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ON THE LIMIT BEHAVIOUR OF RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES

BY

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Abstract. The aim of the paper is to give some new results on weak convergence of random-indexed partial sums to infinitely divisible distributions. To this end we introduce new versions of the Lindeberg condition which allow us to strengthen or generalize results given in [5], [9], [11], [13], [14] and [15].

1. Introduction and notation. Let $(X_{nk})_{n,k\in N}$ be a doubly infinite array (DIA) of random variables (r.v's) such that for every n, the r.v's X_{nk}, $k = 1, 2, \ldots$, are independent, let F_{nk} be the distribution function of X_{nk}, and let $S_{nk} = \sum_{j=1}^{k} X_{nj}$.

We put

$$a_{nk} = \mathrm{E}X_{nk} = \int_{-\infty}^{+\infty} x\,dF_{nk}(x), \qquad \sigma_{nk}^2 = D^2 X_{nk} = \int_{-\infty}^{+\infty} x^2\,dF_{nk}(x) - a_{nk}^2,$$

$$L_{nk} = \sum_{j=1}^{k} a_{nj}, \quad V_{nk}^2 = \sum_{j=1}^{k} \sigma_{nj}^2, \qquad b_{nk} = \max_{1\leqslant j\leqslant k} \sigma_{nj}^2,$$

while for $Y_{nk} = X_{nk} - a_{nk}$, $k = 1, 2, \ldots; n = 1, 2, \ldots$, we write

$$\varphi_{nk}(t) = \mathrm{E}\exp\{itY_{nk}\} = \int_{-\infty}^{+\infty} \exp\{itx\}\,dF_{nk}(x+a_{nk}), \; f_{nk}(t) = \prod_{j=1}^{k} \varphi_{nj}(t).$$

Now let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.v's such that N_n is for every n independent of X_{nk}, $k \geqslant 1$. We assume that the distribution function of N_n is determined by the values

$$p_{nk} = P[\mathrm{N}_n = k], \; k = 1, 2, \ldots; \qquad \sum_{k=1}^{\infty} p_{nk} = 1.$$

Under these assumptions on N_n, the distribution function of

$$S_{nN_n} = \sum_{k=1}^{N_n} X_{nk}$$

depends on N_n, and

$$\mathrm{E}S_{nN_n} = \sum_{k=1}^{\infty} p_{nk} L_{nk} = A_n, \quad \mathrm{E}L_{nN_n} = \sum_{k=1}^{\infty} p_{nk} L_{nk} = A_n,$$

$$\mathrm{E}V_{nN_n}^2 = \sum_{k=1}^{\infty} p_{nk} V_{nk}^2 = \varrho_n, \quad D^2 L_{nN_n} = \sum_{k=1}^{\infty} p_{nk} L_{nk}^2 - A_n^2 = \Delta_n^2,$$

$$D^2 S_{nN_n} = \sum_{k=1}^{\infty} p_{nk} V_{nk}^2 + \sum_{k=1}^{\infty} p_{nk} L_{nk}^2 - A_n^2 = \varrho_n + \Delta_n^2 = \sigma_n^2.$$

Furthermore, let H be a bounded, nondecreasing function such that $H(-\infty) = 0$, $0 \leqslant H(x) \leqslant 1$ for all x, and $H(+\infty) = 1$. We write

$$g_t(x) = \begin{cases} (\exp\{itx\} - 1 - itx)/x^2 & \text{for } x \neq 0, \\ -t^2/2 & \text{for } x = 0, \end{cases}$$

$$f(t) = \exp\left\{\int_{-\infty}^{+\infty} g_t(x)\, dH(x)\right\}. \tag{1}$$

Then $f(t)$ is a characteristic function (ch. f.) of an infinitely divisible distribution with zero mean and unit variance (see, e.g., [4], Theorem 5.5.3).

For the moment, let us assume that $N_n = k_n$ almost surely (a.s.) for all n, where $\{k_n\}$ is a sequence of positive integers such that $k_n \to \infty$, $n \to \infty$. The following conditions are well known:

The classical Feller condition:

$$b_{nk_n}/V_{nk_n}^2 \to 0, \quad n \to \infty. \tag{F}$$

The classical Lindeberg condition:

$$\frac{1}{V_{nk_n}^2} \sum_{k=1}^{k_n} \int_{|x| \geqslant \varepsilon V_{nk_n}} x^2\, dF_{nk}(x + a_{nk}) \to 0, \quad n \to \infty, \tag{L}$$

for every $\varepsilon > 0$.

The condition

$$\frac{1}{V_{nk_n}^2} \sum_{k=1}^{k_n} \int_{x \leqslant y V_{nk_n}} x^2\, dF_{nk}(x + a_{nk}) \to H(y), \qquad n \to \infty, \tag{$\mathrm{L_H}$}$$

for every continuity point y of H, which we shall call the generalized Lindeberg condition (this condition is similar to the one used by Loève in the "bounded variances limit theorem" (cf. [3])).

Furthermore, the following facts hold (cf. [3], p. 293-295):

$$(\mathrm{L}) \Leftrightarrow \{(\mathrm{F}) \text{ and } (S_{nk_n} - L_{nk_n})/V_{nk_n} \xrightarrow{\mathrm{D}} \mathcal{N}_{0,1},\ n \to \infty\}, \tag{A}$$

where $\mathcal{N}_{a,b}$ denotes a normal r.v. with mean a and variance b, and $\xrightarrow{D}$ denotes *convergence in distribution* (*weak convergence*);

(B) $$(L_H) \Leftrightarrow \{E \exp\{it(S_{nk_n} - L_{nk_n})/V_{nk_n}\} \to f(t),\ n \to \infty\},$$

provided that (F) is satisfied.

