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PERSISTENCE OF SOME ITERATED PROCESSES

BY

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Abstract. We study the asymptotic behaviour of the probability that a stochastic process $(Z_t)_{t \geqslant 0}$ does not exceed a constant barrier up to time T (a so-called *persistence probability*) when Z is the composition of two independent processes $(X_t)_{t \in I}$ and $(Y_t)_{t \geqslant 0}$. To be precise, we consider $(Z_t)_{t \geqslant 0}$ defined by $Z_t = X \circ |Y_t|$ if $I = [0, \infty)$ and $Z_t = X \circ Y_t$ if $I = \mathbb{R}$.

For continuous self-similar processes $(Y_t)_{t \geqslant 0}$, the rate of decay of persistence probability for Z can be inferred directly from the persistence probability of X and the index of self-similarity of Y. As a corollary, we infer that the persistence probability for iterated Brownian motion decays asymptotically like $T^{-1/2}$.

If Y is discontinuous, the range of Y possibly contains gaps, which complicates the estimation of the persistence probability. We determine the polynomial rate of decay for X being a Lévy process (possibly two-sided if $I = \mathbb{R}$) or a fractional Brownian motion and Y being a Lévy process or random walk under suitable moment conditions.

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1. INTRODUCTION

1.1. Statement of the problem. The one-sided exit problem consists of finding the asymptotic behaviour of

$$P\left(Z_t \leqslant 1, \forall t \in [0, T]\right) \tag{1.1}$$

as $T \to \infty$ for a given stochastic processes $Z = (Z_t)_{t \geqslant 0}$. The probability in (1.1) is often called *persistence* or *survival probability* up to time T. For many processes, the persistence probability decreases polynomially (modulo terms of lower order), i.e., for some $\theta > 0$, it follows that

$$P\left(Z_t \leqslant 1, \forall t \in [0, T]\right) = T^{-\theta + o(1)}, \quad T \to \infty.$$

θ is called the *persistence* or *survival exponent*.

Of course, (1.1) is a classical problem that has been studied for some particular processes such as random walks, Brownian motion with moving boundaries, integrated processes such as integrated Brownian motion, fractional Brownian motion, and other Gaussian processes. Research on persistence probabilities has been motivated by the study of the inviscid Burgers equation [25] and zeros of random polynomials [12]. We refer to the recent survey [6] for details, applications and references. Apart from pure theoretical interest, persistence probabilities appear in many applications. For instance, the one-sided exit problem arises in various physical models such as reaction diffusion systems and granular media, see the survey [9] for a comprehensive review.

In this article, we consider the one-sided exit problem for processes $Z = (X \circ |Y_t|)_{t \geqslant 0}$, where $X = (X_t)_{t\geqslant 0}$ and $Y = (Y_t)_{t\geqslant 0}$ are independent stochastic processes and $Z = (X \circ Y_t)_{t\geqslant 0}$ if $X = (X_t)_{t\in\mathbb{R}}$ ($\circ$ denotes function composition). Such processes will be referred to as *iterated processes*. Starting with the work [10], the study of iterated Brownian motion has attracted a lot of interest. Moreover, there are interesting connections between the exit times of iterated processes and the solution of certain fourth-order PDEs ([2], [23]). The asymptotic behaviour of the survival probabilities of subordinated Brownian motion is also relevant for the study of Green functions (see [18]). However, the one-sided exit problem for iterated processes has not been studied systematically so far. Here we investigate how the persistence exponent of $X \circ |Y|$ and $X \circ Y$ are related to that of the outer process X and properties of the inner process Y. The relevant scenario affecting the survival probability can be identified so that the results are quite intuitive. For small deviation probabilities (i.e., two-sided exit problems), this problem has been investigated in [5].

Finally, let us introduce some notation and conventions: If $f, g : \mathbb{R} \to \mathbb{R}$ are two functions, we write $f \precsim g$ $(x \to \infty)$ if $\limsup_{x\to\infty} f(x)/g(x) < \infty$ and $f \asymp g$ if $f \precsim g$ and $g \precsim f$. Moreover, $f \sim g$ $(x \to \infty)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. If $(X_t)_{t\geqslant 0}$ is a stochastic process, it will often be convenient to write $X(t)$ instead of X_t. If $(X_n)_{n\in\mathbb{N}}$ is a discrete time process, we set $X_t = X_{\lfloor t \rfloor}$, where $\lfloor t \rfloor := \sup\{k \in \mathbb{Z} : k \leqslant t\}$. Moreover, we say that $(X_t)_{t\in I}$ is *self-similar of index H* if $(Y_{ct})_{t\geqslant 0} \overset{d}{=} (c^H Y_t)_{t\geqslant 0}$ for all $c > 0$, where $\overset{d}{=}$ denotes equality in distribution.

1.2. Main results. First, we consider processes $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$, where Y is self-similar and continuous. In this setup, the following result can be established without much difficulty:

THEOREM 1.1. *Let $(X_t)_{t\geqslant 0}$ be a stochastic process with*

$$P\left(X_t \leqslant 1, \forall t \in [0, T]\right) \asymp T^{-\theta}, \quad T \to \infty,$$

for some $\theta > 0$. Let $(Y_t)_{t\geqslant 0}$ be an independent stochastic process which is self-similar of index H, has continuous paths, and for some $\rho > \theta$ it follows that

$$P\left(|Y_t| \leqslant \epsilon, \forall t \in [0, 1]\right) \precsim \epsilon^{\rho}, \quad \epsilon \downarrow 0. \tag{1.2}$$

Then

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) \asymp T^{-\theta H}, \quad T \to \infty.$$

Moreover, if $P(X_t \leqslant 1, \forall t \in [0,T]) \sim cT^{-\theta}$ *for some* $c > 0$, *it follows that*

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) \sim cA\,T^{-\theta H}, \quad T \to \infty,$$

where $A := E\big[(\sup\{|Y_t| : t \in [0,1]\})^{-\theta}\big] < \infty$.

We remark that the assumption in (1.2) is very weak since this so-called small deviation probability usually decays exponentially fast as $\epsilon \downarrow 0$. Moreover, the result can be explained quite intuitively: by self-similarity of Y, typical fluctuations of $|Y|$ up to time T are of order T^H. The rare event that X stays below one until time T^H is then of order $T^{-\theta H}$. The assumption (1.2) prevents a contribution of the event that Y stays close to the origin to the persistence exponent of $Z = X \circ |Y|$. In short, the persistence probability of Z is determined by a rare event for X and a typical scenario for Y.

The assumption of continuity of the inner process Y allows us to write

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) = P\big(X_t \leqslant 1, \forall t \in [0, (-I_T) \vee M_T]\big),$$

where I and M denote the infimum and supremum process of Y, respectively. This will simplify the proof of the upper bound of Theorem 1.1 very much. If Y is discontinuous, the equality sign has to be replaced by $\geqslant$ in the preceding equation. It is then by far a more challenging task to find the survival exponent of $X \circ |Y|$ since the gaps in the range of $|Y|$ have to be taken into account. We prove the following theorem for X being a Lévy process and Y being a random walk or a Lévy process (in the sequel, we always exclude trivial processes $X \equiv 0$).

THEOREM 1.2. *Let* $(X_t)_{t\geqslant 0}$ *be a centred Lévy process with* $E[\exp(|X_1|^{\alpha})]$ *finite for some* $\alpha > 0$. *Let* $(Y_t)_{t\geqslant 0}$ *denote an independent random walk or Lévy process with* $E[\exp(|Y_1|^{\beta})] < \infty$ *for some* $\beta > 0$. *It follows that*

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) = T^{-\theta + o(1)}, \quad T \to \infty,$$

where $\theta = 1/4$ *if* $E[Y_1] = 0$, *and* $\theta = 1/2$ *if* $E[Y_1] \neq 0$.

Again, the results are intuitive: If $E[Y_1] = 0$, the random walk oscillates and typical fluctuations up to time N are of magnitude $\sqrt{N}$. Since the persistence exponent θ of a centred Lévy process with second finite moments is $1/2$ (details in Section 3), it is very plausible that the persistence exponent of $X \circ |Y|$ is at least $1/4$ if the gaps in the range of the random walk are not too large. If $E[Y_1] > 0$, then $E[Y_N]/N \to E[Y_1]$ by the law of large numbers and one expects the survival exponent of $X \circ |Y|$ to be $1/2$ by the same reasoning.

The methods to prove Theorem 1.2 can be extended to the case when the outer process is a fractional Brownian motion.

THEOREM 1.3. *Let $(X_t)_{t\geqslant 0}$ denote a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Let $(Y_t)_{t\geqslant 0}$ denote a Lévy process or a random walk such that $E[\exp(|Y_1|^\beta)] < \infty$ for some $\beta > 0$. It follows that*

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) = T^{-\theta+o(1)}, \quad T \to \infty,$$

where $\theta = (1-H)/2$ if $E[Y_1] = 0$, and $\theta = 1 - H$ if $E[Y_1] \neq 0$.

Note that the outer processes in Theorems 1.2 and 1.3 share the property of stationary increments. We provide an example showing that an analogous result can fail without this property.

