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ON LÉVY (SPECTRAL) MEASURES OF INTEGRAL FORM ON BANACH SPACES

BY

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Dedicated to my Teacher and Master Professor KAZIMIERZ URBANIK on the occasion of his sixtieth birthday

Abstract. In order to be a Lévy measure some necessary and sufficient conditions are given for measures of integral form. In particular, a complete proof for elements from classes L_α, $\alpha > 0$, is presented. Also some other examples are quoted.

It is well-known that a measure M on a Hilbert space H is a Lévy (spectral) measure iff M integrates the function $\min(1, \|x\|^2)$ over H. On the contrary, on general Banach spaces this condition is neither necessary nor sufficient. Moreover, there is no function g such that the integrability of $g(\|x\|)$ with respect to M would be a necessary and sufficient one for M to be a Lévy measure on an arbitrary Banach space ([1], Chapter III, Theorem 6.3). On the other hand, studying random integrals of the deterministic real-valued function with respect to Lévy processes or stable measures we have to show that some mixtures of (Lévy) measures are Lévy measures as well [5–8]. Some of the previous proofs have appealed to random integral arguments to conclude that a measure is Lévy. Here our justifications go throughout the series of independent Banach space valued random variables and some type of "comparison principle" (cf. Proposition 1). Proposition 2 shows that if the λ-mixture of $T_t G$, $t > 0$, is Lévy, then so is G. The opposite implication is considered in Proposition 3. Finally, in Section 4 some examples are discussed.

1. NOTATION AND SOME BASIC FACTS

Let E be a real separable Banach space with the norm $\|\cdot\|$, the topological dual E' and the bilinear form $\langle\cdot,\cdot\rangle$ between E' and E. Let $\mathrm{ID}(E)$ denote the set of all infinitely divisible measures on E. A σ-finite Borel measure M on E, such that $M(\{0\}) = 0$ and the function

$$\varphi_M(y) = \exp \int_{E\setminus\{0\}} [e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle_{B_1}(x)]\, M(dx), \qquad y \in E',$$

is a characteristic function (of a probability measure $\tilde{e}(M)$), is called *Lévy (spectral) measure* (cf. [1], p. 117–118, where $\tilde{e}(M)$ is denoted by $c_1 \operatorname{Pois} M$). The importance of Lévy measures follows from the fact that $\mu \in \mathrm{ID}$ iff $\mu = \delta_x * \gamma * \tilde{e}(M)$, where $x \in E$, γ is the symmetric Gaussian measure and M is a Lévy (spectral) measure. The triple x, γ and M is uniquely determined by μ (cf. [1], Theorem 6.2, p. 136).

In the sequel $\mathcal{M}(E)$ denotes the set of all Lévy measures on Banach space E, $B_r := \{x \in E \colon \|x\| \leqslant r\}$, $r > 0$, is a ball and $\mathcal{B}_0$ is the family of all Borel subsets of $E_0 := E \setminus \{0\}$. If H is a Hilbert space, then

$$(1.0) \qquad M \in \mathcal{M}(H) \quad \text{iff} \int_H \min(1, \|x\|^2)\, M(dx) < \infty.$$

But on the general Banach space formula (1.0) is not longer true. However, some sufficient conditions are available. Namely, we have the following (cf. [1], Chapter III, Theorems 4.7 and 6.3):

$$(1.1) \qquad M(E) < \infty \text{ implies } M \in \mathcal{M}(E) \text{ and } e(M) := e^{-M(E)} \sum_{k=0}^{\infty} M^{*k}/k! \in \mathrm{ID}(E);$$

$$(1.2) \qquad \int_E \min(1, \|x\|)\, M(dx) < \infty \text{ implies } M \in \mathcal{M}(E);$$

$$(1.3) \qquad \text{if } 0 \leqslant N \leqslant M \text{ and } M \in \mathcal{M}(E), \text{ then } N, M-N \text{ are in } \mathcal{M}(E);$$

$$(1.4) \qquad \text{if } M, N \in \mathcal{M}(E), \text{ then } M+N \in \mathcal{M}(E).$$

On the other hand, the following properties are necessary:

(1.5) $M \in \mathcal{M}(E)$ implies $M(B_r^c) < \infty$ for each $r > 0$, i.e. M is finite outside every neighbourhood of zero;

(1.6) $M \in \mathcal{M}(E)$ implies $\int_E \min(1, \langle y, x\rangle^2)\, M(dx) < \infty$ for each $y \in E'$, because $\Pi_y M \in \mathcal{M}(R)$, where $\Pi_y \colon E \to R$ is given by $\Pi_y x := \langle y, x\rangle$.