However, in the general case, for sums with random indices such results are not known. Following the classical work [9], many authors (see, e.g., [8], [6], [7], p. 471-475, [1], [2], [13], [11], [15], [10], [12], [14]) have investigated the limit behaviour of the distribution of random-sums. Among them, [15] introduces the random Lindeberg condition and proves its sufficiency for the asymptotic normality of $(S_{nN_n} - L_{nN_n})/V_{nN_n}$. The necessity of that condition, as we know, was not yet proved.

The aim of this paper is to extend the results (A) and (B) to partial sums with random indices. For this purpose we introduce new versions of the Lindeberg condition which allow us to strengthen or generalize results given in [5], [9], [11], [13], [14] and [15].

2. A generalized random Lindeberg condition.

Definition. A DIA (X_{nk}) is said to satisfy the generalized random Lindeberg condition (RL_H) if

(RL_H) $$\frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{x \leqslant yV_{nN_n}} x^2\, dF_{nk}(x+a_{nk}) \xrightarrow{P} H(y), \qquad n \to \infty,$$

for every continuity point y of H.

It is easy to see that if we put $H(x) = I(x \geqslant 0)$, then

$$f(t) = \exp\{\int_{-\infty}^{+\infty} g_t(x)\, dH(x)\} = \exp\{-t^2/2\},$$

i.e. $f(t)$ is the ch. f. of a normal distribution with zero mean and unit variance, and (RL_H) gives the random Lindeberg condition (RL) introduced in [15], i.e.

(RL) $$E\left\{\frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{|x| \geqslant \varepsilon V_{nN_n}} x^2\, dF_{nk}(x+a_{nk})\right\} \to 0, \qquad n \to \infty,$$

for every $\varepsilon > 0$. However, we shall use the following equivalent form of (RL)

(RL_0) $$\frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{|x| \geqslant \varepsilon V_{nN_n}} x^2\, dF_{nk}(x+a_{nk}) \xrightarrow{P} 0, \qquad n \to \infty,$$

for every $\varepsilon > 0$. This version of (RL) was used in [11].

Furthermore, if $H(x) = I(x \geqslant c)$, where $c \neq 0$, then

(2) $$f(t) = \exp\left\{\frac{1}{c^2}(\exp\{itc\} - 1 - itc)\right\},$$

i.e. $f(t)$ is the ch. f. of a Poisson type distribution with zero mean and unit variance, and $(\mathrm{RL_H})$ takes the form

$$(\mathrm{RL_C}) \qquad \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int\limits_{|\frac{x}{V_{nN_n}} - c| \geqslant \varepsilon} x^2\, dF_{nk}(x+a_{nk}) \xrightarrow{\mathrm{P}} 0, \qquad n \to \infty,$$

for every $\varepsilon > 0$. Other infinitely divisible distributions may be obtained by suitable choice of the function H (see, e.g., [4]).

Moreover, in what follows, we shall use the random Feller condition

$$(\mathrm{RF}) \qquad b_{nN_n}/V_{nN_n}^2 \xrightarrow{\mathrm{P}} 0, \qquad n \to \infty,$$

introduced in [12]. Of course, (RF) follows from $(\mathrm{RL_0})$ as for every $\varepsilon > 0$

$$b_{nN_n}/V_{nN_n}^2 \leqslant \varepsilon^2 + \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int\limits_{|x| \geqslant \varepsilon V_{nN_n}} x^2\, dF_{nk}(x+a_{nk}) \xrightarrow{\mathrm{P}} \varepsilon^2, \qquad n \to \infty,$$

(cf. [15], the proof of Lemma 2). However, in the general case, $(\mathrm{RL_H})$ does not imply (RF) (e.g. $(\mathrm{RL_H})$ with $H(x) = I(x \geqslant c)$, $c \neq 0$).

Remark 1. Let us observe that if $N_n = k_n$ a.s. for all n, where $\{k_n\}$ is a sequence of positive integers such that $k_n \to \infty$, $n \to \infty$, then $(\mathrm{RL_H})$ reduces to $(\mathrm{L_H})$, while $(\mathrm{RL_0})$ and (RF) reduce to (L) and (F), respectively.

The following result extends (B) to sums with random indices and at the same time generalizes the main result of [15] (Theorem 1).

PROPOSITION 1. *Let (X_{nk}) be a DIA of r.v.'s such that for all n, the r.v.'s X_{nk}, $k = 1, 2, \ldots$, are independent, and $\{N_n\}$ be a sequence of positive integer-valued r.v.'s such that N_n is for each n independent of X_{nk}, $k = 1, 2, \ldots$ Suppose that (RF) is satisfied. Then*

$$\mathrm{E}\exp\{it(S_{nN_n} - L_{nN_n})/V_{nN_n}\} \to f(t), \qquad n \to \infty,$$

where $f(t)$ is given by (1), if and only if (RL_H) holds.