Up to now, the outer process $X = (X_t)_{t\geqslant 0}$ had the index set $[0,\infty)$, so it was only possible to evaluate X over the range of the absolute value of the inner process Y. In order to consider the one-sided exit problem for $X \circ Y$, we consider two-sided processes $X = (X_t)_{t\in\mathbb{R}}$, where

$$X_t := \begin{cases} X_t^+, & t \geqslant 0, \\ X_{-t}^-, & t < 0, \end{cases} \tag{1.3}$$

and $(X^+)_{t\geqslant 0}$ and $(X_t^-)_{t\geqslant 0}$ are stochastic processes. We refer to X^+ and X^- as to the branches of X. We prove that the previous results can be extended in a natural way for two-sided processes.

THEOREM 1.4. *Let $(X_t)_{t\in\mathbb{R}}$ be a two-sided process with*

$$P\left(X_t \leqslant 1, \forall t \in [-T,T]\right) \asymp T^{-\theta}$$

for some $\theta > 0$. Let $(Y_t)_{t\in\mathbb{R}}$ denote an independent self-similar process of index H and continuous paths such that $E[|I|^{-\theta}] + E[M^{-\theta}] < \infty$, where $I = \inf\{Y_t : t \in [-1,1]\}$ and $M = \sup\{Y_t : t \in [-1,1]\}$. Then

$$P\big(X(Y_t) \leqslant 1, \forall t \in [-T,T]\big) \asymp T^{-H\theta}, \quad T \to \infty.$$

If we know the precise asymptotics of $P\left(X_t \leqslant 1, \forall t \in [-T_1, T_2]\right)$ for $T_1, T_2 \to \infty$, we can get a more precise result, see Theorem 4.1. As a corollary, we compute the exact asymptotics of the persistence probability of n-times iterated two-sided Brownian motions (Corollary 4.2).

The result corresponding to Theorem 1.2 in the two-sided setup is the following:

THEOREM 1.5. *Let $(X_t)_{t\in\mathbb{R}}$ denote a two-sided Lévy process with branches X^+, X^- such that $E[X_1^\pm] = 0$ and $E[\exp(|X_1^\pm|^\alpha)] < \infty$ for some $\alpha > 0$. Let $(Y_t)_{t\geqslant 0}$ denote another Lévy process or random walk independent of X such that $E[\exp(|Y_1|^\beta)] < \infty$ for some $\beta > 0$. Then*

$$P\big(X(Y_t) \leqslant 1, \forall t \in [0,T]\big) = T^{-1/2+o(1)}, \quad T \to \infty.$$

Theorem 1.5 shows that the persistence exponent is equal to $1/2$ no matter if $E[Y_1] = 0$ or not; see Remark 4.1 for an explanation.

The remainder of the article is organised as follows. In Section 2, we assume that the inner process Y is a continuous self-similar process. We compute the survival exponent of $X \circ |Y|$ (Theorem 1.1) and provide an example. Next, we turn to discontinuous processes Y. The persistence exponent of $X \circ |Y|$ is found for X being a Lévy process or fractional Brownian motion and Y being a random walk or Lévy process (Theorems 1.2 and 1.3) in Section 3. Finally, we extend the previous results to two-sided processes (Theorems 1.4 and 1.5) in Section 4.

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2. TAKING THE SUPREMUM OVER THE RANGE OF A CONTINUOUS SELF-SIMILAR PROCESS

If $Y = (Y_t)_{t \geqslant 0}$ is a stochastic process, denote by $\mathcal{F}_t^Y := \sigma(Y_s : 0 \leqslant s \leqslant t)$ the filtration generated by Y up to time t. Let us now prove Theorem 1.1 announced in the Introduction.

P r o o f o f T h e o r e m 1.1. Let us write $Y_t^* := \sup_{s \in [0,t]} |Y_s|$. Note that our assumption (1.2) implies that $(Y_1^*)^{-\theta}$ is integrable, see Lemma 2.1 below.

U p p e r b o u n d. By assumption, there are constants $C, T_0 > 0$ such that for any $T > T_0$ we have $P(\sup_{t \in [0,T]} X_t \leqslant 1) \leqslant CT^{-\theta}$. Clearly, we can choose C so large that the inequality holds for all $T > 0$. By continuity of Y, the fact that $Y_0 = 0$ (by self-similarity), and independence of X and Y, and self-similarity of Y, we have

$$\begin{aligned} P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) &= E\left[P(X_t \leqslant 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y)\right] \\ &\leqslant CE[(Y_T^*)^{-\theta}] = CE[(Y_1^*)^{-\theta}]T^{-\theta H}. \end{aligned}$$

L o w e r b o u n d. Let $C > 0$. Note that

$$\begin{aligned} P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) &\geqslant P(Y_T^* \leqslant CT^H, \{X_t \leqslant 1, \forall t \in [0, Y_T^*]\}) \\ &\geqslant P(Y_T^* \leqslant CT^H)\, P(X_t \leqslant 1, \forall t \in [0, CT^H]) \\ &= P(Y_1^* \leqslant C) P(X_t \leqslant 1, \forall t \in [0, CT^H]). \end{aligned}$$

Continuity of Y and the fact that $Y_0 = 0$ imply that $P(Y_1^* \leqslant C) > 0$ for C large enough. This proves the lower bound.

If $P(\sup_{t \in [0,T]} X_t \leqslant 1) \sim cT^{-\theta}$, we can find for all $\epsilon > 0$ small enough a constant $T_0(\epsilon)$ such that $(c - \epsilon)T^{-\theta} \leqslant P(\sup_{t \in [0,T]} X_t \leqslant 1) \leqslant (c + \epsilon)T^{-\theta}$ for all

$T \geqslant T_0(\epsilon)$. Hence,

$$\begin{aligned}
&T^{\theta H}\, P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big)\\
&\geqslant T^{\theta H}\, E\big[1_{\{Y_T^* \geqslant T_0(\epsilon)\}}\, P(X_t \leqslant 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y)\big]\\
&\geqslant (c-\epsilon) T^{\theta H}\, E[1_{\{Y_T^* \geqslant T_0(\epsilon)\}} (Y_T^*)^{-\theta}] = (c-\epsilon)\, E[1_{\{Y_1^* \geqslant T_0(\epsilon) T^{-H}\}} (Y_1^*)^{-\theta}].
\end{aligned}$$

As $T \to \infty$, monotone convergence implies for all $\epsilon > 0$ small enough that

$$\liminf_{T\to\infty} T^{\theta H} P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) \geqslant (c-\epsilon) E[(Y_1^*)^{-\theta}],$$

i.e., $\liminf_{T\to\infty} T^{\theta H} P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) \geqslant c E[(Y_1^*)^{-\theta}]$.

For the proof of the upper bound, note that

$$\begin{aligned}
&P\left(X_t \leqslant 1, \forall t \in [0, Y_T^*]\right)\\
&\qquad \leqslant P\left(Y_T^* \leqslant T_0\right) + E\big[1_{\{Y_T^* \geqslant T_0\}}\, P(X_t \leqslant 1, \forall t \in [0, Y_T^*] | \mathcal{F}_T^Y)\big]\\
&\qquad \leqslant P(Y_1^* \leqslant T_0 T^{-H}) + (c+\epsilon) E[1_{\{Y_T^* \geqslant T_0\}} (Y_T^*)^{-\theta}].
\end{aligned}$$

The assumption on the small deviation probability of Y implies that

$$T^{\theta H} P\left(Y_1^* \leqslant T_0(\epsilon) T^{-H}\right) \precsim T^{H(\theta-\rho)} T_0(\epsilon)^{\rho} \to 0, \qquad T \to \infty.$$

Hence, as in the proof of the lower bound, we obtain

$$\limsup_{T\to\infty} T^{\theta H} P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big) \leqslant (c+\epsilon) E[(Y_1^*)^{-\theta}],$$

which completes the proof upon letting $\epsilon \downarrow 0$. ■

REMARK 2.1. *The proof reveals that the lower bounds of Theorem* 1.1 *are also valid without continuity of paths of Y and the assumption* (1.2) *on the small deviations of Y. Moreover, we remark that the proof can be easily adapted to cover the case of $P(\sup_{t\in[0,T]} X_t \leqslant 1) = T^{-\theta+o(1)}$.*

As already mentioned in the proof, the small deviation probability in (1.2) is linked to integrability of $\sup\{|Y_t| : t \in [0,1]\}$. For convenience and later reference, let us state this fact without proof in the following lemma.

LEMMA 2.1. *Let Z be a random variable such that $Z > 0$ a.s. and $P\left(Z \leqslant \epsilon\right) \precsim \epsilon^{\rho}$ as $\epsilon \downarrow 0$ for some $\rho > 0$. Then for $\eta \in (0, \rho)$ it follows that $E\left[Z^{-\eta}\right] < \infty$. Conversely, if $E\left[Z^{-\eta}\right] < \infty$ for some $\eta > 0$, then $P\left(Z \leqslant \epsilon\right) \precsim \epsilon^{\eta}$ as $\epsilon \downarrow 0$.*

Let us also mention that the assumption (1.2) can fail: If Z is a random variable, set $Y_t := t^H \cdot Z$ for some $H > 0$. The process Y is self-similar, continuous and $P\left(|Y_t| \leqslant \epsilon, \forall t \in [0,1]\right) = P\left(|Z| \leqslant \epsilon\right)$. If $P\left(Z = 0\right) > 0$, (1.2) obviously does not hold. If we consider another continuous self-similar process $(\hat{Y}_t)_{t\geqslant 0}$ independent of Z, then (1.2) also fails for the product $(Y_t\hat{Y}_t)_{t\geqslant 0}$ (again self-similar). This way, one obtains a large class of processes that does not satisfy condition (1.2).

