PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 15 (1995), pp. 165–193

LABYRINTH DIMENSION OF BROWNIAN TRACE*

BY

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Abstract. Suppose that X is a two-dimensional Brownian motion. The trace $X[0, 1]$ contains a self-avoiding continuous path whose Hausdorff dimension is equal to 2.

1. Introduction. The question studied in this paper has been inspired by a series of recent articles of K. Hattori, T. Hattori and Kusuoka (see [10]–[13]) on self-avoiding processes on fractals. Various dimensions of such processes are of interest to physicists. It is clear that the Hausdorff dimension of any self-avoiding path within a fractal set cannot be larger than that of the fractal itself. Can the two dimensions be equal? The answer depends on the fractal set. We will consider self-avoiding paths inside a Brownian trace.

Recall that a set is called a *Jordan arc* if it is homeomorphic to a line segment.

DEFINITION. The supremum of Hausdorff dimensions of all Jordan arcs contained in a set K will be called the *labyrinth dimension* of K.

THEOREM 1. *Suppose that X is a 2-dimensional Brownian motion. The labyrinth dimension of $X([0, 1])$ is equal to 2 a.s.*

Our proof of Theorem 1 actually yields a much stronger statement. Taylor [18] proved that the exact Hausdorff measure for 2-dimensional Brownian trace is given by

$$\phi(u) = u^2 \log(1/u) \log\log\log(1/u)$$

(see also [16]). For the definition of the exact Hausdorff measure see, e.g., [17].

THEOREM 1′. *Let X be a 2-dimensional Brownian motion. With probability 1, for every $\varepsilon > 0$ there exists a Jordan arc $\Gamma \subset X([0, 1])$ such that the ϕ-measure of $X([0, 1])\setminus\Gamma$ is less than ε.*

* Research partially supported by NSF grant DMS 91-00244 and AMS Centennial Research Fellowship.

It is elementary to construct a set with Hausdorff dimension 2 whose labyrinth dimension is equal to 1.

Our proof of Theorem 1′ uses two non-trivial facts about Brownian paths. First, the exact Hausdorff measure of the set of all double points is different from the exact Hausdorff measure of the whole Brownian path [16], [18]. Second, 2-dimensional Brownian paths have no double cut points [6]. Since the last result is one of the key ingredients of our proof, we will take this occasion to correct some results about cut points originally published in [4]. Theorem 2.2 of Burdzy [4] is correct but its proof is false. Theorem 2.3 of the same paper is false. Corollary 2.1 is true but its proof is false. See Theorems 2 and 3 below for the correct statements. The proof of Corollary 2.3 in [4] is based on incorrect Theorem 2.3. The corollary is in fact true – its proof may be based on Theorem 4 below. The error appears on p. 1029, l.-2. in [4] – this form of the scaling property may be applied only in the 3-dimensional space.

Let X be the standard d-dimensional Brownian motion. We will say that $X([0, 1])$ has a *(global) cut point* if for some $t \in (0, 1)$ we have

$$X([0, t)) \cap X((t, 1]) = \emptyset \quad \text{and} \quad X(t) \neq X(s) \quad \text{for all } s \neq t,\ s \in [0, 1].$$

One-dimensional Brownian motion does not have cut points because it does not have points of increase (see the original proof of Dvoretzky et al. [9] or more recent proofs of Adelman [1] or Burdzy [5]). If $d \geqslant 4$, Brownian motion has no double points [8], so it obviously has cut points. It turns out that they also exist for $d = 2$ and 3. Here is a slightly stonger statement.

THEOREM 2. *Suppose that $d = 2$ or 3. With probability 1, for every $\varepsilon > 0$, there exists $t \in (0, \varepsilon)$ such that $X([0, t)) \cap X((t, 1]) = \emptyset$ and $X(t) \neq X(s)$ for $s \neq t$, $s \in [0, 1]$.*

Suppose that $K \subset \mathbf{R}^d$ and $x \in K$ (we will be mostly concerned with $K = X([0, 1])$). We will say that the *order of ramification of x in K* is less than or equal to n if for every $\varepsilon > 0$ we can find (at most) n points $y_1, \ldots, y_n$ different from x such that the connected component of $K \setminus \{y_1, \ldots, y_n\}$ which contains x has diameter less than ε (see [3]). The order of ramification of x is equal to n if it is less than or equal to n but not less than or equal to $n-1$. The definitions of infinite and countable order of ramification are similar. Theorem 2 clearly implies

COROLLARY 1. *Suppose that $d = 2$ or 3. With probability 1, the order of ramification of $X(0)$ in $X([0, 1])$ is equal to 1.*

When we replace $X(0)$ with an arbitrary $X(t)$ in this corollary, the conclusion no longer holds in 2 dimensions.