Proof. First we note that

$$\begin{aligned} \mathrm{E}\exp\{it(S_{nN_n} - L_{nN_n})/V_{nN_n}\} &= \sum_{k=1}^{\infty} p_{nk} f_{nk}(t/V_{nk}) \\ &= \sum_{k=1}^{\infty} p_{nk} \exp\Big\{\sum_{j=1}^{k} [\varphi_{nj}(t/V_{nk}) - 1]\Big\} + A_n(t), \end{aligned}$$

where, by (RF),

$$\text{(a)} \qquad \begin{aligned} |A_n(t)| &= \Big|\sum_{k=1}^{\infty} p_{nk} \Big[\exp\Big\{\sum_{j=1}^{k} \log\varphi_{nj}(t/V_{nk})\Big\} - \exp\Big\{\sum_{j=1}^{k} (\varphi_{nj}(t/V_{nk}) - 1)\Big\}\Big]\Big| \\ &\leqslant (t^4/4)\exp\{t^2/2 + t^2/4\}\, \mathrm{E} b_{nN_n}/V_{nN_n}^2 \to 0, \qquad n \to \infty, \end{aligned}$$

as $b_{nN_n}/V_{nN_n}^2 \leqslant 1$ a.s. for all n. Moreover, we observe that

$$\text{(b)} \qquad \sum_{k=1}^{\infty} p_{nk} \exp\Big\{\sum_{j=1}^{k} [\varphi_{nj}(t/V_{nk})-1]\Big\} = \mathrm{E}\exp\Big\{\sum_{j=1}^{N_n} [\varphi_{njN_n}(t)-1]\Big\},$$

where $\varphi_{njN_n}(t)$ is an r.v. taking values $\varphi_{njk}(t) = \varphi_{nj}(t/V_{nk})$, $k = 1, 2, \ldots$ Hence, we see that the right hand side of (b) is equal to

$$\mathrm{E}\exp\Big\{\frac{1}{V_{nN_n}^2} \sum_{j=1}^{N_n} \int_{-\infty}^{+\infty} g_t(x/V_{nN_n})\, x^2\, dF_{nj}(x+a_{nj})\Big\} = \mathrm{E}\exp\{R_n(t)\}, \quad \text{say}.$$

Thus, taking into account (a), we get

$$\mathrm{E}\exp\{it(S_{nN_n}-L_{nN_n})/V_{nN_n}\} = \mathrm{E}\exp\{R_n(t)\} + A_n(t) \to f(t), \; n \to \infty,$$

if and only if

$$\text{(c)} \qquad \mathrm{E}\exp\{R_n(t)\} \to f(t), \; n \to \infty.$$

Therefore, it is enough to show that

$$\text{(d)} \qquad R_n(t) \xrightarrow{\mathrm{P}} \int_{-\infty}^{+\infty} g_t(x)\, dH(x), \qquad n \to \infty,$$

if and only if $(\mathrm{RL_H})$ holds.

Now, for arbitrary positive ε_1, ε_2 and ε_3, choose an integer m sufficiently large and a subdivision $x_0 < x_1 < \ldots < x_m$, all continuity points of H, so that

$$|B(m)| = \Big|\sum_{j=1}^{m} g_t(x_{j-1})[H(x_j)-H(x_{j-1})] - \int_{-\infty}^{+\infty} g_t(x)\, dH(x)\Big| < \varepsilon_1,$$

$$\max_{1 \leqslant j \leqslant m} |x_j - x_{j-1}| < \varepsilon_2, \qquad |g_t(x)| < \varepsilon_3 \text{ for } x \leqslant x_0 \text{ or } x > x_m.$$

Hence it follows, for $n = 1, 2, \ldots$; $m = 1, 2, \ldots$, that;

$$|C_n(m)| = \Big|\frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{-\infty}^{+\infty} g_t(x/V_{nN_n})\, x^2\, dF_{nk}(x+a_{nk}) -$$
$$- \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \sum_{j=1}^{m} \int_{x_{j-1} < x/V_{nN_n} \leqslant x_j} g_t(x/V_{nN_n})\, x^2\, dF_{nk}(x+a_{nk})\Big| < \varepsilon_3 \text{ a.s.};$$

$$|D_n(m)| = \Big|\frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \sum_{j=1}^{m} \int_{x_{j-1} < x/V_{nN_n} \leqslant x_j} g_t(x/V_{nN_n})\, x^2\, dF_{nk}(x+a_{nk})$$
$$- \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \sum_{j=1}^{m} g_t(x_{j-1}) \int_{x_{j-1} < x/V_{nN_n} \leqslant x_j} x^2\, dF_{nk}(x+a_{nk})\Big| < \varepsilon_2 M_t \text{ a.s.},$$

where $M_t = \sup_x |dg_t(x)/dx| < \infty$. Moreover, we have

$$R_n(t) = \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{-\infty}^{+\infty} g_t(x/V_{nN_n}) x^2 dF_{nk}(x+a_{nk})$$
$$= \int_{-\infty}^{+\infty} g_t(x) dH(x) + B(m) +$$
$$+ \sum_{j=1}^{m} g_t(x_{j-1}) \Big\{ \frac{1}{V_{nN_n}^2} \sum_{k=1}^{N_n} \int_{x_{j-1} < x/V_{nN_n} \le x_j} x^2 dF_{nk}(x+a_{nk}) -$$
$$- [H(x_j) - H(x_{j-1})] \Big\} + D_n(m) + C_n(m).$$

Hence, by the estimates given above, we see that (d) holds if and only if $(\mathrm{RL_H})$ is satisfied. The proof is complete.

