, this ensures the a.s. s^*-convergence of $E^{\mathcal{F}_{\tau_m}}(f_\infty)$ to f_∞. Thus, as in the proof of Lemma 3.1, (ii) is a consequence of (i), the triangle inequality, and the classical Lévy theorem. ■

The following is an $L^1_{X^*}[X](\mathcal{F})$-version of Proposition 3.1.

PROPOSITION 4.2. *Let* $(f_n)_{n \geqslant 1}$ *be a pramart in* $L^1_{X^*}[X](\mathcal{F})$ *satisfying the following condition*:

(M)* *There exist a sequence* (g_n) *with* $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ *and a function* $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ *such that*

$$\int_\Omega \|g_n - f_\infty\| \, dP \to 0.$$

Then (f_n) *converges strongly a.s. to* f_∞.

P r o o f. By the cluster point approximation theorem, we can choose an increasing sequence (τ_m) in T with $\tau_m \geqslant m$ such that

$$\limsup_n \|f_n(\omega) - f_\infty(\omega)\| = \lim_m \|f_{\tau_m}(\omega) - f_\infty(\omega)\| \text{ a.s.} \tag{4.1}$$

Then, by Lemma 4.2,

$$\lim_m \|f_{\tau_m}(\omega) - f_\infty(\omega)\| \leqslant \limsup_m \sup_{k \geqslant \tau_m} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)\| \text{ a.s.}$$

On the other hand, as (f_n) is a pramart, by repeating *mutatis mutandis* the techniques of Millet and Sucheston developed in [9], Theorem 4.1, we continue to have

$$\lim_m \sup_{k \geqslant \tau_m} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)\| = 0 \text{ a.s.} \tag{4.2}$$

This equation together with (4.1) give

$$\lim_n \|f_n(\omega) - f_\infty(\omega)\| = 0 \text{ a.s.} \quad ■$$

Unlike the first case, it is interesting to note that Proposition 4.2 need not hold if one replaces condition (M)* with

(SM)* There exists a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

$$\forall x \in X,\ \lim_n \inf_{g \in \mathrm{co}\{f_i : i \geqslant n\}} \int_\Omega |\langle x, g\rangle - \langle x, f_\infty\rangle| \, dP = 0$$

(just consider a regular martingale associated with the member of $L^1_{X^*}[X](\mathcal{F})$ that is not σ-measurable, see Remark 4.2 and its proof). However, we can prove:

PROPOSITION 4.3. *Let $(f_n)_{n\geqslant 1}$ be a pramart in $L^1_{X^*}[X](\mathcal{F})$ satisfying the above condition* (SM)*. *Then*

$$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \ \textit{a.s.} \quad \textit{and} \quad (f_n)\ w^*\textit{-converges to } f_\infty \textit{ a.s.}$$

P r o o f. Let f_∞ be a function as given in condition (SM)*. Then to each $\ell \geqslant 1$ there corresponds a sequence (g_n) with $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$ such that

$$\lim_n \int_\Omega |\langle x_\ell, g_n\rangle - \langle x_\ell, f_\infty\rangle| \, dP = 0. \tag{4.3}$$

Further, by the cluster approximation theorem, one can choose an increasing sequence (τ_m) in T with $\tau_m \geqslant m$ for all $m \geqslant 1$ such that

$$\limsup_{n\to\infty} \|f_n(\omega) - E^{\mathcal{F}_n}(f_\infty)(\omega)\| = \lim_{m\to\infty} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)\|. \tag{4.4}$$

By (4.3), it is possible to apply Lemma 3.1 to the real-valued adapted sequences $(\langle x_\ell, f_n\rangle)$ $(\ell \geqslant 1)$, which gives

$$|\langle x_\ell, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)\rangle| \leqslant \sup_{k\geqslant \tau_m} |\langle x_\ell, f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_k)(\omega)\rangle|$$

a.s. for every $m \geqslant 1$ and every $\ell \geqslant 1$. Taking the supremum on $\ell \geqslant 1$ we get

$$\|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_\infty)(\omega)\| \leqslant \sup_{n\geqslant \tau_m} \|f_{\tau_m}(\omega) - E^{\mathcal{F}_{\tau_m}}(f_n)(\omega)\| \tag{4.5}$$

a.s. for every $m \geqslant 1$. Since (f_n) is a pramart, equation (4.2) together with (4.4) and (4.5) imply

$$\lim_{n\to\infty} \|f_n(\omega) - E^{\mathcal{F}_n}(f_\infty)(\omega)\| = 0 \text{ a.s.}, \tag{4.6}$$

and then, by the classical Lévy theorem,

$$\lim_{n\to\infty} \langle x, f_n(\omega)\rangle = \langle x, f_\infty(\omega)\rangle \text{ a.s.} \quad \text{for all } x \in X. \tag{4.7}$$

Since

$$\sup_{n\geqslant 1} \|E^{\mathcal{F}_n}(f_\infty)(\omega)\| \leqslant E^{\mathcal{F}_n}\|f_\infty\|(\omega) < \infty \ \text{ a.s.},$$

equality (4.6) implies

$$\sup_{n\geqslant 1}\|f_n(\omega)\|<\infty \text{ a.s.}$$

Using the separability of X, (4.7) and the pointwise boundedness of (f_n) we infer, by a routine argument, that

$$\big(f_n(\omega)\big)\ w^*\text{-converges to } f_\infty(\omega) \text{ a.s.}$$

Thus the proof is completed. ∎

A version of Proposition 4.3 for $L^1_{X^*}[X](\mathcal{F})$-bounded mil is available in [2], Corollary 3.1 (see also [10]).