For further references let explicitly state that (cf. (1.3) and (1.4))

(1.7) $M \in \mathcal{M}(E)$ iff $M^0 := M + M^- \in \mathcal{M}(E)$, where $M^-(A) := M(-A)$ for $A \in \mathcal{B}_0$.

Thus we can consider symmetric measures M only. The measure M^0 in (1.7) is called the *symmetrization* of a measure M. We complete this introductory section with a lemma which will be repeatedly applied later on.

Lemma 1. *If M is a symmetric Lévy measure supported by the unit ball B_1, ξ_n's are E-valued independent rv's with distributions*

$$e(M|_{\{x:(n+1)^{-1} < \|x\| \leqslant n^{-1}\}}) \quad \textit{for } n = 1, 2, \ldots,$$

and ξ has the distribution $\tilde{e}(M)$, then $\sum_n \xi_n$ converges to ξ in L_p-norm for each $p > 0$.

Proof. By Lemma 4.4 and Theorem 2.10, Chapter III, in [1], we infer that $\sum_n \xi_n$ converges a.s. to ξ. From [1], Corollary 3.3, we infer that ξ has all exponential moments, in particular all p-moments. From the Lévy inequality and Theorem 2.11 in [1] we conclude the proof of Lemma 1.

2. FUNDAMENTAL INEQUALITIES

Let λ be a measure on $R^+ = (0, \infty)$ and m a Borel measure on E. Then the measure $m^{(\lambda)}$, defined on Borel sets A by

$$m^{(\lambda)}(A) = \int_E \int_{R^+} 1_A(tx)\,\lambda(dt)\,m(dx) = \int_{R^+} m(t^{-1}A)\,\lambda(dt), \tag{2.1}$$

is the λ-mixture of measures $(T_t m)(\cdot) := m(t^{-1}\cdot)$, $t \in R^+$. Measures of the form (2.1) appeared in the study of stable measures (cf. [6], p. 272–273, or [1], p. 165), random integrals (cf. [7], p. 250, or [4], Theorems 1.3 and 3.2) and in fractional calculus in probability theory (cf. [8]). In all of these circumstances one has to determine whether $m^{(\lambda)}$ is a Lévy measure for a particular given measure λ. Here (in Section 3) we will discuss this question as well as the opposite one in general.

Proposition 1. *For $1 \leqslant j \leqslant k$ let λ_j be finite measures on R^+ with mean values v_j and let m_j be finite Borel measures on E with zero mean values. Then*

$$\int_E \|x\|\, T_{a_1} m_1 * \ldots * T_{a_k} m_k(dx) \leqslant \int_E \|x\|\, m_1^{(\lambda_1)} * \ldots * m_k^{(\lambda_k)}(dx) \leqslant c_k \int_E \|x\|\, m_1 * \ldots * m_k(dx),$$

where $a_i = v_i \prod_{j \neq i} \lambda_j(R^+)$ for $1 \leqslant i \leqslant k$ and

$$c_k = 2 \int_{R^+} \ldots \int_{R^+} \max(t_1, \ldots, t_k)\,\lambda_1(dt_1)\ldots\lambda_k(dt_k).$$

Proof. Let A_k be the middle term in the above inequality. By Lemma 2.12, p. 108, from [1], we get

$$A_k = \int_{R^+} \ldots \int_{R^+} \int_E \ldots \int_E \|t_1 x_1 + \ldots + t_k x_k\|\, m_1(dx_1)\ldots m_k(dx_k)\,\lambda_1(dt_1)\ldots\lambda_k(dt_k)$$
$$\leqslant c_k \int_E \|x\|\, m_1 * \ldots * m_k(dx),$$

which gives the right-hand side inequality. Since the norm of an integral is not greater than the integral of the norm of a function, we have

$$A_k \geqslant \int_E \ldots \int_E \Big\| \int_{R^+} \ldots \int_{R^+} (t_1 x_1 + \ldots + t_k x_k)\,\lambda_1(dt_1)\ldots\lambda_k(dt_k) \Big\|\, m_1(dx_1)\ldots m_k(dx_k)$$
$$= \int_E \ldots \int_E \|a_1 x_1 + \ldots + a_k x_k\|\, m_1(dx_1)\ldots m_k(dx_k),$$

which is the left-hand side inequality in Proposition 1.