To conclude this section, let us give a simple application of Theorem 1.1.

EXAMPLE 2.1. If X and Y are independent Brownian motions, it follows that $H = 1/2$ and it is well known by the reflection principle that

$$P\left(X_t \leqslant 1, \forall t \in [0,T]\right) = P\left(|B_T| \leqslant 1\right) \sim \sqrt{2/\pi}\, T^{-1/2}, \quad T \to \infty.$$

Since

$$P\left(|Y_t| \leqslant \epsilon, \forall t \in [0,1]\right) \leqslant C \exp\left(-(\pi^2/8)\, \epsilon^{-2}\right), \quad \epsilon > 0,$$

$(\sup_{t\in[0,1]} |Y_t|)^{-\eta}$ is integrable for every $\eta > 0$ by Lemma 2.1. Hence, Theorem 1.1 implies that the persistence exponent $X \circ |Y|$ is $1/4$.

More generally, if W and $B^{(1)}, \ldots, B^{(n)}$ are independent Brownian motions, it follows for any $n \geqslant 1$ that

$$P\big(W(|B^{(1)}| \circ \ldots \circ |B_t^{(n)}|) \leqslant 1, \forall t \in [0,T]\big) \sim c_n T^{-2^{-(n+1)}}, \quad T \to \infty,$$

with $c_n = \sqrt{2/\pi}\, \prod_{k=1}^{n} E[(W_1^*)^{-2^{-k}}]$ for $n \geqslant 1$ and $W_1^* = \sup_{t\in[0,1]} |W_t|$.

3. TAKING THE SUPREMUM OVER THE RANGE OF DISCONTINUOUS PROCESSES

The goal of this section is to find the asymptotics of

$$P\big(X(|S_n|) \leqslant 1, \forall n = 1, \ldots, N\big) \quad \text{and} \quad P\big(X(|Y_t|) \leqslant 1, \forall t \in [0,T]\big),$$

respectively. Here $X = (X_t)_{t\geqslant 0}$ is a centred Lévy process or a fractional Brownian motion, S is a random walk, and Y is a Lévy process. First, we recall known results on survival probabilities of Lévy processes and prove a slight generalisation. If X is a centred Lévy process with $E\left[X_1^2\right] < \infty$, it follows that

$$P\left(X_t \leqslant 1, \forall t \in [0,T]\right) \sim c\, T^{-1/2}\, l(T), \quad T \to \infty,$$

where l is slowly varying at infinity and $c > 0$; see, e.g., [7] or [13], Section 4.4. Our goal is to show that the function l may be chosen asymptotically constant, which is suggested by the analogous result for random walks: If $(S_n)_{n\geqslant 1}$ is a centred random walk with finite variance, then $P(\sup_{n=1,\ldots,N} S_n \leqslant 0) \sim cN^{-1/2}$ by [17], Theorem XII.7.1a. However, to the author's knowledge, an analogous result for Lévy processes has not been stated in the literature so far.

Clearly, $P(\sup_{t\in[0,T]} X_t \leqslant 1) \leqslant P(\sup_{n=1,\ldots,\lfloor T\rfloor} X_n \leqslant 1) \asymp T^{-1/2}$ since the process $(X_n)_{n\geqslant 1}$ is a centred random walk with finite variance. Moreover, if $E[X_1^{2+\epsilon}] < \infty$ for some $\epsilon > 0$, then also $P(\sup_{n=1,\ldots,\lfloor T\rfloor} X_n \leqslant 1) \gtrsim T^{-1/2}$

by [4], Proposition 2.1. The next theorem states the precise asymptotic decay of the probability $P(\sup_{t\in[0,T]} X_t \leqslant 1)$ as $T\to\infty$ under the assumption of finite variance. The idea to approximate the integral over $P(X_t>0)$ by the sum over $P(X_n>0)$ in the proof below is due to Ron Doney.

THEOREM 3.1. *Let $(X_t)_{t\geqslant 0}$ be a centred Lévy process with $E[X_1^2]<\infty$. For any $x>0$, there is a constant $c(x)>0$ such that*

$$P(X_t\leqslant x, \forall t\in[0,T]) \sim c(x)\,T^{-1/2},\qquad T\to\infty.$$

Proof. Let τ_x be the first hitting time of the set (x,∞), $x>0$. According to eq. 4.4.7 in [13], it follows that

$$1-E[e^{-q\tau_x}]\sim U(x)\kappa(q),\quad q\downarrow 0, \tag{3.1}$$

where U is a renewal function (see [13], eq. 4.4.6) and

$$\kappa(u)=\exp\left(\int_0^\infty \frac{e^{-t}-e^{-ut}}{t}\,P(X_t>0)\;dt\right),\qquad u\geqslant 0.$$

Since $\int_0^\infty t^{-1}(e^{-t}-e^{-ut})\,dt=\log u$ for $u>0$ (a Frullani integral), we have

$$\kappa(u)=\sqrt{u}\,\exp\left(\int_0^\infty \frac{e^{-t}-e^{-ut}}{t}\big(P(X_t>0)-1/2\big)\,dt\right). \tag{3.2}$$

Let $h(t):=P(X_t>0)-1/2$. We will show that

$$\lim_{\lambda\downarrow 0}\int_0^\infty \frac{e^{-\lambda t}-e^{-t}}{t}\,h(t)\,dt=\int_0^\infty \frac{1-e^{-t}}{t}\,h(t)\,dt=:A,$$

where $A\in\mathbb{R}$. This implies that $\kappa(\lambda)\sim\sqrt{\lambda}e^{-A}$ as $\lambda\downarrow 0$. By a Tauberian theorem (see [17], Theorem XIII.5.4), we conclude from (3.1) that

$$P(\tau_x>T)\sim\frac{U(x)e^{-A}}{\Gamma(1/2)}\,T^{-1/2}=\frac{U(x)e^{-A}}{\sqrt{\pi}}\,T^{-1/2},\qquad T\to\infty,$$

so the theorem follows.

To prove that A is finite, we approximate the term $P(X_t>0)$ by $P(X_n>0)$ for $t\in(n,n+1]$, which allows us to use classical results from fluctuation theory of random walks to show that the integral in (3.2) converges as $u\to 0$. To this end, note that, for $u\in(0,1)$,

$$\begin{aligned}
0\leqslant\int_0^\infty \frac{e^{-ut}-e^{-t}}{t}\;|h(t)|\;dt &\leqslant \int_0^1 \frac{1-e^{-t}}{t}\;|h(t)|\;dt\\
&\quad+\sum_{n=1}^\infty\int_n^{n+1}\frac{e^{-ut}-e^{-t}}{t}\,\big(|P(X_t>0)-P(X_n>0)|+|h(n)|\big)\,dt\\
&\leqslant c+\sum_{n=1}^\infty n^{-1}\sup_{t\in[n,n+1]}|P(X_t>0)-P(X_n>0)|+\sum_{n=1}^\infty n^{-1}\,|h(n)|.
\end{aligned}$$

By Theorem 3 in [24], the series $\sum_{n=1}^{\infty} n^{-1}\big(P(X_n>0)-1/2\big)=\sum_{n=1}^{\infty} n^{-1}h(n)$ converges absolutely if $E[X_1]=0$ and $E[X_1^2]\in(0,\infty)$. Next, using results on the speed of convergence in the CLT, we show that the first series also converges. To this end, let $t\in(n,n+1]$. By the independence and stationarity of increments of X, we have

(3.3)

$$\begin{aligned} P(X_t\leqslant 0)-P(X_n\leqslant 0) &= \int_{-\infty}^{\infty}\big(P(X_n\leqslant -y)-P(X_n\leqslant 0)\big)\,P(X_{t-n}\in dy) \\ &= \int_{-\infty}^{\infty}\Big(F_n\big(-y/\sqrt{n}\big)-F_n(0)\Big)\,P(X_{t-n}\in dy), \end{aligned}$$

where $F_n(x):=P\big(X_n/\sqrt{n}\leqslant x\big)$. Let Φ denote the cdf of a standard Gaussian variable. With $\Delta_n:=\sup\{|F_n(x)-\Phi(x)|:x\in\mathbb{R}\}$, we infer for $y\in\mathbb{R}$ that

$$\begin{aligned} &\big|F_n\big(y/\sqrt{n}\big)-F_n(0)\big| \\ &\quad\leqslant \big|F_n\big(y/\sqrt{n}\big)-\Phi\big(y/\sqrt{n}\big)\big|+\big|\Phi\big(y/\sqrt{n}\big)-\Phi(0)\big|+|\Phi(0)-F_n(0)| \\ &\quad\leqslant 2\Delta_n+\big|\Phi\big(y/\sqrt{n}\big)-\Phi(0)\big|\leqslant 2\Delta_n+\sqrt{2/\pi}\,|y|\,/\sqrt{n}. \end{aligned}$$