The following modification of Proposition 1 is sometimes useful in applications.

PROPOSITION. 2. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that there exists a sequence* $\{s_n\}$ *of positive real numbers such that*

$$b_{nN_n}/s_n^2 \xrightarrow{P} 0, \quad n \to \infty, \tag{3}$$

$$V_{nN_n}^2/s_n^2 \xrightarrow{P} D, \quad n \to \infty, \tag{4}$$

where D *is a strictly positive real number.*

Then

$$\mathrm{E}\exp\{it(S_{nN_n} - L_{nN_n})/s_n\} \to f(t\sqrt{D}), \quad n \to \infty, \tag{5}$$

where

$$f(t\sqrt{D}) = \exp\{D \int_{-\infty}^{+\infty} g_t(x) dH(x/\sqrt{D})\}, \tag{6}$$

if and only if

$$\frac{1}{s_n^2} \sum_{k=1}^{N_n} \int_{x \le ys_n} x^2 dF_{nk}(x+a_{nk}) \xrightarrow{P} DH(y/\sqrt{D}), \quad n \to \infty, \tag{7}$$

for every continuity point y *of* $H(\cdot/\sqrt{D})$.

Moreover, if (4) *is satisfied with* $D = 0$, *then*

$$(S_{nN_n} - L_{nN_n})/s_n \xrightarrow{P} 0, \quad n \to \infty. \tag{8}$$

Proof. First we note that (3) is equivalent to (RF) and (5) is equivalent to $\mathrm{E}\exp\{it(S_{nN_n} - L_{nN_n})/V_{nN_n}\} \to f(t)$, $n \to \infty$, as (4) is satisfied with $D > 0$. Thus, the first part of Proposition 2 will be proved if we show that $(\mathrm{RL_H})$ is equivalent to (7).

Assume that (RL_H) is satisfied. Then

$$\text{(a)}\quad \frac{1}{s_n^2}\sum_{k=1}^{N_n}\int_{x\leqslant ys_n} x^2\,dF_{nk}(x+a_{nk})$$
$$=\frac{D}{V_{nN_n}^2}\sum_{k=1}^{N_n}\int_{x\leqslant yU_nV_{nN_n}} x^2\,dF_{nk}(x+a_{nk})+Z_n\frac{1}{V_{nN_n}^2}\sum_{k=1}^{N_n}\int_{x\leqslant ys_n} x^2\,dF_{nk}(x+a_{nk}),$$

where

$$U_n = 1/\sqrt{V_{nN_n}^2/s_n^2}\xrightarrow{P} 1/\sqrt{D}\quad \text{and}\quad Z_n = V_{nN_n}^2/s_n^2-D\xrightarrow{P} 0$$

for $n\to\infty$.

Hence the first term of (a) converges in probability to $DH(y/\sqrt{D})$, and the second one is less than $|Z_n|$, where $|Z_n|\xrightarrow{P} 0$, $n\to\infty$, which gives (7).

On the other hand, if (7) holds, then in a similar way we can prove that (RL_H) is satisfied.

Thus, the proof of Proposition 2 will be completed if we show that (8) follows from $V_{nN_n}^2/s_n^2\xrightarrow{P} 0$, $n\to\infty$. But this is so, as

$$P\{|S_{nN_n}-L_{nN_n}|\geqslant \varepsilon s_n] = \sum_{k=1}^{\infty} p_{nk}P[|S_{nk}-L_{nk}|\geqslant \varepsilon s_n]$$
$$\leqslant (1/\varepsilon^2)\sum_{k=1}^{\infty} p_{nk}V_{nk}^2/s_n^2 = (1/\varepsilon^2)\,\mathrm{E}\{V_{nN_n}^2/s_n^2\},$$

$V_{nN_n}^2/s_n^2\xrightarrow{P} 0$, $n\to\infty$, and we may assume that

$$\text{(9)}\qquad V_{nN_n}^2/s_n^2\leqslant 1\quad \text{a.s. for all } n.$$

Of course, condition (9) implies no loss of generality, because if (Y_{nk}) does not satisfy (9), then we can set $Z_{nk} = Y_{nk}I(V_{nk}^2/s_n^2\leqslant 1)$, $k = 1, 2, \ldots$; $n = 1, 2, \ldots$, and the DIA(Z_{nk}) will satisfy (9). Furthermore,

$$P[\bigcup_{k=1}^{N_n}\{Y_{nk}\neq Z_{nk}\}]\leqslant P[V_{nN_n}^2/s_n^2>1]\to 0,\qquad n\to\infty,$$

as $V_{nN_n}^2/s_n^2\xrightarrow{P} 0$, $n\to\infty$, i.e.

$$S_{nN_n}^Y/s_n = (S_{nN_n}-L_{nN_n})/s_n\quad \text{and}\quad S_{nN_n}^z/s_n = \sum_{k=1}^{N_n} Z_{nk}/s_n$$

have the same limit law. This completes the proof.

From Proposition 2 we can deduce the following generalization of Theorem 2 in [15].

THEOREM 1. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that* (RF) *is* *satisfied, and*

(10) $$(V_{nN_n}^2-\varrho_n)/\sigma_n^2 \xrightarrow{\mathrm{P}} 0, \quad n\to\infty,$$

(11) $$d_n = \Delta_n/\sigma_n \to d, \quad n\to\infty,$$

where $\varrho_n = \mathrm{E}V_{nN_n}^2$, $\Delta_n^2 = D^2 L_{nN_n}$, $\varrho_n+\Delta_n^2 = \sigma_n^2$, *and* $0 \leqslant d < 1$.