We conclude this subsection with some simple properties of λ-mixtures $m^{(\lambda)}$, which will be needed later on. Namely we have:

(2.2) $(m^{(\lambda)})^0 = (m^0)^{(\lambda)}$, where m^0 is the symmetrization of m;

(2.3) if $m_1 \leqslant m_2$ or $\lambda_1 \leqslant \lambda_2$, then $m_1^{(\lambda_1)} \leqslant m_2^{(\lambda_2)}$;

(2.4) $(\sum_j m_j)^{(\lambda)} = \sum_j m_j^{(\lambda)}$ and $m^{(\sum_j \lambda_j)} = \sum_j m^{(\lambda_j)}$;

(2.5) $m^{(\delta_a)} = T_a m$ and $m^{(\sum_j \alpha_j \delta_{a_j})} = \sum_j \alpha_j T_{a_j} m$;

(2.6) $am^{(\lambda)} = (am)^{(\lambda)} = m^{(a\lambda)}$ for $a \in R^+$.

In formulas (2.2)–(2.6), m and m_j's are measures on E, and λ and λ_j's are measures on R^+.

3. MIXTURES OF MEASURES

As before, E denotes a Banach space, $\mathscr{B}_0$ is a σ-algebra of Borel subset of $E_0 := E \setminus \{0\}$ and $\mathscr{M}(E)$ is the family of all Lévy measures on E.

PROPOSITION 2. *Let g be a measure on R^+ and G be a measure on $\mathscr{B}_0$, both non-zero, such that $G^{(g)} \in \mathscr{M}(E)$, i.e., $G^{(g)}$ is a Lévy measure. Then*

(a) $$\int_{R^+} G(B^c_{t^{-1}})\, g(dt) = \int_{E^0} g(t : t > \|x\|^{-1})\, G(dx) < \infty,$$

(b) *g and G are Lévy measures on R^+ and E, respectively.*

Proof. From (1.5) and (2.1) we get

$$G^{(g)}(B^c_1) = \int_{R^+} G(B^c_{t^{-1}})\, g(dt) = \int_{E_0} g(t : t > \|x\|^{-1})\, G(dx) < \infty,$$

which gives (a). Moreover, G and g are finite outside some neighbourhoods of zero in E and R^+. In fact, they are finite outside every neighbourhood of zero. Note that

$$\int_{a^{-1}}^{\infty} G(B^c_{t^{-1}})\, g(dt) < \infty \quad \text{for each } a > 0.$$

Thus there is a $t_0 \in (a^{-1}, \infty)$ such that $G(B_{t_0^{-1}}) < \infty$, i.e., $G(B^c_a) < \infty$ for $a > 0$. Similarly, we show this for g. Furthermore, $\Pi_y G^{(g)} \in \mathscr{M}(R)$ for all $y \in E'$ (cf. (1.6)), and (1.0) gives

$$\int_R \min(1, s^2)\, \Pi_y G^{(g)}(ds) = \int_E \int_{R^+} \min(1, t^2 \langle y, x\rangle^2)\, g(dt)\, G(dx) < \infty.$$

Hence there are $x_0 \in E$ and $y_0 \in E'$ such that $w = \langle y_0, x_0\rangle \neq 0$ and g integrates $\min(1, t^2 w^2)$ over R^+, i.e., $g \in \mathscr{M}(R^+)$.