In view of (3.3), we obtain

$$\begin{aligned} |P(X_t\leqslant 0)-P(X_n\leqslant 0)| &\leqslant 2\Delta_n+\sqrt{2/\pi}E\left[|X_{t-n}|\right]/\sqrt{n} \\ &\leqslant 2\Delta_n+\sqrt{2/\pi}E\left[|X_1|\right]/\sqrt{n}, \end{aligned}$$

where we have used the fact that $0\leqslant t-n\leqslant 1$ and that $(|X_t|)_{t\geqslant 0}$ is a submartingale in the last inequality. Since $E[X_1^2]<\infty$, $\sum_{n=1}^{\infty}\Delta_n/n$ is finite by Theorem 1 in [14]. Hence, the first series above is also finite, so by dominated convergence, we get

$$A=\lim_{u\downarrow 0}\int_0^{\infty}\frac{e^{-ut}-e^{-t}}{t}\big(P(X_t>0)-1/2\big)\,dt\in\mathbb{R}. \quad ■$$

Having determined the asymptotic behaviour of the survival probability for X, let us continue to give some heuristics concerning the survival exponent of $X\circ|S|$. If $E[S_1]=0$ and $E[S_1^2]=1$, it follows from the invariance principle that

$$\lim_{N\to\infty} P\big(|S_n|\leqslant\sqrt{N}\,x,\forall n=1,\dots,N\big)=P\left(|B_t|\leqslant x,\forall t\in[0,1]\right),\qquad x>0,$$

where B denotes a standard Brownian motion. Intuitively, one would therefore expect that

$$P\big(X(|S_n|) \leqslant 1, \forall n = 1, \dots, N\big) \asymp P\big(X_t \leqslant 1, \forall t \in [0, \sqrt{N}]\big) \asymp N^{-1/4},$$

at least if the points $|S_1|, \dots, |S_N|$ are sufficiently "dense" in $\big[0, \sqrt{N}\big]$. Under a subexponential moment condition on the random walk, we show that the persistence exponent is indeed $1/4$. For simplicity of notation, we denote by $\mathcal{X}(\gamma)$ the class of non-degenerate random variables X with $E[e^{|X|^\gamma}] < \infty$, where $\gamma > 0$.

Before proving the upper bound of Theorem 1.2, we need the following auxiliary result which follows directly from Lemma 5 in [15]:

LEMMA 3.1. *Let $(S_n)_{n\geqslant 1}$ denote a centred random walk with $E[S_1^2] \in (0, \infty)$ and let $M_n := \max\{S_1, \dots, S_n\}$. There is a constant C such that*

$$P(M_N \leqslant x) \leqslant 1 \wedge C(x+1)N^{-1/2}, \quad x \geqslant 0, N \geqslant 1.$$

Let $(X_t)_{t\geqslant 0}$ denote a centred Lévy process such that $E[X_1^2] < \infty$. Since $(X_n)_{n\geqslant 1}$ is a random walk, it follows for all $T \geqslant 1, x \geqslant 0$ that

$$P(X_t \leqslant x, \forall t \in [0, T]) \leqslant 1 \wedge C(x+1)\lfloor T \rfloor^{-1/2}. \tag{3.4}$$

We are now ready to establish the upper bounds of Theorem 1.2. In the proof of the upper bound, we need stretched exponential moments in order to ensure that the probability of a gap of size $(C \log N)^\gamma$ in the set $\{|S_1|, \dots, |S_N|\}$ is asymptotically irrelevant, i.e., of lower order than $N^{-1/2}$. This allows us (at the cost of a lower order term) to consider the supremum of the process X over the whole interval from zero to the maximum of the absolute value of the random walk up to time N instead of the set $\{0, |S_1|, \dots, |S_N|\}$.

Proof of Theorem 1.2 (upper bound). Let us first observe that it suffices to prove the upper bound for the case when the inner process Y is a random walk. Indeed, if Y is a Lévy process, $(Y_n)_{n\geqslant 1}$ is a random walk and we infer for all $T > 0$ that

$$P\big(X(|Y_t|) \leqslant 1, \forall t \in [0, T]\big) \leqslant P\big(X(|Y_n|) \leqslant 1, \forall n = 1, \dots, \lfloor T \rfloor\big). \tag{3.5}$$

Therefore, we consider the persistence probability of $X \circ |S|$, where $S_n = \xi_1 + \dots + \xi_n$, and $\xi_1, \xi_2, \dots$ are i.i.d. Let us begin to develop a method to deal with the gaps in the range of the random walk. The idea is to fill the gaps in the range, which will only result in a term of lower order if the gaps are not too large. Let $t(1) \leqslant t(2) \leqslant \dots$, $N \geqslant 2$, $k \geqslant 0$, $x, y > 0$ (below, we set $t(k) := S_k$). Using the equality $(X_{T-t} - X_T)_{t\in[0,T]} \overset{d}{=} (-X_t)_{t\in[0,T]}$, observe that

$$\begin{aligned}
&P\big(\bigcap_{n=1}^{N} \{ \sup_{t\in[t(n)-k,t(n)+k]} X_t \leqslant x + ky\}\big) \\
&\qquad\qquad\qquad \leqslant P\big(\bigcap_{n=1}^{N} \{ \sup_{t\in[t(n)-(k+1),t(n)+k+1]} X_t \leqslant x + (k+1)y\}\big) \\
&\qquad\qquad\qquad\quad + \sum_{n=1}^{N-1} P(\sup_{t\in[0,1]} X_{t(n)-k-t} - X_{t(n)-k} \geqslant y) \\
&\qquad\qquad\qquad\quad + \sum_{n=1}^{N-1} P(\sup_{t\in[0,1]} X_{t(n)+k+t} - X_{t(n)+k} \geqslant y) \\
&\qquad\qquad\qquad \leqslant P\big(\bigcap_{n=1}^{N} \{ \sup_{t\in[t(n)-(k+1),t(n)+k+1]} X_t \leqslant x + (k+1)y\}\big) \\
&\qquad\qquad\qquad\quad + 2NP(\sup_{t\in[0,1]} |X_t| \geqslant y).
\end{aligned}$$

(Here and below, the interval $[t(n) - k, t(n) + k]$ stands for $[0, t(n) + k]$ whenever $t(n) - k < 0$.) Let $p_N := P\left(\sup_{n=1,\dots,N} X(|S_n|) \leqslant 1\right)$. Conditioning on $S_1, \dots, S_N$ and using the previous inequality with $x = 1$ and $y = (2\log N)^{1/\alpha}$ iteratively for $k = 0, \dots, L$, we obtain

$$\begin{aligned}
\text{(3.6)}\qquad p_N &\leqslant P\big(\bigcap_{n=0}^{N} \{X_t \leqslant 1 + (2\log N)^{1/\alpha}, \forall t \in [|S_n| - 1, |S_n| + 1]\}\big) \\
&\quad + 2(N+1)P\big(\sup_{t\in[0,1]} |X_t| \geqslant (2\log N)^{1/\alpha}\big) \\
&\leqslant \dots \leqslant P\big(\bigcap_{n=0}^{N} \{X_t \leqslant 1 + L(2\log N)^{1/\alpha}, \forall t \in [|S_n| - L, |S_n| + L]\}\big) \\
&\quad + L \cdot 2(N+1)P\big(\sup_{t\in[0,1]} |X_t| \geqslant (2\log N)^{1/\alpha}\big).
\end{aligned}$$

Since $X_1 \in \mathcal{X}(\alpha)$, we infer by Doob's inequality for submartingales that $C_1 := E\big[\sup\{e^{|X_t|^\alpha} : t \in [0,1]\}\big] < \infty$, and therefore

$$\text{(3.7)}\qquad P\big(\sup_{t\in[0,1]} |X_t| \geqslant (2\log N)^{1/\alpha}\big) \leqslant C_1 e^{-2\log N} = C_1\, N^{-2}.$$

Let $S_N^* := \max\{|S_1|, \dots, |S_N|\}$. Setting

$$L = L_N = C(\log N)^\gamma \quad \text{and} \quad u_N := 2^{1/\alpha} C(\log N)^{\gamma+1/\alpha},$$

we see from (3.6) and (3.7) that

$$
\begin{aligned}
(3.8)\qquad p_N &\leqslant P\Big(\bigcap_{n=0}^{N}\{\sup_{t\in[|S_n|-L,|S_n|+L]} X_t \leqslant 1+u_N\}\Big) + 2CC_1(\log N)^{\gamma}N^{-1}\\
&\leqslant P\left(X_t \leqslant 2u_N, \forall\, t\in[0,S_N^*]\right) + P\left(A_N\right) + C_2(\log N)^{\gamma}N^{-1},
\end{aligned}
$$

where A_N is the event that the set $\{0,|S_1|,\ldots,|S_N|\}$ contains a gap larger than $L = C(\log N)^{\gamma}$. In particular, the event A_N implies that the random walk must have a jump larger than L up to time N. If $S_1 = \xi_1 \in \mathcal{X}(\beta)$, take $\gamma = 1/\beta$ and note that

(3.9)

$$
\begin{aligned}
P\left(A_N\right) &\leqslant P\big(\max_{n=1,\ldots,N}|\xi_n| \geqslant C(\log N)^{1/\beta}\big) \leqslant NP\big(|\xi_1| \geqslant C(\log N)^{1/\beta}\big)\\
&\leqslant Ne^{-C^{\beta}\log N}\, E[e^{|\xi_1|^{\beta}}] = o(N^{-1/2}),
\end{aligned}
$$

where the last equality holds for C large enough. Now combining (3.8) and (3.9), we arrive at

$$
(3.10)\qquad p_N \leqslant P\big(X_t \leqslant 2^{1+1/\alpha}C(\log N)^{1/\beta+1/\alpha}, \forall\, t\in[0,S_N^*]\big) + o(N^{-1/2}).
$$

We need to distinguish the cases $E\left[S_1\right]=0$ and $E\left[S_1\right]\neq 0$.