THEOREM 3. (i) *Suppose that $d = 2$. With probability 1, for almost every $t \in (0, 1)$, the order of ramification of $X(t)$ in $X([0, 1])$ is infinite.*

(ii) *Suppose that $d = 3$. With probability 1, for almost every $t \in (0, 1)$, the order of ramification of $X(t)$ in $X([0, 1])$ is equal to 2.*

We will not supply a proof of Theorem 3 (ii). The proof of Theorem 2.3 in [4] applies here with only minor modifications needed for the 3-dimensional process. As we have already mentioned, there is no problem with scaling in the 3-dimensional case.

We would like to pose the following

PROBLEM. Suppose that X is a 2-dimensional Brownian motion. Is it true that with probability 1, for almost every $t \in (0, 1)$, the order of ramification of $X(t)$ in $X([0, 1])$ is countable?

We will say that $x = X(t)$ is a *local* cut point for $X([0, 1])$ if $X(t) \neq X(s)$ for all $s \neq t$, $s \in [0, 1]$, and for some $\varepsilon > 0$ we have $X((t-\varepsilon, t)) \cap X((t, t+\varepsilon)) = \emptyset$. We have rather easy

THEOREM 4. *Suppose that $d = 2$ or 3. The set of local cut points is dense in $X([0, 1])$, with probability* 1.

A version of the following result is needed in the proof of Theorem 2. We state it in a form which seems to have some interest of its own. The ball and sphere with center x and radius r will be denoted by $B(x, r)$ and $\partial B(x, r)$, respectively.

PROPOSITION 1. *With probability* 1, *for infinitely many integers k, the 3-dimensional Brownian motion does not return to $\partial B(0, k-1)$ after hitting $\partial B(0, k)$.*

We will give a very elementary proof of Proposition 1. See [14] for a number of related results.

The next section contains the proofs. We start by proving Proposition 1 and Theorems 2–4 since these proofs are much shorter than that of Theorem 1′.

We would like to explain the main idea of the proof of Theorem 1. Recall that the Hausdorff dimension of a Brownian path is equal to 2. We will take $X[0, 1]$, remove some small pieces of this path, and reconnect the remaining ones to obtain a self-avoiding path with large Hausdorff measure. In order to visualize this process, consider first a smooth curve which makes a single loop in the shape of the letter α. We can remove a tiny neighborhood of the intersection point and replace it with two short non-intersecting line segments so that we obtain a self-avoiding path containing most of the original path. In this case we had to add two extra line segments but we do not have to do it in the case of the Brownian path because such a path does not have double cut points, and hence there are always some paths around an intersection point. We can limit the changes to a small subset of the path as the set of all intersection points has a smaller exact Hausdorff measure than that for the whole path. A Brownian path contains infinitely many loops which contain and intersect other loops and this presents a combinatorial problem much harder than the one suggested by the example with the α-shaped curve. This is the main reason why the proof of Theorem 1 is quite long.

We would like to thank Omer Adelman for the most helpful discussions related to cut points. We are grateful to Robin Pemantle and Vic Reiner for supplying a proof of Lemma 2.7 which is one of the key steps in the proof of Theorem 1.

2. Proofs. See [7] for a review of h-process techniques used in our paper. The distribution of an h-process will be denoted by $P_h^{\cdot}$.

We will use the following "scaling property" of h-processes.

LEMMA 2.1. *Suppose that $c \in (0, \infty)$, $D \subset \mathbf{R}^d$ is a Greenian domain, h is a positive superharmonic function in D, and μ is a measure supported in $\bar{D}$. Let Ω be the space of paths continuous until their lifetime and let $\mathscr{F}$ be the Borel σ-field in Ω. Suppose that $A \in \mathscr{F}$ and define*

$$h_c(z) = h(z/c) \text{ for } z \in cD, \qquad \mu_c(B) = \mu(B/c) \text{ for } B \subset cD,$$

$$A_c = \{\omega \in \Omega \colon \exists \omega_1 \in A \text{ such that } \omega(t) = c\omega_1(t/c^2) \text{ for all } t\}.$$

Then $P_{h_c}^{\mu_c}(A_c) = P_h^{\mu}(A)$.