To complete the proof of part (b) we can assume that G is symmetric and concentrated on the unit ball B_1 (cf. (1.7) and (1.1)). Also, without loss of the generality, we can assume that g is concentrated on a bounded set A in R^+ and that $g(A) = 1$, because $G^{(g)} \geqslant G^{(g|_A)} \in \mathscr{M}(E)$ and $aG^{(g|_A)} = G^{(aG|_A)}$ (cf. (2.3), (2.6) and (1.3)). Consequently, $G^{(g_1)}$ is a symmetric Lévy measure concentrated on the ball B_r, where $r := \sup A < \infty$ and $g_1 := g|_A$. Taking

$$I_n := B_{n^{-1}} \setminus B_{(n+1)^{-1}}, \quad G_n := G|_{I_n}$$

and independent E-valued rv's ξ_n with distribution $e(G_n^{(g_1)})$, we see that $\sum_n \xi_n$ converges in L_1-norm (cf. Lemma 1). Applying Proposition 1 for $\lambda_1 = \ldots = \lambda_k = g_1$ and $m_1 = m_2 = \ldots = m_k = \sum_{n=j}^{l} G_n$, we obtain

$$a \int_E \|x\| \big(\sum_{n=j}^{l} G_n\big)^{*k}(dx) \leqslant \int_E \|x\| \big(\sum_{n=j}^{l} G_n^{(g_1)}\big)^{*k}(dx) \quad \text{for } k \in N,$$

where a is the mean-value of g_1. Since $G_n^{(g_1)}(E) = G_n(E)$, we conclude that

$$E \big\| \sum_{n=j}^{l} \eta_n \big\| \leqslant a^{-1} E \big\| \sum_{n=j}^{l} \xi_n \big\| \quad \text{for all } j, l \in N,$$

where η_n are E-valued independent rv's such that $L(\eta_n) = e(G_n)$. Thus $\sum_n \eta_n$ converges in L_1-norm and $G = \sum_n G_n \in \mathscr{M}(E)$, which completes the proof of Proposition 2.

Now we will determine when $G^{(g)}$ is a Lévy measure if so is G. However, we should be aware that in full generality the answer may depend on the geometry (the norm) of the Banach space E. Let us consider the following example: take $g(dt) = t^{-(p+1)} dt$ on R^+ and finite measures m on the unit sphere S in E, i.e., for $A \in \mathscr{B}_0$,

$$m^{(g)}(A) = \int_S \int_0^\infty 1_A(tu) t^{-(p+1)} dt\, m(du).$$

It is known that $m^{(g)}$ is a Lévy measure (an exponent of p-stable distribution) for all finite m's on S if and only if E is of stable type p (cf. [1], Theorem 7.9, p. 165, or, for a partial answer, [6], Theorem 2). In view of this example, sufficient conditions for $G^{(g)}$ to belong to $\mathscr{M}(E)$ will be given for some measures g only.

PROPOSITION 3. (1) *If g is finite and concentrated on* $(0, T]$, *then $G^{(g)}$ is a Lévy measure if so is G.*

(2) *Let g be concentrated on* $(0, T]$ *such that, for some sequence $a_n \downarrow 0$ in R^+, we have*

$$a_1 = T, \quad c := \sum_n a_n < \infty \quad \textit{and} \quad b := \sup_n g(a_{n+1}, a_n] < \infty.$$

Then, for $G \in \mathscr{M}(E)$ concentrated on B_1, we have $G^{(g)} \in \mathscr{M}(E)$.

(3) *Let G be a finite measure concentrated on B_1^c and g be a measure on R^+. If*

(i) $$\int_{B_1^c} g(s: s > \|x\|^{-1})\, G(dx) < \infty$$

and

(ii) $$\int_{B_1^c} \|x\| \int_0^{\|x\|^{-1}} t g(dt)\, G(dx) < \infty,$$

then $G^{(g)}$ is a Lévy measure on E.

Proof. In view of (2.2) and (1.7) we assume that all G's are symmetric measures. We will prove each of these cases separately.

Case (1). By (2.6) we may assume additionally that g is a probability measure. Since $G|_{B_1^c}$ is finite (see (1.5)), and, for finite G, the measure $G^{(g)}$ is also finite, we restrict our consideration to $G \in \mathscr{M}(E)$ and concentrated on B_1.