Case $E\left[S_1\right]=0$. Let $g_N := 2^{1+1/\alpha}C(\log N)^{1/\beta+1/\alpha}$. First, note that

$$
\begin{aligned}
(3.11)\qquad P\left(X_t \leqslant g_N, \forall\, t\in[0,S_N^*]\right) &\leqslant P\big(\sup_{n=1,\ldots,N}|S_n| \leqslant \sqrt{N/\log N}\big)\\
&\quad + P\big(X_t \leqslant g_N, \forall\, t \leqslant \sqrt{N/\log N}\big).
\end{aligned}
$$

By Corollary 4.6 in [1] (or Theorem 4 in [21]), we have

$$
(3.12)\qquad \lim_{N\to\infty} a_N^{-2}\,\log P\big(S_N^* \leqslant \sqrt{N}/a_N\big) = -\pi^2/8
$$

whenever $0 < a_N \to \infty$ and $a_N^2/N \to 0$. This shows that

$$
(3.13)\qquad P\big(\max_{n=1,\ldots,N}|S_n| \leqslant \sqrt{N/\log N}\big) = N^{-\pi^2/8+o(1)} = o(N^{-1/4}).
$$

Finally, (3.10), (3.11), (3.13), and (3.4) imply that

$$
\begin{aligned}
P\big(\sup_{n=1,\ldots,N} X(|S_n|) \leqslant 1\big) &\leqslant P\big(X_t \leqslant g_N, \forall t \leqslant \sqrt{N/\log N}\big) + o(N^{-1/4})\\
&\precsim (\log N)^{1/\alpha+1/\beta+1/4}\, N^{-1/4}.
\end{aligned}
$$

C a s e $E[S_1] \neq 0$. Similarly, with g_N as above, note that

$$\begin{aligned} P\left(X_t \leqslant g_N, \forall\, t \in [0, S_N^*]\right) &\leqslant P\left(X_t \leqslant g_N, \forall\, t \leqslant |S_N|\right) \\ &\leqslant P\left(X_t \leqslant g_N, \forall\, t \leqslant N\,|E[S_1]|/2\right) + P\left(|S_N| \leqslant N\,|E[S_1]|/2\right). \end{aligned}$$

Write $\tilde{S}_n := S_n - nE[S_1]$ and note that

$$\begin{aligned} P\left(|S_N| \leqslant N\,|E[S_1]|/2\right) &\leqslant P(N\,|E[S_1]| - |\tilde{S}_N| \leqslant N|E[S_1]|/2) \\ &\leqslant P(|\tilde{S}_N| \geqslant |E[S_1]|\,N/2) \leqslant 4\frac{E[\tilde{S}_N^2]}{E[S_1]^2\,N^2} = C_1 N^{-1}. \end{aligned}$$

As above, in combination with (3.10) and (3.4), we conclude that

$$\begin{aligned} p_N &\leqslant P\big(X_t \leqslant C_2(\log N)^{1/\beta+1/\alpha}, \forall\, t \leqslant N\,|E[S_1]|/2\big) + o(N^{-1/2}) \\ &\precsim (\log N)^{1/\alpha+1/\beta} N^{-1/2}. \quad \blacksquare \end{aligned}$$

Let us now prove the lower bound of Theorem 1.2. We only prove the lower bound if the inner process Y is a Lévy process. If Y is a random walk, the proof is almost identical.

P r o o f o f T h e o r e m 1.2 (l o w e r b o u n d).

C a s e $E[Y_1] = 0$. By independence of X and Y, we have

$$P\big(\sup_{t\in[0,T]} X(|Y_t|) \leqslant 1\big) \geqslant P(\sup_{t\in[0,c\sqrt{T}]} X_t \leqslant 1)\, P\big(\sup_{t\in[0,T]} |Y_t| \leqslant c\sqrt{T}\big).$$

Note that, by Doob's inequality applied to the submartingale $(|Y_t|)_{t\geqslant 0}$, we obtain

$$P\big(\sup_{t\in[0,T]} |Y_t| \leqslant c\sqrt{T}\big) = 1 - P\big(\sup_{t\in[0,T]} |Y_t| > c\sqrt{T}\big) \geqslant 1 - \frac{EY_T^2}{c^2 T} = 1/2$$

for $c := \sqrt{2\,E[Y_1^2]}$. We have used the equality $E[Y_t^2] = t \cdot EY_1^2$ for a square integrable Lévy martingale. This proves the lower bound if $E[Y_1] = 0$.

C a s e $E[Y_1] \neq 0$. As before, for any $c > |E[Y_1]|$, we have

$$P\big(\sup_{t\in[0,T]} X(|Y_t|) \leqslant 1\big) \geqslant P(\sup_{t\in[0,cT]} X_t \leqslant 1)\, P(\sup_{t\in[0,T]} |Y_t| \leqslant cT).$$

Next, since $|Y_t| \leqslant |Y_t - E[Y_t]| + |E[Y_t]|$ and $E[Y_t] = E[Y_1] \cdot t$ for a Lévy process, it follows that

$$\begin{aligned} P(\sup_{t\in[0,T]} |Y_t| \leqslant cT) &\geqslant P\big(\sup_{t\in[0,T]} |Y_t - E[Y_t]| \leqslant (c - |E[Y_1]|)T\big) \\ &\geqslant 1 - \frac{E\big[\,|Y_T - E[Y_T]|^2\,\big]}{(c - |E[Y_1]|)^2\,T^2} = 1 - \frac{E\big[\,|Y_1 - E[Y_1]|^2\,\big]}{(c - |E[Y_1]|)^2\,T} \to 1 \end{aligned}$$

as $T \to \infty$. We have again used Doob's inequality, and the last equality follows from the fact that $E\big[\,|Y_T - E\,[Y_T]|^2\,\big] = E\big[\,|Y_1 - E\,[Y_1]|^2\,\big] \cdot T$. This completes the proof of the lower bound. ■

REMARK 3.1. *The proof reveals that under the assumptions of Theorem* 1.2, *if* $E\,[Y_1] = 0$, *it follows that*

$$N^{-1/4} \precsim P\big(\sup_{n=1,\ldots,N} X(|S_n|) \leqslant 1\big) \precsim N^{-1/4}\,(\log N)^{1/\alpha+1/\beta+1/4}.$$

Note that the lower bounds of Theorem 1.2 *hold if* $E[X_1^2] + E[Y_1^2] < \infty$.

The upper bound of Theorem 1.2 *can be improved if* X *is a symmetric Lévy process and* Y *is a subordinator. Assume without loss of generality that* $Y_1 \geqslant 0$ *a.s. Then* $Z := X \circ Y$ *is a symmetric Lévy process. In particular*,

$$P(\sup_{t\in[0,T]} Z_t \leqslant 1) \precsim P(\sup_{n\in[0,\lfloor T\rfloor]} Z_n \leqslant 1) \asymp T^{-1/2}$$

without any additional assumption of moments (see, e.g., [11], Proposition 1.4). *This observation suggests that Theorem* 1.2 *remains true under much weaker integrability conditions.*

In (3.9), *we have seen that the probability of a gap of size* $C(\log N)^{1/\beta}$ *up to time* N *can be made of arbitrarily small polynomial order by increasing the constant* C *under the assumption that* $S_1 \in \mathcal{X}(\beta)$. *However*, *if we only assume that* $E\,[|S_1|^p]$ *is finite for some* $p \geqslant 2$, *it does not seem easy to get a polynomial upper bound on this probability. Moreover*, *it is easy to see that a gap of size* $(\log N)^\gamma$ *is much more likely in that case. For simplicity*, *assume that* $E[S_1^2] < \infty$ *and that* $P\,(S_1 > x) \asymp x^{-p}$ *as* $x \to \infty$ *with* $p > 2$. *The event that the random walk jumps above* L *and stays above the level* S_1 *after that up to time* N *clearly implies that the set* $\{0, |S_1|\,, \ldots, |S_N|\}$ *has a gap of size* L. *Hence*, *the probability of a gap of size* L *is bounded from below by*

$$P(S_1 \geqslant L, \sup_{n=2,\ldots,N} S_n - S_1 \geqslant 0) = P\,(S_1 \geqslant L)\,P(\sup_{n=1,\ldots,N-1} S_n \geqslant 0),$$

and if $L = C(\log N)^\gamma$, *the product is of order* $(\log N)^{-p\gamma} N^{-1/2} = N^{-1/2+o(1)}$.