Then

(12) $$\mathrm{E}\exp\{it(S_{nN_n}-L_{nN_n})/\sigma_n\} \to f(t\sqrt{1-d^2}), \quad n\to\infty,$$

where

(13) $$f(t\sqrt{1-d^2}) = \exp\{(1-d^2)\int_{-\infty}^{+\infty} g_t(x)\,dH(x/\sqrt{1-d^2})\},$$

if and only if $(\mathrm{RL_H})$ *holds.*

Moreover, if (11) *is satisfied with* $d = 1$, *then* (8) *holds.*

Proof. It is easy to see that $V_{nN_n}^2/\sigma_n^2 \xrightarrow{\mathrm{P}} 1-d^2$, $n\to\infty$, as (10) and (11) hold, i.e. (4) is satisfied with $s_n^2 = \sigma_n^2$ for all n and either $D = 0$ if $d = 1$ or $D > 0$ if $0 \leqslant d < 1$. Thus, the proof of Theorem 1 immediately follows from the proof of Proposition 2.

We now give conditions, expressed in terms of statistics such that (5) holds.

PROPOSITION 3. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that there exists a sequence* $\{s_n\}$ *of positive real numbers such that* (3) *holds, and such that*

(14) $$W_{nN_n}^2/s_n^2 \xrightarrow{L_1} D \text{ and } \sigma^2 V_{nN_n}^2/s_n^4 \to 0, \quad n\to\infty,$$

where

$$W_{nN_n}^2 = \sum_{k=1}^{N_n} Y_{nk}^2, \quad Y_{nk} = X_{nk}-a_{nk}, \quad k = 1, 2, \ldots;\ n = 1, 2, \ldots,$$

and D is a strictly positive real number.

Then (5) *holds if*

(15) $$\frac{1}{s_n^2}\sum_{k=1}^{N_n} Y_{nk}^2 I(Y_{nk} \leqslant ys_n) \xrightarrow{\mathrm{P}} DH(y/\sqrt{D}),\ n\to\infty,$$

for every continuity point y of $H(\cdot/\sqrt{D})$.

Moreover, if (14) *holds with* $D = 0$, *then* (8) *is satisfied.*

Proof. We shall verify the assumptions of Propositions 2. Suppose that (X_{nk}) is a DIA satisfying assumptions (3) and (4) of Proposition 2. Then we may assume that

(a) $$V_{nN_n}^2/s_n^2 \leqslant D+1 \quad \text{a.s. for all } n.$$

Condition (a) implies no loss of generality, because if (X_{nk}) does not satisfy (a), then we can set $Z_{nk} = X_{nk} I(V_{nk}^2/s_n^2 \leqslant D+1)$, $k = 1, 2, \ldots$; $n = 1, 2, \ldots$, and the DIA (Z_{nk}) will satisfy (3), (4) and (a). Hence, under (a), condition (7) is equivalent to

$$\text{(b)} \qquad \frac{1}{s_n^2} \sum_{k=1}^{N_n} \int_{x \leqslant ys_n} x^2 \, dF_{nk}(x+a_{nk}) \xrightarrow{L_1} DH(y/\sqrt{D}), \qquad n \to \infty,$$

in a sense that if (X_{nk}) satisfies (7) but does not (b), then there exists an equivalent DIA (e.g., (Z_{nk})) for which (b) holds.

Now, since $\mathrm{E} V_{nN_n}^2/s_n^2 = \mathrm{E} W_{nN_n}^2/s_n^2$ for all n, then (4) is satisfied as (14) holds, and for the proof of Proposition 3 we only need to show that (15) implies (7). Since

$$\text{(c)} \qquad \mathrm{E}\left\{\frac{1}{s_n^2} \sum_{k=1}^{N_n} Y_{nk}^2 I(Y_{nk} \leqslant ys_n)\right\} = \mathrm{E}\left\{\frac{1}{s_n^2} \sum_{k=1}^{N_n} \int_{x \leqslant ys_n} x^2 \, dF_{nk}(x+a_{nk})\right\}$$

for all n, then the proof will be completed if we show that (15) implies (b). But

$$\frac{1}{s_n^2} \sum_{k=1}^{N_n} Y_{nk}^2 I(Y_{nk} \leqslant ys_n) \leqslant W_{nN_n}^2/s_n^2 \qquad \text{a.s. for all } n,$$

which by (14) is uniformly integrable. This fact combined with (15) and (c) implies the result.

Remark 2. It is known that assumption (14) is equivalent to

$$\text{(16)} \qquad W_{nN_n}^2/s_n^2 \xrightarrow{\mathrm{P}} D \text{ and } \sigma^2 V_{nN_n}^2/s_n^4 \to 0, \qquad n \to \infty,$$

and

$$\text{(16')} \qquad \{W_{nN_n}^2/s_n^2\} \quad \text{is uniformly integrable.}$$

Moreover, it is easy to see that (14) can be replaced by (16) and

$$\text{(17)} \qquad \{\{\max_{1 \leqslant k \leqslant N_n} Y_{nk}^2\}/s_n^2\} \quad \text{is uniformly integrable,}$$

as for a DIA (X_{nk}) one can find an equivalent DIA (e.g.,

$$Z_{nk} = (X_{nk} - a_{nk}) I(W_{n,k-1}^2/s_n^2 \leqslant D+1),$$

$k = 1, 2, \ldots; n = 1, 2, \ldots$) satisfying (14).