Let $I_n := B_{1/n} \setminus B_{1/(n+1)}$, $G_n := G|_{I_n}$ and let η_n be independent E-valued rv's with distributions $e(G_n)$ for $n = 1, 2, \ldots$ From Lemma 1, $\sum_n \eta_n$ converges in L_1-norm. Let ξ_n, $n \in N$, be independent, E-valued rv's with distributions $e(G_n^{(g)})$. Applying Proposition 1 to $\lambda_1 = \ldots = \lambda_k = g$ and $m_1 = m_2 = \ldots = m_k = \sum_{n=j}^{l} G_n$, we obtain

$$\int_E \|x\| \big(\sum_{n=j}^{l} G_n^{(g)}\big)^{*k}(dx) \leqslant 2T \int_E \|x\| \big(\sum_{n=j}^{l} G_n\big)^{*k}(dx).$$

Since $G_n^{(g)}(E) = G_n(E)$, hence summing over k we get

$$\Big\|\sum_{n=j}^{l} \xi_n\Big\| \leqslant 2T \Big\|\sum_{n=j}^{l} \eta_n\Big\| \quad \text{for all } j, l \in N,$$

i.e., $\sum_n \xi_n$ converges in L_1-norm to an infinitely divisible rv with Lévy measure $\sum_n G_n^{(g)} = G^{(g)}$, which completes the proof of case (1).

Remark 1. An alternative proof for the case (1) is also possible by a random integral approach. Note that the random integral $\int_{(0,T]} t\, dY(\tilde{g}(t))$ exists for $D_E[0, T]$-valued rv Y with stationary independent increments, $Y(0) = 0$ a.s. and $\tilde{g}(t) := g(s: s \leqslant t)$. Its Lévy measure equals $G^{(g)}$ (see [4, 5]).

Case (2). Let $L_n := (a_{n+1}, a_n]$ and $g_n := g|_{L_n}$, $n \in N$. Assuming additionally that G is a finite measure we get

$$\int_E \|x\|\, e(G^{(g_n)})(dx) < e^{-G(E)g(L_n)} \sum_{k=1}^{\infty} \frac{c_k}{k!} \int_E \|x\|\, G^{*k}(dx)$$

from Proposition 1 and formula (1.1). Since

$$c_k = 2g^k(L_n)\int_0^{a_n}\left[1-\left(g(s: s \leqslant t)/g(L_n)\right)^k\right]dt \leqslant 2a_n g^k(L_n),$$

we obtain (cf. [1], Lemma 2.7, p. 103)

$$\int_E \|x\|\, e(G^{(g_n)})(dx) \leqslant 2a_n \int_E \|x\|\, e\left(g(L_n)G\right)(dx) \leqslant 2a_n \int_E \|x\|\, e(bG)(dx).$$

Taking η_n to be E-valued independent rv's with distribution $e(G^{(g_n)})$, we obtain

$$E\left\|\sum_{n=j}^{l} \eta_n\right\| \leqslant 2\sum_{n=j}^{l} a_n \int_E \|x\|\, e(bG)(dx).$$

Hence $\sum_n \eta_n$ converges in L_1-norm, $\sum_n G^{(g_n)} = G^{(g)} \in \mathscr{M}(E)$, and

$$\int_E \|x\|\, e(G^{(g)})(dx) \leqslant 2c\int_E \|x\|\, e(bG)(dx),$$

for G finite and concentrated on B_1.

If $G \in \mathscr{M}(E)$ and is concentrated on B_1, then, taking $G_n := G|_{I_n}$ ($I_n := B_{n^{-1}} \setminus B_{(n+1)^{-1}}$), we have $G_n^{(g)} \in \mathscr{M}(E)$ and the above inequality holds for $G_n^{(g)}$. Hence and from Lemma 1 we conclude that $\sum_n G_n^{(g)} = G^{(g)} \in \mathscr{M}(E)$ and

$$\int_E \|x\|\, \tilde{e}(G^{(g)})(dx) \leqslant 2c\int_E \|x\|\, \tilde{e}(bG)(dx),$$

which completes the proof of case (2).