Moreover, *let us mention that even for a deterministic increasing sequence* $(s_n)_{n\geqslant 1}$ *such that* $s_N \to \infty$ *as* $N \to \infty$ *and a Brownian motion* $(B_t)_{t\geqslant 0}$, *it is not obvious to find conditions on* $(s_n)_{n\geqslant 1}$ *such that*

$$P\big(\sup_{n=1,\ldots,N} B(s_n) \leqslant 1\big) \asymp P(\sup_{t\in[0,s_N]} B_t \leqslant 1) \asymp s_N^{-1/2}.$$

We refer to [3] for related results.

Let us now prove Theorem 1.3. Recall that fractional Brownian motion with Hurst parameter $H \in (0,1)$ (abbreviated as fBm(H) hereafter) is a centred Gaussian process $(X_t)_{t\in\mathbb{R}}$ with covariance

$$E\,[X_t X_s] = \frac{1}{2}\,(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad s,t \in \mathbb{R}.$$

X has stationary increments and is self-similar: $(X_{ct})_{t\in\mathbb{R}} \stackrel{d}{=} (c^H X_t)_{t\in\mathbb{R}}$.

P r o o f o f T h e o r e m 1.3. Let X be an fBm(H). First, recall from [22] that $P\,(X_t \leqslant 1, \forall t \in [0,T]) = T^{-(1-H)+o(1)}$.

U p p e r b o u n d. We can almost repeat the proof of Theorem 1.2. Let $c = E\,[M_1]$, where $M_1 := \sup\{X_t : t \in [0,1]\}$, and recall that

$$\begin{aligned} P\big(\sup_{t\in[0,1]} |X_t| > (4\log N)^{1/2}\big) &\leqslant 2P\big(M_1 > (4\log N)^{1/2}\big) \\ &\leqslant C_1 \exp\Big(-\big((4\log N)^{1/2} - c\big)^2/2\Big) \\ &= N^{-2+o(1)}, \end{aligned}$$

by the Gaussian concentration inequality (see, e.g., [19]), which is the equivalent of (3.7). Moreover, since $(X_{t+T} - X_T)_{t\in[0,T]}$ and $(X_{T-t} - X_T)_{t\in[0,T]}$ are equal in law to $(X_t)_{t\in[0,T]}$, we can proceed as in the proof of Theorem 1.2 (cf. (3.10)) to obtain

$$P\big(\sup_{n=1,\ldots,N} X(|S_n|) \leqslant 1\big) \leqslant P\big(X_t \leqslant C(\log N)^{1/\beta+1/2}, \forall t \in [0, S_N^*]\big) + o(N^{-1}).$$

Set $g_N := C(\log N)^{1/\beta+1/2}$. If $E\,[S_1] = 0$, in view of (3.12) and the self-similarity of X, we get

$$\begin{aligned} P\,(X_t \leqslant g_N, \forall t \in [0, S_N^*]) &\leqslant P\big(X_t \leqslant g_N, \forall t \in [0, \sqrt{N/\log N}]\big) + o(N^{-1}) \\ &= P\big(X_t \leqslant 1, \forall t \in \big[0, g_N^{-1/H}\sqrt{N/\log N}\big]\big) + o(N^{-1}) \\ &= \big(g_N^{-1/H}\sqrt{N/\log N}\big)^{-(1-H)+o(1)} + o(N^{-1}) = N^{-(1-H)/2+o(1)}. \end{aligned}$$

If $E\,[S_1] \neq 0$, a similar argument yields the upper bound. The proof of the lower bound poses no difficulty and is omitted. ∎

REMARK 3.2. *In Theorems* 1.2 *and* 1.3, *the outer process X had stationary increments in both cases. One might wonder if this assumption can be relaxed. In view of Theorem* 1.1, *one might guess that if X has a persistence exponent $\theta > 0$ and $E\,[S_1] = 0$, it would follow that*

$$P\big(\sup_{n=1,\ldots,N} X(|S_n|) \leqslant 1\big) = N^{-\theta/2+o(1)}, \qquad N \to \infty,$$

under suitable moment conditions. However, this turns out to be false in general. As an example, consider a sequence $\tilde{X}_1, \tilde{X}_2, \ldots$ *of independent random variables with* $P(\tilde{X}_n = 2) = 1 - P(\tilde{X}_n = 0) = 1/(n+1)$ *for* $n \geqslant 1$ *and define* $X = (X_t)_{t\geqslant 0}$ *by*

$$X_t = \begin{cases} \tilde{X}_n & \text{if } t = (2n-1)/2 \text{ for some } n \in \mathbb{N}, \\ 0 & \text{elsewhere}. \end{cases}$$

Obviously, X *does not have stationary increments. Moreover, it is not hard to check that*

$$P(\sup_{t\in[0,T]} X_t \leqslant 1) \asymp P(\tilde{X}_1 = 0, \ldots, \tilde{X}_{\lfloor T\rfloor} = 0) = \prod_{n=1}^{\lfloor T\rfloor} \big(1 - 1/(n+1)\big) \asymp T^{-1}.$$

If $(S_n)_{n\geqslant 1}$ *is a symmetric simple random walk, by construction it follows that* $X(|S_n|) = 0$ *for all* n*, i.e.,* $P\big(X(|S_n|) \leqslant 1, \forall n \geqslant 1\big) = 1$.

4. TWO-SIDED PROCESSES

In Sections 2 and 3, the outer process $X = (X_t)_{t\geqslant 0}$ had the index set $[0,\infty)$, so it was only possible to evaluate X over the range of the absolute value of the inner process Y. In this section, we work with two-sided processes $X = (X_t)_{t\in\mathbb{R}}$ allowing us to consider the one-sided exit problem for the process $X \circ Y$.

In Section 4.1, we assume that X is a two-sided process defined in (1.3) and that the inner process Y is a self-similar continuous process before turning to the case of random walks and Lévy processes in Section 4.2.

4.1. Continuous self-similar processes. Let us first prove Theorem 1.4. As a corollary, we obtain the persistence exponent of iterated Brownian motions and iterated fractional Brownian motions.

Proof of Theorem 1.4. The lower bound can be proved as in Theorem 1.1, so we give only the proof of the upper bound. Denote by I and M the infimum and maximum process of Y, i.e., $I_t = \inf_{u\in[-t,t]} Y_u$ and $M_t = \sup_{u\in[-t,t]} Y_u$. By assumption, we can choose a constant C such that for all $T > 0$

$$P\left(X_t \leqslant 1, \forall t \in [-T,T]\right) \leqslant C\, T^{-\theta}.$$

Since Y is independent of X and has continuous paths, we have

$$\begin{aligned} P\big(X(Y_t) \leqslant 1, \forall t \in [0,T]\big) &= P\left(X_t \leqslant 1, \forall t \in [I_T, M_T]\right) \\ &\leqslant P\big(X_t \leqslant 1, t \in [-(|I_T| \wedge M_T), |I_T| \wedge M_T]\big) \\ &\leqslant C\, E[(|I_T| \wedge M_T)^{-\theta}]. \end{aligned}$$

In view of the self-similarity, we have

$$E[(|I_T| \wedge M_T)^{-\theta}] \leqslant T^{-\theta H} \cdot \big(E[(-I_1)^{-\theta}] + E[M_1^{-\theta}]\big),$$

and the last expectation is finite by assumption. This completes the proof. ∎

Let us apply Theorem 1.4 to iterated fractional Brownian motions.

COROLLARY 4.1. *Let* $\big(Y_n(t)\big)_{t\in\mathbb{R}}$ *be an* fBm(H_n) *for every* $n \geqslant 1$, *all independent. For* $t \in \mathbb{R}$, *set* $X_1(t) := Y_1(t)$ *and* $X_n(t) := X_{n-1} \circ Y_n(t)$. *Let* $\theta_1 = 1$ *and* $\theta_n = H_2 \cdot \ldots \cdot H_n$. *It follows that*

$$P\big(X_n(t) \leqslant 1, \forall t \in [-T,T]\big) = T^{-\theta_n + o(1)}, \quad T \to \infty,\ n \geqslant 1.$$

P r o o f. By Theorem 3 in [22], if B^H is an fBm(H), we have

$$P\left(B^H(t) \leqslant 1, \forall t \in [-T,T]\right) = T^{-1+o(1)}, \quad T \to \infty.$$

In view of the self-similarity, this equals $P\big(\sup_{t\in[-1,1]} B^H(t) \leqslant \epsilon\big) = \epsilon^{1/H+o(1)}$ as $\epsilon \downarrow 0$. Hence, by symmetry,

$$E\big[(-\inf\{B_t^H : t \in [-1,1]\})^{-\eta}\big] + E\big[(\sup\{B_t^H : t \in [-1,1]\})^{-\eta}\big] < \infty$$

for any $\eta < 1/H$ by Lemma 2.1. Since $\theta_n \leqslant 1$ for all n, the assertion follows now easily by induction in view of Theorem 1.4. ∎

If we know the precise behaviour of $P\left(X_t \leqslant 1, \forall t \in [-T_1, T_2]\right)$ as $T_1, T_2 \to \infty$, we can get a stronger result than Theorem 1.4. In particular, if X has independent branches such as in the case of two-sided Brownian motion, the next theorem allows us to determine the exact asymptotics of the persistence probability (see Corollary 4.2).