Now we give simple consequences of Propositions 1, 2 and 3, which extend or strengthen results given in [5], [14], and [15].

THEOREM 2 (cf. [15], Theorem 1). *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Then*

$$(S_{nN_n} - L_{nN_n})/V_{nN_n} \xrightarrow{\mathrm{D}} \mathcal{N}_{0,1}, \qquad n \to \infty,$$

and (RF) *hold if and only if* (RL_0) *is satisfied.*

Proof. The "only if" part was proved in Proposition 1. Knowing that (RL_0) implies (RF) we see, by Proposition 1, that the central limit theorem holds if (RL_0) is satisfied.

THEOREM 3 (cf. [14], Theorem 2). *Let (X_{nk}) and $\{N_n\}$ be as in Proposition* 1. *Suppose that there exists a sequence $\{s_n\}$ of positive real numbers such that* (4) *is satisfied with $D > 0$. Then*

$$(18) \qquad (S_{nN_n} - L_{nN_n})/s_n \xrightarrow{\mathrm{D}} \mathcal{N}_{0,D}, \quad n \to \infty,$$

and (3) *hold if and only if*

$$(\mathrm{a}) \qquad \frac{1}{s_n^2} \sum_{k=1}^{N_n} \int_{|x| \geqslant \varepsilon s_n} x^2 \, dF_{nk}(x + a_{nk}) \xrightarrow{\mathrm{P}} 0, \quad n \to \infty,$$

for every $\varepsilon > 0$.

THEOREM 4 (cf. [15], Theorem 2). *Let (X_{nk}) and $\{N_n\}$ be as in Proposition* 1. *Suppose that* (10) *and* (11) *are satisfied with $0 \leqslant d < 1$. Then*

$$(S_{nN_n} - L_{nN_n})/\sigma_n \xrightarrow{\mathrm{D}} \mathcal{N}_{0,1-d^2}, \quad n \to \infty,$$

and (RF) *hold if and only if* (RL_0) *is satisfied.*

Proof. The result immediately follows from Theorem 1.

THEOREM 5 (cf. [5], Theorem 2.3). *Let (X_{nk}) and $\{N_n\}$ be as in Proposition* 1. *Suppose that there exists a sequence $\{s_n\}$ of positive real numbers such that* (16) *and* (17) *are satisfied with $D > 0$. Then* (18) *and* (3) *hold if and only if*

$$(19) \qquad \{\max_{1 \leqslant k \leqslant N_n} |Y_{nk}|\}/s_n \xrightarrow{\mathrm{P}} 0, \quad n \to \infty.$$

Proof. It is enough to note that, under the assumptions of Theorem 5, the condition (15) states that for every $\varepsilon > 0$

$$P\left[\frac{1}{s_n^2} \sum_{k=1}^{N_n} Y_{nk}^2 I(|Y_{nk}| \geqslant \varepsilon s_n) \geqslant \varepsilon^2\right] \to 0, \quad n \to \infty,$$

which is equivalent to (19) as for every $\varepsilon > 0$

$$P\left[\sum_{k=1}^{N_n} Y_{nk}^2 I(|Y_{nk}| \geqslant \varepsilon s_n) \geqslant \varepsilon^2 s_n^2\right] = P\left[\max_{1 \leqslant k \leqslant N_n} |Y_{nk}| \geqslant \varepsilon s_n\right],$$

and, moreover, by (17) and (19),

$$b_{nN_n}/s_n^2 \leqslant \mathrm{E}\left\{\{\max_{1 \leqslant k \leqslant N_n} Y_{nk}^2\}/s_n^2\right\} \to 0, \quad n \to \infty.$$

Remark 3. It is known (cf. [5], Theorem 2.3) that in the case $N_n = k_n$ a.s. for all n, the "if" part of Theorem 5 can be proved under weaker conditions, namely, it is enough to assume that (16) and (19) hold, and

$$\{\{\max_{1\leqslant k\leqslant N_n} |Y_{nk}|\}/s_n\}$$

is uniformly bounded in L_2 norm.

THEOREM 6. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that* (RF) *is satisfied. Then*

$$(S_{nN_n}-L_{nN_n})/V_{nN_n} \xrightarrow{D} \mathscr{P}_c, \quad n\to\infty,$$

where $\mathscr{P}_c$ *is a Poisson type r.v. with ch.f.* $f(t)$ *given by* (2), *if and only if* $(\mathrm{RL_C})$ *holds.*

THEOREM 7. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that there exists a sequence* $\{s_n\}$ *of positive real numbers such that* (3) *and* (4) *are satisfied with* $D>0$. *Then*

$$(S_{nN_n}-L_{nN_n})/s_n \xrightarrow{D} \mathscr{P}_{c\sqrt{D}}, \quad n\to\infty, \tag{20}$$

where $\mathscr{P}_{c\sqrt{D}}$ *is a Poisson type r.v. with ch.f.* $f(t\sqrt{D})$ *given by*

$$f(t\sqrt{D}) = \exp\left\{\frac{1}{c^2}[\exp\{itc\sqrt{D}\}-1-itc\sqrt{D}]\right\}, \tag{21}$$

if and only if for every $\varepsilon>0$,

$$\frac{1}{s_n^2}\sum_{k=1}^{N_n}\int\limits_{|\frac{x}{s_n}-c\sqrt{D}|\geqslant\varepsilon} x^2\,dF_{nk}(x+a_{nk}) \xrightarrow{P} 0, \quad n\to\infty. \tag{22}$$

Proof. Putting $H(x)=I(x\geqslant c)$, the result immediately follows from Proposition 2.