As a direct consequence of Theorem 4.1 and Proposition 4.3 we have:

COROLLARY 4.1. *Let* $(f_n)_{n\geqslant 1}$ *be a pramart in* $L^1_{X^*}[X](\mathcal{F})$ *satisfying the following condition*:

(SM)$^{*+}$ *There exists a* σ*-measurable function* $f_\infty\in L^1_{X^*}[X](\mathcal{F})$ *such that*

$$\lim_n \inf_{g\in \mathrm{co}\{f_i:i\geqslant n\}}\int_\Omega |\langle x,g\rangle-\langle x,f_\infty\rangle|\,dP=0 \quad \textit{for all } x\in X.$$

Then (f_n) *converges strongly a.s. to* f_∞.

Now towards a dual version of Theorem 3.1, it is useful to reformulate Lemma 3.2 for pramarts in $L^1_{X^*}[X](\mathcal{F})$.

LEMMA 4.3. *Let* $(f_n)_{n\geqslant 1}$ *be a pramart in* $L^1_{X^*}[X](\mathcal{F})$, *which satisfies the following two conditions*:

(b)* *For each* $x\in X$, *there exists a sequence* (h_n) *with* $h_n\in \mathrm{co}\{f_i:i\geqslant n\}$, *such that* $(\langle x,h_n\rangle)$ *is uniformly integrable.*

(c)* *There exists a sequence* (h'_n) *with* $h'_n\in\mathrm{co}\{f_i:i\geqslant n\}$ *such that*

$$\sup_{\ell\geqslant 1}\liminf_{n\to\infty}|\langle x_\ell,h'_n\rangle|\in L^1_{\mathbb{R}}(\mathcal{F}).$$

Then $\limsup_n\|f_n(\omega)\|\in L^1_{\mathbb{R}}(\mathcal{F})$. *Consequently*, (f_n) *is pointwise bounded almost surely.*

P r o o f. It is an easy adaptation of Lemma 3.2. ∎

Proceeding as in the proof of Remark 3.1, note that the condition (SM)* implies (b)* and (c)*.

In the next result we provide a version of Proposition 4.3 where the condition (SM)* may be replaced with (b)* and (c)*. The proof follows the same lines as those of Proposition 3.3 with appropriate modifications, but involves w^*-compactness instead of w-compactness *via* Lemma 4.3.

PROPOSITION 4.4. *Under the assumptions of Lemma* 4.3 *there exists a function* $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ *such that*

$$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \ \textit{a.s.} \quad \textit{and} \quad (f_n)\ w^*\textit{-converges to } f_\infty \ \textit{a.s.}$$

P r o o f. Let $\ell \geqslant 1$ be fixed and let (h_n) be a sequence associated with ℓ according to (b)*. As the sequence $(\langle x_\ell, h_n\rangle)$ is uniformly integrable, there exist a subsequence (h_{n_k}) of (h_n) and a function $\varphi_\ell \in L^1_{\mathbb{R}}(\mathcal{F})$ such that

$$\lim_{k\to\infty} \int_\Omega |\langle x_\ell, h_{n_k}\rangle - \varphi_\ell|\, dP = 0. \tag{4.8}$$

Since $h_{n_k} \in \operatorname{co}\{f_i : i \geqslant k\}$ and $(\langle x_\ell, f_n\rangle)_n$ is a pramart, by Proposition 3.1, we have

$$\lim_{n\to+\infty} \langle x_\ell, f_n\rangle = \varphi_\ell \text{ a.s.} \quad \text{for all } \ell \geqslant 1. \tag{4.9}$$

On the other hand, by Lemma 4.3, the sequence (f_n) is pointwise bounded in X^* almost surely; hence it is relatively w^*-sequentially compact (the weak star topology being metrizable on bounded sets). Therefore, for each ω outside a negligible set N, there exist a subsequence of (f_{k_n}) (possibly depending upon ω) and an element $x^*_\omega \in X^*$ such that

$$\big(f_{k_n}(\omega)\big)\ w^*\text{-converges to } x^*_\omega.$$

Define $f_\infty(\omega) := x^*_\omega$ for $\omega \in \backslash\Omega$ and $f_\infty(\omega) := 0$ for $\omega \in N$. Then, taking into account (4.9) we get

$$\lim_{n\to+\infty} \langle x_\ell, f_n\rangle = \langle x_\ell, f_\infty\rangle = \varphi_\ell \text{ a.s.} \quad \text{for all } \ell \geqslant 1. \tag{4.10}$$