THEOREM 4.1. *Let* $(X_t)_{t\geqslant 0}$ *be a stochastic process such that*

$$P\left(X_t \leqslant 1, \forall t \in [-T_1, T_2]\right) \sim c\, T_1^{-\theta^-} T_2^{-\theta^+}, \quad T_1, T_2 \to \infty.$$

Let $(Y_t)_{t\in\mathbb{R}}$ *denote an independent self-similar process of index* H *with* $Y_0 = 0$ *and continuous paths such that for some* $\rho > \theta^+ + \theta^-$ *it follows that*

$$P\left(Y_t \geqslant -\epsilon, \forall t \in [-1,1]\right) + P\left(Y_t \leqslant \epsilon, \forall t \in [-1,1]\right) = O(\epsilon^\rho), \quad \epsilon \downarrow 0.$$

Then $A := E\big[\,|\inf\{Y_t, t \in [-1,1]\}|^{-\theta^-}\left(\sup\{Y_t : t \in [-1,1]\}\right)^{-\theta^+}\big] < \infty$ *and*

$$P\big(X(Y_t) \leqslant 1, \forall t \in [-T,T]\big) \sim A\, c\, T^{-H(\theta^+ + \theta^-)}, \quad T \to \infty.$$

P r o o f. Let $I_T := \inf\{Y_t : t \in [-T,T]\}$, $M_T := \sup\{Y_t : t \in [-T,T]\}$. We know from Lemma 2.1 that $E[|I_1|^{-\eta}] + E[M_1^{-\eta}] < \infty$ for $\eta \in (0,\rho)$. The finiteness of A then follows from Lemma 4.1 below. The rest of the proof is analogous to the one of Theorem 1.1. We only sketch the proof of the upper bound. For $\epsilon > 0$, we can find T_0 such that for all $T_1, T_2 \geqslant T_0$ we have

$$P\left(X_t \leqslant 1, \forall t \in [-T_1, T_2]\right) \leqslant (c+\epsilon)\, T_1^{-\theta^-} T_2^{-\theta^+}.$$

Using the independence of X and Y, we see that

$$\begin{aligned} P\big(X(Y_t) \leqslant 1, \forall t \in [-T,T]\big) &= E\big[P(X_t \leqslant 1, t \in [I_T, M_T] | \mathcal{F}_T^Y)\big] \\ &\leqslant P\left(I_T \geqslant -T_0\right) + P\left(M_T \leqslant T_0\right) + (c+\epsilon) E[|I_T|^{-\theta^-} M_T^{-\theta^+}]. \end{aligned}$$

Next, note that $\limsup_{T\to\infty} T^{H(\theta^+ + \theta^-)} \big(P\left(I_T \geqslant -T_0\right) + P\left(M_T \leqslant T_0\right)\big) = 0$. Indeed, by self-similarity, this follows from the fact $\rho > \theta^+ + \theta^-$ and

$$P(I_1 \geqslant -T_0 T^{-H}) + P(M_1 \leqslant T_0 T^{-H}) \leqslant C T_0^{\rho} T^{-\rho H}.$$

Hence, writing $\theta := \theta^+ + \theta^-$ and using the self-similarity of Y, we conclude that

$$\limsup_{T\to\infty} T^{H\theta}\, P\big(X(Y_t) \leqslant 1, \forall t \in [-T,T]\big) \leqslant (c+\epsilon) E[|I_1|^{-\theta^-} M_1^{-\theta^+}].$$

Letting $\epsilon \downarrow 0$ establishes the desired upper bound. ∎

The following lemma is stated separately for better readability and is needed in the preceding proof.

LEMMA 4.1. *Let X_1, X_2 be nonnegative random variables such that, for some $\alpha_1, \alpha_2 > 0$, $E[X_i^{\alpha_i}] < \infty$ $(i = 1, 2)$. Then for $\beta_i \in (0, \alpha_i)$ it follows that $E[X_1^{\beta_1} X_2^{\beta_2}] < \infty$.*

P r o o f. We have

$$E[X_1^{\beta_1} X_2^{\beta_2}] \leqslant E[X_1^{\beta_1}] + E[X_2^{\beta_2}] + E[X_1^{\beta_1} X_2^{\beta_2} 1_{\{X_1>1, X_2>1\}}].$$

It suffices to show that the last expectation is finite. If $1/p + 1/q + 1/r = 1$, we deduce from a generalised version of Hölder's theorem that

$$E[X_1^{\beta_1} X_2^{\beta_2} 1_{\{X_1>1, X_2>1\}}] \leqslant E[X_1^{\beta_1 p}]^{1/p} E[X_2^{\beta_2 q}]^{1/q} E[1_{\{X_1>1, X_2>1\}}]^{1/r}.$$

With $p = \alpha_1/\beta_1 > 1$, $q = \alpha_2/\beta_2 > 1$ and appropriate r, the claim follows. ∎

Theorem 4.1 allows us to state the precise behaviour of the persistence probability for n-times iterated Brownian motion.

COROLLARY 4.2. *Let $(B_n)_{n\geqslant 1}$ denote a sequence of independent two-sided Brownian motions. Set $W_t^{(1)} := B_1(t)$ and $W_t^{(n)} := B_n\big(W^{(n-1)}(t)\big)$. For every $n \geqslant 1$, let $\theta_n := 2^{-(n-1)}$. It follows that*

$$P(W_t^{(n)} \leqslant 1, \forall t \in [-T,T]) \sim c_n T^{-\theta_n}, \quad T \to \infty,\ n \geqslant 1,$$

where $c_1 = 2/\pi$ and, for $n \geqslant 2$,

$$c_n = \frac{2}{\pi} E\big[|\inf\{W_t^{(n-1)} : t \in [-1,1]\}|^{-1/2} (\sup\{W_t^{(n-1)} : t \in [-1,1]\})^{-1/2}\big].$$

Proof. If B is a two-sided Brownian motion, by the independence of the branches, we have

$$\begin{aligned} P\left(B_t \leqslant 1, \forall t \in [-T_1, T_2]\right) &= P\left(B_t \leqslant 1, \forall t \in [0,T_1]\right) P\left(B_t \leqslant 1, \forall t \in [0,T_2]\right) \\ &\sim \frac{2}{\pi} T_1^{-1/2} T_2^{-1/2} \end{aligned}$$

whenever $T_1, T_2 \to \infty$. The assertion is therefore clear for $n = 1$. By induction, if the assertion holds for some $n \geqslant 1$, we can apply Theorem 4.1 with $X = B_{n+1}$ and $Y = W^{(n)}$. Indeed, $W^{(n)}$ is 2^{-n}-self-similar. Moreover, since $W^{(n)}$ is symmetric and by the induction hypothesis, we have

$$\begin{aligned} &P(W_t^{(n)} \geqslant -\epsilon, \forall t \in [-1,1]) + P(W_t^{(n)} \leqslant \epsilon, \forall t \in [-1,1]) \\ &\quad = 2P(W_t^{(n)} \leqslant \epsilon, \forall t \in [-1,1]) \\ &\quad = 2P(W_t^{(n)} \leqslant 1, \forall t \in [-\epsilon^{-2^n}, \epsilon^{-2^n}]) \sim 2c_n \epsilon^{2^n \theta_n} = 2c_n\epsilon^2, \quad \epsilon \downarrow 0. \end{aligned}$$

Hence, we infer from Theorem 4.1 ($c = 2/\pi, \theta^+ = \theta^- = 1/2, \rho = 2, H = 2^{-n}$) that

$$A_n := E\big[|\inf\{W_t^{(n)} : t \in [-1,1]\}|^{-1/2} (\sup\{W_t^{(n)} : t \in [-1,1]\})^{-1/2}\big] < \infty$$

and

$$P(W_t^{(n+1)} \leqslant 1, t \in [-T,T]) \sim (2/\pi) A_n T^{-2^{-n}}, \quad T \to \infty. \ \blacksquare$$

4.2. Two-sided Lévy processes at random walk or Lévy times. Let us now consider the one-sided exit problem for the process $\big(X(S_n)\big)_{n\geqslant 0}$, where S is again a random walk and X is a two-sided Lévy process, i.e., the branches of X are independent Lévy processes. Theorem 1.5 shows that the persistence exponent is $1/2$ under suitable integrability conditions regardless of the sign of $E\left[S_1\right]$, in contrast to Theorem 1.2. We now give a proof of Theorem 1.5 for the case when the inner process Y is a random walk. As before, the upper bound for the case when Y is a Lévy process follows immediately, whereas the proof of the lower bound is similar and is omitted.

P r o o f o f T h e o r e m 1.5. The lower bound can be established as in the proof of Theorem 1.2 if $E[Y_1]=0$. If $E[Y_1]>0$ (say), due to the fact that $\inf_{n\geqslant 1} S_n$ is a finite random variable a.s., the result follows along similar lines.