Proof. The lemma follows from the scaling properties of the Brownian motion and harmonic functions. ∎

Proof of Proposition 1. Let A_k denote the event that a 3-dimensional Brownian motion Z does not return to $\partial B(0, k-1)$ after hitting $\partial B(0, k)$. Let h_k be the restriction of the function $h(x) = k/|x|$ to the set $D_k = \{|x| > k\}$. Note that h_k is harmonic in D_k with boundary values 1 on $\{|x| = 1\}$ and 0 at infinity. If $|y| = k$, then

$$P(A_k) = P^y\big(T(\partial B(0, k-1)) = \infty\big) = 1 - h_{k-1}(y) = 1 - (k-1)/k = 1/k.$$

Note that the formula is true even for $k = 1$ although our proof does not apply in this case. The process Z conditioned by A_k is a g_{k-1}-process where the conditioning function g_{k-1} is equal to $1 - h_{k-1}$. Suppose that $n \geqslant 1$, $|z| = k+n-1$, and $|y| = k+n$. Then

$$\begin{aligned} P(A_{k+n}^c \mid A_k) &= P_{g_{k-1}}^y\big(T(\partial B(0, k+n-1)) < \infty\big) \\ &= \frac{g_{k-1}(z)}{g_{k-1}(y)} P^y\big(T(\partial B(0, k+n-1)) < \infty\big) \\ &= \frac{1-(k-1)/(k+n-1)}{1-(k-1)/(k+n)} P(A_{k+n}^c) \\ &= \frac{1-(k-1)/(k+n-1)}{1-(k-1)/(k+n)} \big(1 - 1/(k+n)\big) = \frac{n}{n+1}. \end{aligned}$$

Hence

$$P(A_k \cap A_{k+n}) = P(A_k) P(A_{k+n} \mid A_k) = \frac{1}{k(n+1)}.$$

It follows that for all N

$$\sum_{k=1}^{N}\sum_{n=1}^{N-k} P(A_k \cap A_{k+n}) = \sum_{k=1}^{N}\sum_{n=1}^{N-k}\frac{1}{k(n+1)} \leqslant \sum_{k=1}^{N}\frac{1}{k}c_1\log(N-k) \leqslant \sum_{k=1}^{N}\frac{1}{k}c_1\log N$$

$$\leqslant c_2\log^2 N \leqslant c_3\Big(\sum_{k=1}^{N} 1/k\Big)^2 = c_3\Big(\sum_{k=1}^{N} P(A_k)\Big)^2.$$

This and the fact that $\sum_k P(A_k) = \infty$ allow us to use a version of the Borel–Cantelli lemma proved by Kochen and Stone [15] in Theorem (ii). Their theorem implies that A_k's occur for infinitely many k with positive probability. By the 0-1 law for tail events, this happens in fact with probability 1. ∎

Proof of Theorem 2. First we consider the 2-dimensional case. Suppose that Y is an h-process in $\{x \in \mathbf{R}^2\colon |x| < 1\}$. Let the conditioning function h be given by $h(x) = -\log|x|$ in the complex notation. The initial distribution of Y is assumed to be uniform on the unit circle. Then it is easy to see that the process Y converges to 0 at its lifetime a.s. Let C_k denote the event that Y does not return to $\partial B(0, 2^{-k+1})$ after hitting $\partial B(0, 2^{-k})$. It is routine to check that the distribution of $-\log|Y|$ is that of the 3-dimensional Bessel process, i.e., the norm of the 3-dimensional Brownian motion. Hence Proposition 1 implies that, with probability 1, C_k holds for infinitely many k.