THEOREM 8. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 1. *Suppose that there exists a sequence* $\{s_n\}$ *of positive real numbers such that* (3), (16) *and* (17) *are satisfied with* $D>0$. *Then* (20) *holds if and only if*

$$\frac{1}{s_n^2}\sum_{k=1}^{N_n} Y_{nk}^2 I\left(\left|\frac{Y_{nk}}{s_n}-c\sqrt{D}\right|\geqslant\varepsilon\right)\xrightarrow{P} 0, \quad n\to\infty, \tag{23}$$

for every $\varepsilon>0$.

3. Random limit theorems of Robbins type. In this section we give necessary and sufficient conditions for the weak convergence of $(S_{nN_n}-A_n)/\sigma_n$. The results obtained are generalizations or extensions of those in [9], [13] and [15].

Putting

$$h_n(t) = \mathrm{E}\exp\{it(L_{nN_n}-A_n)/\Delta_n\} = \sum_{k=1}^{\infty} p_{nk}\exp\{it(L_{nk}-A_n)/\Delta_n\},$$

$$f_n(t) = \mathrm{E}\exp\{it(S_{nN_n}-A_n)/\sigma_n\} = \sum_{k=1}^{\infty} p_{nk}\exp\{it(L_{nk}-A_n)/\sigma_n\} f_{nk}(t/\sigma_n),$$

$$d_n = \Delta_n/\sigma_n,\ 0 \leqslant d_n \leqslant 1, \quad d = \lim_{n\to\infty} d_n,\ 0 \leqslant d \leqslant 1, \quad h(t) = \lim_{n\to\infty} h_n(t),$$

we have the following generalization of Theorem 3 of [15]:

PROPOSITION 4. *Let* (X_{nk}) *be a DIA of r.v's such that for all* n, *the r.v's* X_{nk}, $k = 1, 2, \ldots$, *are independent, and* $\{N_n\}$ *be a sequence of positive integer-valued r.v's such that* N_n *is for each* n *independent of* X_{nk}, $k = 1, 2, \ldots$ *Suppose that* (RF), (10) *and* (11) *are satisfied with* $0 \leqslant d < 1$. *Then*

$$f_n(t) \to h(td) f(t\sqrt{1-d^2}), \quad n\to\infty, \tag{24}$$

where $f(t\sqrt{1-d^2})$ *is given by* (13), *if and only if* ($\mathrm{RL_H}$) *holds.*

Moreover, if (11) *is satisfied with* $d = 1$, *then*

$$f_n(t) \to h(t), \quad n\to\infty. \tag{25}$$

Proof. First we note that

(a) $f_n(t) = \mathrm{E}[\exp\{itd_n(L_{nN_n}-A_n)/\Delta_n\} f_{nN_n}(t/\sigma_n)]$

$$= f(t\sqrt{1-d^2})\,h_n(td_n) + \mathrm{E}\big[\exp\{itd_n(L_{nN_n}-A_n)/\Delta_n\}\,(f_{nN_n}(t/\sigma_n) - f(t\sqrt{1-d^2}))\big].$$

Then, if $d = 1$, (25) immediately holds as

$$\mathrm{E}\exp\{it(S_{nN_n}-L_{nN_n})/\sigma_n\} = \mathrm{E}f_{nN_n}(t/\sigma_n) \to 1, \quad n\to\infty,$$

i.e. $f_{nN_n}(t/\sigma_n) \xrightarrow{P} 1$, $n\to\infty$, and $f(t\sqrt{1-d^2}) = 1$ for all t.

Moreover, if $0 \leqslant d < 1$, then Theorem 1 states that ($\mathrm{RL_H}$) holds if and only if

$$f_{nN_n}(t/\sigma_n) \xrightarrow{P} f(t\sqrt{1-d^2}), \quad n\to\infty.$$

Hence, by (a), ($\mathrm{RL_H}$) is equivalent to (24), which completes the proof.

An equivalent modification of Proposition 4 expressed in terms of statistics runs as follows:

PROPOSITION 5. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (11) *is satisfied with* $0 \leqslant d < 1$, *and that*

$$b_{nN_n}/\sigma_n^2 \xrightarrow{P} 0, \quad n\to\infty, \tag{26}$$

$$(W_{nN_n}^2 - \varrho_n)/\sigma_n^2 \xrightarrow{P} 0 \text{ and } \sigma^2 V_{nN_n}^2/\sigma_n^4 \to 0, \quad n\to\infty. \tag{27}$$

Then (24) *holds if*

$$\frac{1}{\sigma_n^2}\sum_{k=1}^{N_n} Y_{nk}^2 I(Y_{nk} \leqslant y\sigma_n) \xrightarrow{P} (1-d^2) H(y/\sqrt{1-d^2}), \quad n\to\infty, \tag{28}$$

for every continuity point y *of* $H(\cdot/\sqrt{1-d^2})$.