The proof of the upper bound is also similar to that of Theorem 1.2. Let $p_N := P\big(\sup_{n=1,\dots,N} X(S_n) \leqslant 1\big)$. Repeating the steps given before (3.6), we obtain

$$\begin{aligned} p_N &\leqslant P\Big(\bigcap_{n=0}^{N}\Big\{\sup_{t\in[S_n-L,S_n+L]} X_t \leqslant 1 + L(2\log N)^{1/\alpha}\Big\}\Big) \\ &\quad + 2L(N+1)P\Big(\sup_{t\in[0,1]} \big|X_t^+\big| \geqslant (2\log N)^{1/\alpha}\Big) \\ &\quad + 2L(N+1)P\Big(\sup_{t\in[0,1]} \big|X_t^-\big| \geqslant (2\log N)^{1/\alpha}\Big). \end{aligned}$$

Let us take $L = C(\log N)^{1/\beta}$, and let $\tilde{A}_N$ mean the event that the random set $\{0, S_1, \dots, S_N\}$ contains a gap larger than L. Let $g_N := 2^{1+1/\alpha} C(\log N)^{1/\alpha+1/\beta}$. Since $X_1^+, X_1^- \in \mathcal{X}(\alpha)$, we see in view of (3.7) and (3.8) that

$$\begin{aligned} p_N &\leqslant P\big(X_t \leqslant g_N, \forall\, t \in [I_N \wedge (-L), M_N]\big) + P(\tilde{A}_N) + C_2(\log N)^{1/\beta} N^{-1} \\ &\leqslant P\big(X_t \leqslant g_N, \forall\, t \in [I_N \wedge (-L), M_N]\big) + o(N^{-1/2}). \end{aligned}$$

The last inequality follows from an estimate on $P(\tilde{A}_N)$ as in (3.9).

Let us again consider two cases:

C a s e $E[Y_1] \neq 0$. Assume first that $E[Y_1] > 0$. If $E[Y_1] < 0$, the proof is almost identical. Note that

$$\begin{aligned} P\big(X_t \leqslant g_N, \forall\, t \in [I_N \wedge (-L), M_N]\big) &\leqslant P\left(X_t \leqslant g_N, \forall\, t \in [0, M_N]\right) \\ &\leqslant P\left(M_N \leqslant \delta N\right) + P\big(X_t \leqslant g_N, \forall\, t \in [(2\log N^{1/\alpha}), \delta N]\big) \\ &\leqslant C_2 N^{-1} + C_3 g_N N^{-1/2} \precsim N^{-1/2} (\log N)^{1/\alpha+1/\beta}, \end{aligned}$$

where we have used the fact that $P\left(M_N \leqslant \delta N\right) \leqslant P\left(S_N \leqslant \delta N\right) = o(N^{-1})$ for δ small enough and (3.4) in the second inequality.

C a s e $E[Y_1] = 0$. With g_N as above, it suffices to show that

$$h_N := P\left(X_t \leqslant g_N, \forall\, t \in [I_N, M_N]\right) \precsim N^{-1/2+o(1)}.$$

Let $f_N := \sqrt{N/\log N}$, $N \geqslant 1$. Note that

$$\begin{aligned} h_N &\leqslant P\left(|S_n| \leqslant f_N, \forall n = 1, \dots, N\right) \\ &\quad + P\left(M_N \leqslant f_N, -I_N > f_N, X_t \leqslant g_N, \forall t \in [I_N, M_N]\right) \\ &\quad + P\left(M_N > f_N, -I_N \leqslant f_N, X_t \leqslant g_N, \forall t \in [I_N, M_N]\right) \\ &\quad + P\left(M_N > f_N, -I_N > f_N, X_t \leqslant g_N, \forall t \in [I_N, M_N]\right) \\ &=: J_1(N) + J_2(N) + J_3(N) + J_4(N). \end{aligned}$$

First, recall that $J_1(N) = o(N^{-1/2})$ (cf. (3.12)). It remains to estimate the terms J_2 and J_4. The term J_3 can be dealt with analogously to J_2.

Step 1. Note that

$$\begin{aligned} J_2(N) &\leqslant P(M_N \leqslant N^{1/4}, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \leqslant g_N) \\ &\quad + P(N^{1/4} \leqslant M_N \leqslant f_N, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \leqslant g_N) \\ &=: K_{2,1}(N) + K_{2,2}(N). \end{aligned}$$

Let us now find upper bounds for $K_{2,j}$ for $j = 1, 2$. First, note that

$$K_{2,1}(N) \leqslant P(M_N \leqslant N^{1/4}) P(\sup_{t\in[-f_N,0]} X_t \leqslant g_N).$$

Applying Lemma 3.1, we conclude that

$$K_{2,1}(N) \precsim N^{-1/4} g_N f_N^{-1/2} \asymp N^{-1/2} (\log N)^{1/\alpha+1/\beta+1/4}. \tag{4.1}$$

Let us now find an upper bound on $K_{2,2}$. Set $a(k) := \sum_{l=1}^{k} 2^{-(l+1)} = (1-2^{-k})/2$, $k \geqslant 1$. Since $a(N) \to 1/2$, we can find $\gamma(N)$ such that

$$N^{a(\gamma(N))} \geqslant f_N = \sqrt{N/\log N}.$$

Indeed, this just amounts to

$$a\big(\gamma(N)\big) = \frac{1-2^{-\gamma(N)}}{2} \geqslant \frac{\log f_N}{\log N} = \frac{1}{2} - \frac{\log\log N}{2\log N}, \tag{4.2}$$

i.e.,

$$\gamma(N) \geqslant \frac{1}{\log 2} \log\left(\frac{\log N}{\log\log N}\right).$$

Hence, it suffices to set $\gamma(N) := \lceil (\log\log N)/\log 2 \rceil$.

Next, note that $\{N^{1/4} \leqslant M_N \leqslant f_N\} \subseteq \{N^{a(1)} \leqslant M_N \leqslant N^{a(\gamma(N))}\}$. Hence, we get

$$\begin{aligned} K_{2,2}(N) &\leqslant \sum_{k=1}^{\gamma(N)-1} P(N^{a(k)} \leqslant M_N \leqslant N^{a(k+1)}, -I_N > f_N, \sup_{t\in[I_N,M_N]} X_t \leqslant g_N) \\ &\leqslant \sum_{k=1}^{\gamma(N)-1} P(N^{a(k)} \leqslant M_N \leqslant N^{a(k+1)}) P(\sup_{t\in[-f_N,N^{a(k)}]} X_t \leqslant g_N) \\ &\leqslant P(X_t^- \leqslant g_N, \forall t \in [0, f_N]) \\ &\quad \times \sum_{k=1}^{\gamma(N)-1} P(M_N \leqslant N^{a(k+1)}) P(X_t^+ \leqslant g_N, \forall t \in [0, N^{a(k)}]). \end{aligned}$$

In view of Lemma 3.1, we can find constants C_1 and N_0 such that for $N \geqslant N_0$

$$P(M_N \leqslant N^{a(k+1)}) \leqslant C_1\, N^{a(k+1)-1/2}, \quad k = 1, 2, \ldots$$

Similarly, for all N large enough,

$$P(\sup_{t\in[0,N^{a(k)}]} X_t^+ \leqslant g_N) \leqslant C_2\, g_N N^{-a(k)/2}, \quad k = 1, 2, \ldots$$

Hence, for N large enough, we obtain

$$\begin{aligned} K_{2,2}(N) &\leqslant (C_3 g_N/\sqrt{f_N}) \sum_{k=1}^{\gamma(N)-1} N^{a(k+1)-1/2}\, g_N N^{-a(k)/2} \\ &= C_3 g_N^2 (\log N)^{1/4} N^{-1/4} \sum_{k=1}^{\gamma(N)-1} N^{a(k+1)-a(k)/2-1/2} \\ &= C_4 (\log N)^{2/\alpha+2/\beta+1/4} \big(\gamma(N)-1\big) N^{-1/2}, \end{aligned}$$

since $a(k+1) - a(k)/2 = 1/4$. By definition of $\gamma(N)$, we arrive at $K_{2,2}(N) \precsim (\log\log N)\,(\log N)^{2/\alpha+2/\beta+1/4}\, N^{-1/2}$. Combining this with (4.1), we have

$$J_2(N) \precsim (\log\log N)\,(\log N)^{2/\alpha+2/\beta+1/4}\, N^{-1/2}, \quad N \to \infty. \tag{4.3}$$

S t e p 2. Finally, with g_N as above, note that

$$\begin{aligned} J_4(N) &\leqslant P\left(X_t \leqslant g_N, \forall t \in [-f_N, f_N]\right) \\ &= P(X_t^- \leqslant g_N, \forall t \in [0, f_N])P(X_t^+ \leqslant g_N, \forall t \in [0, f_N]) \\ &\precsim \big(g_N/\sqrt{f_N}\big)^2 \asymp (\log N)^{2/\alpha+2/\beta+1/2}\, N^{-1/2}. \quad \blacksquare \end{aligned}$$

REMARK 4.1. *The proof reveals that the persistence exponent is equal to* $1/2$ *no matter if* $E\,[Y_1] = 0$ *or not for quite different reasons. If* $E\,[Y_1] > 0$, $S_N/N \to E\,[Y_1]$ *by the law of large numbers, so the random walk diverges to* $+\infty$ *with speed* N *and the persistence probability is determined by the right branch* X^+ *of* X.

If $E\,[Y_1] = 0$, *the random walks oscillate and typical fluctuations are of order* $\pm\sqrt{N}$. *The persistence probability up to time* N *is therefore approximately equal to the probability that both* X^+ *and* X^- *stay below one until time* $\sqrt{N}$. *By independence of* X^+ *and* X^-, *this probability is equal to the product of these two probabilities which are each of order* $N^{-1/4}$.

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