Fix an arbitrarily small $\varepsilon \in (0, 1)$ and let K be the smallest k such that C_k holds and $2^{-k} < \varepsilon$. Let $S = T(\partial B(0, 2^{-K}))$. We will condition Y on $\{K = k_0\}$ and $\{Y(S) = y_0\}$ for some fixed k_0 and y_0. The conditioned path consists of two pieces: $V_1 = \{Y(t), 0 \leqslant t < S\}$ and $V_2 = \{Y(t), S \leqslant t < \tau\}$, where τ is the lifetime of Y. Note that V_2 is an h_1-process in $B(0, 2^{-k_0+1})$ starting from y_0 and conditioned to go to 0. Here $h_1(x) = -\log|x| + \log(2^{-k_0+1})$. The process V_2 is independent of V_1 given $\{K = k_0\}$ and $\{Y(S) = y_0\}$. Theorem 2.1 of Burdzy [4] implies that with probability $p > 0$ the process V_2 has a cut point within $B(0, 2^{-k-1})$. In particular, with probability p, there exists $t \in (T, \tau)$ such that $Y(t) \in B(0, 2^{-k-1})$ and $Y([T, t)) \cap Y((t, \tau)) = \emptyset$. The probability p does not depend on k_0 or y_0 by scaling (see Lemma 2.1) and rotation invariance of conditioned Brownian motion. Since $\{K = k_0\}$ implies C_{k_0}, we also have $Y([0, t)) \cap Y((t, \tau)) = \emptyset$ with (conditional) probability not less than p. Let $A(\varepsilon)$ denote the event that there exists $t \in (0, \tau)$ such that $|Y(t)| \leqslant \varepsilon$ and $Y([0, t)) \cap Y((t, \tau)) = \emptyset$. By summing over all k_0 and integrating with respect to y_0 we can prove that $P(A(\varepsilon)) \geqslant p > 0$ for every fixed $\varepsilon > 0$. The sequence $\{A(1/j)\}_{j \geqslant 1}$ is monotone, so $P\big(\bigcap_{j \geqslant 1} A(1/j)\big) \geqslant p$. The event $\{\bigcap_{j \geqslant 1} A(1/j)\}$ belongs to the tail σ-field for the h-process Y, so by the 0-1 law its probability is equal to 1. In other words, with probability 1, for every $\varepsilon > 0$ there exists $t \in (0, \tau)$ such that $|Y(t)| \leqslant \varepsilon$ and $Y([0, t)) \cap Y((t, \tau)) = \emptyset$.

Suppose X is a 2-dimensional Brownian motion starting from 0 and let U_r be the first hitting time of the circle $B(0, r)$ by X. The process $\{X(t), 0 \leqslant t < U_1\}$ is the time-reversal of Y, so its path contains cut points in every neighborhood of 0 a.s. By scaling, this is true for the process stopped at any stopping time $U_{1/k}$, $k \geqslant 1$. Since at least one of these stopping times is less than 1, the trace $X([0, 1])$ contains cut points in every neighborhood of 0 a.s. Note that if $X(t_k)$ is a sequence of cut points converging to 0, then $t_k \to 0$ by continuity of Brownian paths and the fact that Brownian motion never returns to its starting point.

The 3-dimensional result is a straightforward corollary to the 2-dimensional result and the fact that the orthogonal projection of the 3-dimensional Brownian motion on a plane is a 2-dimensional Brownian motion. ∎

Proof of Theorem 3 (i). Consider two independent 2-dimensional Brownian motions X^1 and X^2 starting from 0 and killed at the hitting time of $\partial B(0, 1)$. Let

$$\begin{aligned} S_1^j(k) &= \inf\{t\colon |X^j(t)| = 2^{-k}\}, \\ T_m^j(k) &= \inf\{t > S_m^j(k)\colon |X^j(t)| = 2^{-k-1}\}, \quad m \geqslant 1, \\ S_m^j(k) &= \inf\{t > T_{m-1}^j(k)\colon |X^j(t)| = 2^{-k}\}, \quad m \geqslant 2. \end{aligned}$$

Let $N_j(k)$ be the number of crossings of X^j from $\partial B(0, 2^{-k})$ to $\partial B(0, 2^{-k-1})$, i.e., $N_j(k)$ is equal to the largest m such that $T_m^j(k) < \infty$. A standard calculation shows that the probability that a Brownian motion starting from a point of $\partial B(0, 2^{-k})$ will hit $\partial B(0, 2^{-k-1})$ before hitting $\partial B(0, 1)$ is equal to $p_k \overset{\text{df}}{=} k/(k+1)$. By the strong Markov property applied at $S_m^j(k)$'s, the distribution of $N_j(k)$ is geometric, i.e., $P(N_j(k) = n) = p_k^n(1-p_k)$. The independence of X^1 and X^2 implies

$$P(N_1(k)+N_2(k) = n) = \sum_{j=0}^{n} p_k^{n-j}(1-p_k)\, p_k^j(1-p_k) = (n+1)\, p_k^n(1-p_k)^2.$$

Hence

$$P(N_1(k)+N_2(k) \leqslant n) = \sum_{j=0}^{n} (j+1)\, p_k^j(1-p_k)^2 \leqslant (n+1)(1-p_k)(1-p_k^{n+1}).$$

