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ON THE CLASS OF OPERATOR STABLE DISTRIBUTIONS IN A SEPARABLE BANACH SPACE

BY

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Abstract. This paper characterizes the class of all limit probability measures μ of normalized and centralized convolution powers in a separable Banach space E which are defined by

$$A_n \nu^{*n} * \delta_{x_n} \xrightarrow{w} \mu$$

for some linear and bounded operators A_n and some shifts $x_n \in E$. It is shown that this class coincides with the set of all infinitely divisible laws in E provided that E is infinite dimensional.

1. Introduction. Let ν be a probability measure on a real separable Banach space E and denote by A a linear bounded operator from E into E. The image measure $\lambda = A\nu$ of ν is also a probability measure on E with $\lambda(B) = \nu(A^{-1}B)$ for each Borel set B of E. Denote by ν^n, $n \geqslant 1$, the convolution power of ν, and by δ_x the degenerate distribution concentrated at $x \in E$. Then one can introduce an operator stable distribution μ (in the sense of Sharpe [12]) by the weak convergence

$$A_n \nu^n * \delta_{a_n} \xrightarrow{w} \mu, \quad n \to \infty, \tag{1.1}$$

where A_n are certain linear and bounded operators (normalizing operators), and $a_n \in E$ some (centralizing) elements.

Characterizations of such limit laws μ are well known for $A_n = c_n^{-1} I$, $c_n > 0$, i.e., when A_n are multiples of the identity operator I (cf. [9]). Following the pioneering work of Sharpe [12] many authors studied operator stable distributions in finite dimensional spaces (for instance, [3]–[5], [7]). The difficulty in infinite dimensional spaces results from the fact that in this case it is not possible to show an elementary convergence theorem of operator types like in finite dimensional spaces; useful results in this area are derived in [10], where the strong operator topology plays an outstanding role.

It is well known that the limit law μ defined in (1.1) is always infinitely divisible (see Proposition 1). The basic aim of the present paper is to verify that any infinitely divisible distribution μ stands for a limit law of (1.1) provided that A_n and ν are suitably chosen and E is infinite dimensional. By virtue of this assertion it is clear that one needs additional assumptions on the operators A_n

in order to show stronger properties of operator stable distributions. Such assertions were discussed in [14] and [16], where only mild conditions on A_n are imposed (see Proposition 5). An earlier paper of Krakowiak [6] contains also characterizations of operator stable distributions on Banach spaces under significantly stronger conditions. In [15] one can find extensions of results given in [14] to the case of operator normed and centered sums of E-valued random variables defined on stationary sequences. Further results for independent random vectors are derived in [8].

2. Main results. Let E be a real separable Banach space with norm $\|\cdot\|$. $P(E)$ denotes the set of probability measures on E, and $J(E)$ stands for the subclass of all infinitely divisible distributions, i.e., distributions μ having the property $\mu = \mu_n^n$ for all $n \geqslant 1$ and certain $\mu_n \in P(E)$. We make sometimes use of the abbreviation $\mu_n = \mu^{1/n}$ for the n-fold root in the sense of convolution. By $L(E)$ we denote the set of linear bounded operators from E into E. For the image of an arbitrary $A \in L(E)$ we briefly write $\operatorname{Im} A$.

We begin with the following well-known result (see also [14]):

PROPOSITION 1. *Let* (1.1) *be satisfied for some* $A_n \in L(E)$, $a_n \in E$, *and certain* $\nu \in P(E)$. *Then* $\mu \in J(E)$ *holds and* $A_n \nu$ *is asymptotically degenerate, i.e., there are* $y_n \in E$ *such that*

$$A_n \nu * \delta_{y_n} \xrightarrow{w} \delta_0 .$$

Now we turn to the major problem of finding the class of all possible limit laws $\mu \in J(E)$ occurring in (1.1). Recall that in the case of subsequences this class coincides with $J(E)$. This follows from Doeblin's result concerning the existence of a universal law belonging to the set of partial attraction of any $\mu \in J(R^1)$. Here we state the generalized version of this theorem for arbitrary Banach spaces given in [2] (see also [11]).

PROPOSITION 2. *Let E be a separable Banach space. Then there exists a fixed distribution* $\nu \in P(E)$ *such that to each* $\mu \in J(E)$ *there correspond normalizing constants* $c_n > 0$, *centralizing elements* $a_n \in E$, *and a subsequence of integers* $k_n \to \infty$ *which satisfy*

$$(c_n^{-1} I)\nu^{k_n} * \delta_{a_n} \xrightarrow{w} \mu \quad \text{as } n \to \infty .$$

Next we state our basic result the proof of which can be found in the next section.

THEOREM 3. *Let E be infinite dimensional. Then there exists a fixed distribution* $\nu \in P(E)$ *such that to each* $\mu \in J(E)$ *there correspond operators* $A_n \in L(E)$ *which satisfy*

$$A_n \nu^n \xrightarrow{w} \mu \quad \text{as } n \to \infty .$$

Now we give further results, remarks and corollaries closely related to this theorem.

Remark 1. (i) The probability measure ν mentioned in the proof of Theorem 3 is infinitely divisible. But it is easy to select another distribution ν which does not belong to $J(E)$.