This implies the scalar $\mathcal{F}$-measurability of f_∞. Furthermore, we have

$$\|f_\infty\| \leqslant \sup_{\ell\geqslant 1} \liminf_{n\to+\infty} |\langle x_\ell, f_n\rangle| \text{ a.s.},$$

which, in view of (c)*, shows that $\|f_\infty\|$ is integrable. Thus $f_\infty \in L^1_{X^*}[X](\mathcal{F})$. Replacing in (4.8) φ_ℓ by $\langle x_\ell, f_\infty\rangle$ (because of the second equality of (4.10)) we get

$$\lim_{k\to\infty} \int_\Omega |\langle x_\ell, h_{n_k}\rangle - \langle x_\ell, f_\infty\rangle|\, dP = 0.$$

The desired conclusion follows then from Proposition 4.3. ■

Now we are ready to state the analogue of Theorem 3.1 in the framework of $L^1_{X^*}[X](\mathcal{F})$-space, which is a direct consequence of Propositions 4.1–4.4 and Theorem 4.1. It can also be seen as a pramart version of Proposition 5.2 in [10].

THEOREM 4.2. *Let* $(f_n)_{n\geqslant 1}$ *be a pramart in* $L^1_{X^*}[X](\mathcal{F})$. *Suppose the following conditions are satisfied*:

(a)* *There exists an* $\mathcal{S}\big(\mathcal{R}(X^*_w)\big)$*-tight sequence* (g_n) *with* $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$.

(b)* *For each* $\ell \geqslant 1$, *there exists a sequence* (h_n) *with* $h_n \in \mathrm{co}\{f_i : i \geqslant n\}$ *such that* $(\langle x_\ell, g_n\rangle)$ *is uniformly integrable.*

(c)* *There exists a sequence* (h'_n) *with* $h'_n \in \mathrm{co}\{f_i : i \geqslant n\}$ *such that*

$$\sup_{\ell\geqslant 1} \liminf_{n\to\infty} |\langle x_\ell, h'_n\rangle| \in L^1_{\mathbb{R}}(\mathcal{F}).$$

Then there exists a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ *such that*

$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

Conditions (b)* *and* (c)* *can be replaced with the following*:

(b$'$)* *There exists a uniformly integrable sequence* (h_n) *with* $h_n \in \mathrm{co}\{f_i : i \geqslant n\}$.

P r o o f. By Proposition 4.5, there exists a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ such that

(i) $$\|f_n - E^{\mathcal{F}_n}(f_\infty)\| \to 0 \text{ a.s.},$$

(ii) $$(f_n)\ w^*\text{-converges to } f_\infty \text{ a.s.}$$

We have to show that f_∞ is σ-measurable. To this purpose, let (g_n) be as given in the condition (a)*. Since (f_n) is pointwise bounded, so is the sequence (g_n). Hence (g_n) is $\mathcal{S}\big(cwk(X^*_w)\big)$-tight, since it is $\mathcal{S}\big(\mathcal{R}(X^*_w)\big)$-tight. Further, (g_n) w^*-converges almost surely to f_∞ (by (i)). Consequently, by Proposition 4.1, f_∞ is σ-measurable. By Theorem 4.1 and (ii), the proof is complete. ∎

Theorem 4.2 extends Theorem 3.1 to the space $L^1_{X^*}[X](\mathcal{F})$. Along the way, we get the following $L^1_{X^*}[X](\mathcal{F})$-extension of Corollary 3.2.

COROLLARY 4.2. *Let* $(f_n)_{n\geqslant 1}$ *be a pramart in* $L^1_{X^*}[X](\mathcal{F})$ *such that the conditions* (a$''$)*, (b)*, *and* (c)* *hold*, *where*

(a$''$)* *There exists an* $\mathcal{R}(X^*_w)$*-tight sequence* (g_n) *with* $g_n \in \mathrm{co}\{f_i : i \geqslant n\}$.

Then there exists a function $f_\infty \in L^1_{X^*}[X](\mathcal{F})$ *such that*

$$(f_n) \text{ converges strongly a.s. to } f_\infty.$$

Conditions (b)* *and* (c)* *can be replaced with* (b$'$)*.

P r o o f. Arguing as in Remark 3.3, we show that every $\mathcal{C}$-tight sequence in $L^0_{X^*}[X](\mathcal{F})$ is $\mathcal{S}(\mathcal{C})$-tight, so that the condition (a$''$)* implies (a)*. ∎

Finally, it is worth mentioning that it is possible to obtain dual versions of Theorems 3.2 and 3.3 and their corollaries, but for the sake of brevity we refrain from giving the details.

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