Case (3). Note that, for $A \in \mathscr{B}_0$,

$$G^{(g)}(A) = \int_{B_1^c}\int_{(\|x\|^{-1},\infty)} 1_A(tx)\, g(dt)\, G(dx) + \int_{B_1^c}\int_{(0,\|x\|^{-1}]} 1_A(tx)\, g(dt)\, G(dx)$$

is a sum of two measures, ν_1 and ν_2, say. Because of assumption (i), the measure ν_1 is finite and $\nu_1 \in \mathscr{M}(E)$. The measure ν_2 is concentrated on B_1 and, by (ii),

$$\int_E \|x\|\, \nu_2(dx) = \int_{B_1^c} \|x\| \int_0^{\|x\|^{-1}} t g(dt)\, G(dx) < \infty.$$

Consequently, by (1.2) and (1.4), $G^{(g)} \in \mathscr{M}(E)$, which completes the proof of case (3) and Proposition 3.

4. EXAMPLES

A. For $\alpha > 0$, let $\nu_\alpha(dt) = (\log t^{-1})^{\alpha-1} t^{-1}\, dt$ be a measure on (0, 1]. Taking $a_n := \exp(-n^{1/\alpha})$, $n = 0, 1, 2, \ldots$, we get

$$\sum_n a_n < \infty \quad \text{and} \quad \nu_\alpha(a_{n+1}, a_n] = \alpha^{-1} < \infty.$$

Condition (i) in case (3) of Proposition 3 or (a) in Proposition 2 for ν_α means the following:

$$\int_{B_1^c} \int_{\|x\|^{-1}}^{1} (\log t^{-1})^{\alpha-1} t^{-1} dt G(dx) = \alpha^{-1} \int_{B_1^c} \log^\alpha \|x\| \, G(dx) < \infty.$$

With the above restriction on G, condition (ii) in case (3) of Proposition 3 is fulfilled because

$$\lim_{\|x\|\to\infty} \|x\|/\log^\alpha \|x\| \int_0^{\|x\|^{-1}} (\log t^{-1})^{\alpha-1} dt = \lim_{u\to\infty} u^{-\alpha} e^u \int_u^\infty e^{-t} t^{\alpha-1} dt = 0.$$

This and Propositions 2 and 3 give

Corollary 1. *Let* $\alpha > 0$ *and*

$$G_\alpha(A) = \int_E \int_0^1 1_A(tx)(\log t^{-1})^{\alpha-1} t^{-1} dt G(dx)$$
$$= \int_E \int_0^\infty 1_A(e^{-t}x) t^{\alpha-1} dt G(dx) \quad \textit{for } A \in \mathscr{B}_0.$$

Then G_α *is a Lévy measure on* E *iff so is* G *and*

$$\int_{B_1^c} \log^\alpha \|x\| \, G(dx) < \infty.$$

Remark 2. Thu ([8], Theorem 4.3) claims the result as above. The proof is a combination of random integral arguments from [7] and property (1.3) of $\mathscr{M}(E)$. However, inequality (4.12) in [8] needs a correction and applying Corollary 4.2 (in Cases 1 and 2) one requires that $G \in G_{[\alpha]+1}(X)$, not only $G \in G_\alpha(X)$.

Remark 3. Taking $a(t) = \exp(-t^{1/\alpha})$ in Theorem 4 of Hong [3], one gets Corollary 1 for Hilbert spaces. Since a part of Hong's proof depends on the Three-Series-Theorem, it is not obvious that his arguments can be extended to arbitrary Banach spaces.

B. For $\beta > 0$ let us put $g_\beta(dt) = t^{-(\beta+1)} dt$ on $(0, 1]$. For $0 < \beta < 1$ and $a_n = n^{-1/\beta}$ we get $\sum_n a_n < \infty$, $g_\beta(a_{n+1}, a_n] = \beta^{-1}$. If G is a Lévy measure concentrated on B_1, then $G^{(g)} \in \mathscr{M}(E)$ by case (2) of Proposition 2. If G is supported by B_1^c and finite, then from assumptions (i) and (ii) of case (3) and (a) in Proposition 2 if follows that

$$\int_{B_1^c} \|x\|^\beta G(dx) < \infty.$$

From this and Propositions 2 and 3 we obtain

Corollary 2. *Let* $0 < \beta < 1$ *and*

$$G_\beta(A) = \int_E \int_0^1 1_A(tx) t^{-(\beta+1)} dt G(dx) \quad \textit{for } A \in \mathscr{B}_0.$$