Proof. First we note that if (11) holds with $0 \leqslant d < 1$, then (27) is equivalent to

$$(W_{nN_n}^2-\varrho_n)/\sigma_n^2 \xrightarrow{L_1} 0 \text{ and } \sigma^2 V_{nN_n}^2/\sigma_n^4 \to 0, \quad n\to\infty.$$

Hence, $W_{nN_n}^2/\sigma_n^2 \xrightarrow{L_1} 1-d^2$, $n\to\infty$. Thus, the proof of Proposition 5 may be obtained by using Proposition 3 and the considerations analogous to those in the proof of Proposition 4.

As consequences of Propositions 4 and 5 we get the following strengthenings of Theorem 3 in [15].

THEOREM 9. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (10) *and* (11) *are satisfied with* $0 \leqslant d < 1$. *Then*

$$f_n(t) \to h(td)\exp\left\{-\frac{t^2}{2}(1-d^2)\right\}, \quad n\to\infty, \tag{29}$$

and (RF) *hold if and only if* ($\mathrm{RL_0}$) *is satisfied.*

THEOREM 10. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (11) *and* (27) *are satisfied with* $0 \leqslant d < 1$. *Then* (29) *and* (26) *hold if and only if*

$$\{\max_{1\leqslant k\leqslant N_n} |Y_{nk}|\}/\sigma_n \xrightarrow{P} 0, \quad n\to\infty. \tag{30}$$

Moreover, using Proposition 4 one can easily obtain the following results:

THEOREM 11. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (RF), (10) *and* (11) *are satisfied with* $0 \leqslant d < 1$. *Then*

$$f_n(t) \to h(td) f(t\sqrt{1-d^2}), \quad n\to\infty,$$

where $f(t\sqrt{D})$, $D = 1-d^2$, *is given by* (21), *if and only if* ($\mathrm{RL_C}$) *holds.*

Remark 4. Suppose that (10) and (11) are satisfied with $0 \leqslant d < 1$. Then

$$V_{nN_n}^2 \xrightarrow{P} \infty \quad \text{if and only if } \sigma_n^2 \to \infty \quad (n\to\infty).$$

COROLLARY 1 (Robbins' theorem [9]). *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of independent and identically distributed r.v's with* $\mathrm{E}X_1 = a$, $\mathrm{D}^2 X_1 = b^2$, *and* $\{N_n\}$ *be a sequence of positive integer-valued r.v's such that* N_n *is for each* n

independent of X_k, $k \geq 1$. *Suppose that*

(a) $$N_n \xrightarrow{P} \infty \quad \text{or} \quad \sigma_n^2 \to \infty \quad (n \to \infty),$$

(b) $$(N_n - \alpha_n)/\sigma_n^2 \xrightarrow{P} 0, \quad n \to \infty,$$

where $\alpha_n = \mathrm{E} N_n$, $n = 1, 2, \ldots$

Then

$$\lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} g_n(ta\beta_n/\sigma_n) \exp\left\{-\frac{t^2}{2}\left(1 - \frac{a^2 \beta_n^2}{\sigma_n^2}\right)\right\},$$

where $\beta_n^2 = \mathrm{D}^2 N_n$, $g_n(t) = \mathrm{E}\exp\{it(N-\alpha_n)/\beta_n\}$, $n = 1, 2, \ldots$

Corollary 2. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* $\mathrm{E} X_{nk} = a_{nk} = 0$ $(k = 1, 2, \ldots;\ n = 1, 2, \ldots,)$, *and that* (RF) *and*

(a) $$V_{nN_n}^2 / \mathrm{E} V_{nN_n}^2 \xrightarrow{P} 1, \quad n \to \infty,$$

are satisfied. Then $f_n(t) \to f(t)$, $n \to \infty$, *where* $f(t)$ *is given by* (1), *if and only if* $(\mathrm{RL_H})$ *holds.*

Corollary 3. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (RF), (10) *and* $\Delta_n^2 = o(\sigma_n^2)$ *with* $n \to \infty$, *are satisfied. Then* $f_n(t) \to f(t)$, $n \to \infty$, *if and only if* $(\mathrm{RL_H})$ *holds.*

Corollary 4. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that*

$$(L_{nN_n} - A_n)/\Delta_n \xrightarrow{D} \mathcal{N}_{0,1}, \quad n \to \infty,$$

and (10) *and* (11) *are satisfied with* $0 \leq d < 1$. *Then*

$$f_n(t) \to \exp\left\{-\frac{t^2}{2}\right\}, \quad n \to \infty,$$

and (RF) *hold if and only if* $(\mathrm{RL_0})$ *is satisfied.*

Corollary 5. *Let* (X_{nk}) *and* $\{N_n\}$ *be as in Proposition* 4. *Suppose that* (RF) *and* (10) *are satisfied, and that*

(a) $$\lim_{n \to \infty} \varrho_n/\Delta_n = s, \quad 0 < s < \infty.$$

Then

$$f_n(t) \to h\left(\frac{t}{\sqrt{1+s}}\right) f\left(t\sqrt{\frac{s}{1+s}}\right), \quad n \to \infty,$$

if and only if $(\mathrm{RL_H})$ *holds.*

Remark 5. If $s = 0$ (which holds if $\varrho_n = o(\Delta_n^2)$ with $n \to \infty$), then $f_n(t) \to h(t)$, $n \to \infty$. If $s = \infty$ (which holds if $\Delta_n^2 = o(\varrho_n)$ with $n \to \infty$), then $f_n(t) \to f(t)$, $n \to \infty$ (see Corollary 3).

The results given above generalize results of [9], [13] and [15].

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