(ii) The operators A_n constructed in the proof are not invertible. But there exist also invertible operators A_n fulfilling Theorem 3. This assertion results from the fact that every operator A_n belongs to the closure of the set of invertible operators in the norm topology of $L(E)$.

The following surprising result was stimulated by Theorem 3 and the main theorem of Kucharczak and Urbanik [8].

PROPOSITION 4. *Assume that there exists a sequence Q_n of projectors on E such that $Q_i Q_j = 0$, $i \neq j$, and $\operatorname{Im} Q_n$ is isomorphic to E for every $n \geqslant 1$. Then to any $\mu \in J(E)$ there correspond $\lambda \in P(E)$ and a sequence $B_n \in L(E)$ such that*

(2.1) $$\mu = B_n \lambda^n \quad n = 1, 2, \ldots$$

Remark 2. (i) From the assumption of Proposition 4 it follows necessarily that E is infinite dimensional.

(ii) The Kucharczak and Urbanik paper [8] shows that $B_n \lambda$ and μ coincide but for a shift depending on n.

(iii) The problem arises to describe the set of all real separable Banach spaces which satisfy the assumptions of Proposition 4. Obviously, this set contains all spaces E which have a basis.

To formulate more specific properties of operator stable distributions we need some additional assumptions on A_n. In particular, we are interested in the representation

(2.2) $$\mu = B_n \mu^n * \delta_{b_n}, \quad n = 1, 2, \ldots,$$

for certain $B_n \in L(E)$, $b_n \in E$. For example, we define:

(*) To every $m > 1$ there correspond $K > 1$ and $n_0 \geqslant 1$ such that $\|A_k A_n^{-1}\| < K$ for all n, k satisfying $n \geqslant n_0$ and $m/n \leqslant k \leqslant mn$.

(For a first application of this condition see [13].) Then we have (see [16])

PROPOSITION 5. *Assume that (1.1) is true for some $A_n \in L(E)$, $a_n \in E$, and certain $\nu \in P(E)$. Assume further that μ is full. If the A_n are invertible for every n and (*) is satisfied, then (2.2) holds true for some $B_n \in L(E)$, $b_n \in E$. Moreover, the B_n are invertible and satisfy also (*).*

3. Proofs. We begin with the following

LEMMA. *There exists a countable subclass $J_0 = \{\varrho_n\}$ of $J(E)$ which is dense within $J(E)$ such that every element ϱ_n is concentrated on a finite dimensional subspace of E.*

Proof. It is well known that the family of infinitely divisible laws $\{e(\gamma)$: γ a bounded measure on $E\}$ forms a dense set in $J(E)$ (cf. [9], Proposition 5.7.2).

Approximating γ by measures concentrated on a finite set of points, it is easy to see that there exists even a subset J_0 with the required properties.

Proof of Theorem 3. We show this assertion in two steps.

(i) Construction of ν. Consider the family $\{\varrho_n\}$ as in the Lemma and put $J_1 = \{\varrho_m^{1/n}, m, n = 1, 2, \ldots\}$. Choose an arrangement $\lambda_n = \varrho_{m_n}^{1/l_n}$, $n \geq 1$, of J_1 for certain sequences l_n and m_n of natural numbers, that is, $J_1 = \{\lambda_n\}$ holds true. It is obvious that for every sequence j_n the sequence $\varrho_{j_n}^{1/n}$ is a subsequence of λ_n. Suppose λ_n is concentrated on E_n. By the Lemma we have $\dim E_n < \infty$. Next we choose a sequence of projectors Q_n of E such that $Q_i Q_j = 0$, $i \neq j$, and, moreover, the sequence of subspaces $F_j = \operatorname{Im} Q_n$ has the property $\dim F_j = \dim E_j$. Denote by B_n an isomorphism from F_n onto E_n and define

$$\nu = \prod_{n=1}^{\infty}{}^{*} ((c_n^{-1}) B_n^{-1}) \lambda_n,$$

where $c_n > 0$ are positive numbers tending to infinity sufficiently quickly so that this convolution product converges weakly. Note that $B_n^{-1}\lambda_n$ is well defined since λ_n is concentrated on E_n.

(ii) Selection of A_n and convergence. Let $\mu \in J(E)$ be fixed and select a sequence j_n satisfying $\varrho_{j_n} \xrightarrow{w} \mu$. Further, let k_n be a sequence of natural numbers such that $\lambda_{k_n} = \varrho_{j_n}^{1/n}$. So we may define $A_n = c_{k_n} B_{k_n} Q_{k_n}$ and obtain $A_n \nu = \lambda_{k_n}$. Thus

$$A_n \nu^n = \varrho_{j_n} \xrightarrow{w} \mu.$$

Proof of Proposition 4. Let U_n be a fixed isomorphism from $\operatorname{Im} Q_n$ onto E for $n = 1, 2, \ldots$ For a given $\mu \in J(E)$ we put

$$\lambda = \prod_{n=1}^{\infty}{}^{*} ((c_n^{-1}) U_n^{-1}) \mu^{1/n},$$

where $c_n > 0$ is again a sequence of positive numbers tending to infinity sufficiently quickly so that the convolution product converges weakly. Moreover, we put $B_n = c_n U_n Q_n$. Thus we get $B_n \lambda = \mu^{1/n}$ and $B_n \lambda^n = \mu$, as asserted.

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