Then G_β is a Lévy measure iff so is G and

$$\int_{B_1^c} \|x\|^\beta G(dx) < \infty.$$

Remark 4. Taking in Corollary 2 a finite measure m on the unit sphere S of E we obtain measures

$$m_\beta(A) = \int_S \int_0^\infty 1_A(tx) t^{-(\beta+1)} dt m(dx) \quad \text{for } A \in \mathscr{B}_0.$$

which are always Lévy measures (corresponding to stable distributions with the exponent $\beta \in (0, 1)$.

C. For $\gamma > 0$ let us put $\nu_\gamma(dt) = (\log t^{-1})^{\gamma-1} dt$ on $(0, 1]$. Since ν_γ are finite measures ($\nu_\gamma(0, 1] = \Gamma(\gamma)$), (a) of Proposition 2 is fulfilled. Thus Proposition 3, case (1), and Proposition 2 give the following

COROLLARY 3. *Let $\gamma > 0$ and*

$$G_\gamma(A) = \int_E \int_0^1 1_A(tx)(\log t^{-1})^{\gamma-1} dt G(dx)$$
$$= \int_E \int_0^1 1_A(e^{-t} x) e^{-t} t^{\gamma-1} dt G(dx) \quad \textit{for } A \in \mathscr{B}_0.$$

Then G_γ is a Lévy measure iff so is G.

Remark 5. Lévy measures G_α from Corollary 1 correspond to infinitely divisible measures from the class L_α distributions (cf. [8]). These are subclasses of the class $L = L_1$ of selfdecomposable distributions. Similarly, measures G_γ from Corollary 3 are Lévy measures of distributions from classes $\mathscr{U}_\gamma$. The class $\mathscr{U}_1 = \mathscr{U}$ coincides with limit distributions of non-linearly deformed rv's (s-selfdecomposable distributions; cf. [4], Section 2).

5. FINAL COMMENTS

(1) All results (Propositions 1–3) are also valid if in the definition of $m^{(\lambda)}$ (see (2.1)) we replace $1_A(tx)$ by $1_A(f(t)x)$, where f is a real-valued measurable function on R^+. Simply, the measure λ should be replaced by the measure $f\lambda \equiv \lambda f^{-1}$ in (2.1). There is a need to have analogous characterizations for operator-valued functions (cf. [5] for measures from $\mathscr{U}_\beta(Q)$ with $\beta < 0$). However, some of the present methods of proofs do not cover such a generality.

(2) The integrability of $\log^\alpha(1+\|x\|)$ (or $\|x\|^\beta$) over B_1^c with respect to $G \in \mathscr{M}(E)$ is equivalent to

$$\int_E \log^\alpha(1+\|x\|)\tilde{e}(G)(dx) < \infty \quad \left(\text{or } \int_E \|x\|^\beta \tilde{e}(M)(dx) < \infty\right)$$

(cf., for instance, [2], Corollary 3.4).

REFERENCES

[1] A. Araujo and E. Giné, *The central limit theorem for real and Banach space valued random variables*, John Wiley, New York 1980.

[2] A. de Acosta, *Exponential moments of vector valued random series and triangular arrays*, Ann. Probability 8 (1980), p. 381–389.

[3] N. N. Hong, *On convergence of some random series and integrals*, Probab. Math. Statistics 8 (1987), p. 151–154.

[4] Z. J. Jurek, *Relations between the s-selfdecomposable and selfdecomposable measures*, Ann. Probability 13 (1985), p. 592–608.

[5] — *Random integral representations for classes of limit distributions similar to Lévy class L_0*, Probab. Th. Rel. Fields 78 (1988), p. 473–490.

[6] Z. J. Jurek and K. Urbanik, *Remarks on stable measures on Banach spaces*, Coll. Math. 38 (1978), p. 269–276.

[7] Z. J. Jurek and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrsch. verw. Gebiete 62 (1983), p. 247–262.

[8] N. van Thu, *An alternative approach to multiply selfdecomposable probability measures on Banach spaces*, Probab. Th. Rel. Fields 72 (1986), p. 35–54.

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