In particular, if we take $n = c\log k$ for some constant $c < \infty$, we obtain

$$P(N_1(k)+N_2(k) \leqslant c\log k) \leqslant (c\log k+1)(1-p_k)(1-p_k^{c\log k+1}). \tag{2.1}$$

For $k > k_0$ we have

$$\begin{aligned} (c\log k+1)\log(k/(k+1)) &= (c\log k+1)\log(1-1/(k+1)) \geqslant -\frac{2}{k+1}(c\log k+1) \\ &\geqslant -k^{-1/2}/2 \geqslant \log(1-k^{-1/2}). \end{aligned}$$

Exponentiating yields

$$(k/(k+1))^{c\log k+1} \geqslant 1-k^{-1/2}$$

and, therefore

$$1-p_k^{c\log k+1} = 1-\big(k/(k+1)\big)^{c\log k+1} \leqslant k^{-1/2}.$$

This and (2.1) give

$$\sum_{k>k_0} P\big(N_1(k)+N_2(k) \leqslant c\log k\big) \leqslant \sum_{k>k_0} (c\log k+1)\big(1-k/(k+1)\big)k^{-1/2} < \infty.$$

By the Borel–Cantelli lemma, for any fixed $c<\infty$, only a finite number of events $\{N_1(k)+N_2(k) \leqslant c\log k\}$ occur.

Let $K = \bigcup_{k\geqslant 1} \partial B(0, 2^{-k})$ and let

$$U_1^j(n) = \inf\{t > 1/n\colon X^j(t) \in K\},$$

$$U_m^j(n) = \inf\{t > U_{m-1}^j(n)\colon X^j(t)\in K,\ |X^j(t)| \neq |X^j(U_{m-1}^j(n))|\}, \qquad m \geqslant 2.$$

Note that we have $U_m^j(n) = U_k^j(i)$ for many distinct values of m, n, k and i. For a family of points $x_m^j(n)$ let $F = F\big(\{x_m^j(n),\ j=1,2,\ m,n\geqslant 1\}\big)$ denote the event $\{X^j(U_m^j(n)) = x_m^j(n),\ j=1,2,\ m,n\geqslant 1\}$. Observe that $x_m^j(n)$'s determine the values of $N_1(k)+N_2(k)$ for all $k \geqslant 1$. Given F, the processes $\{X^j(t), t\in[U_m^j(n), U_{m+1}^j(n)]\}$, $m,n\geqslant 1$, are independent. If we condition on F and suppose that $|x_m^j(n)| = 2^{-k}$, then $\{X^j(t),\ t\in[U_m^j(n), U_{m+1}^j(n)]\}$ is an h-process in $D_k \stackrel{\mathrm{df}}{=} B(0, 2^{-k+1})\backslash\overline{B(0, 2^{-k-1})}$ converging to $x_{m+1}^j(n)\in\partial D_k$.

Fix some integer M. We will say that a set C *belongs to the family* $\mathscr{S}_k$ if

(i) C contains closed loops within both annuli

$$B(0,(7/4)2^{-k})\backslash\overline{B(0,(3/2)2^{-k})} \quad\text{and}\quad B(0,(3/4)2^{-k})\backslash\overline{B(0,(5/8)2^{-k})},$$

and

(ii) for each $j=1,\dots,M$, the set C contains a continuous path connecting the circles $\partial B(0,(9/16)2^{-k})$ and $\partial B(0,(15/8)2^{-k})$ within the wedge

$$\{z = re^{i\alpha}\colon \alpha\in(2j\pi/M,\ 2j\pi/M+\pi/M)\}.$$

Let A_k be the event that the process starts from a point of $\partial B(0,2^{-k})$ and its path stopped at the hitting time of $K\backslash\partial B(0,2^{-k})$ belongs to $\mathscr{S}_k$. Let Q be the union of trajectories of X^1 and X^2. It is standard to prove that, for any harmonic function h in D_k and any point $x\in\partial B(0,2^{-k})$, the probability of A_k for an h-process starting from x is bounded from below by a constant $q>0$ which does not depend on x or h and which, moreover, does not depend on k, by Lemma 2.1. Hence the conditional probability given F that Q does not contain a set in $\mathscr{S}_k$ is greater than $(1-q)^{N_1(k)+N_2(k)}$. Let $c = -2/\log(1-q)$. We can assume that $N_1(k)+N_2(k)\geqslant c\log k$ for all $k\geqslant k_1 = k_1(F)$. Then

$$(1-q)^{N_1(k)+N_2(k)} \leqslant (1-q)^{c\log k} = k^{c\log(1-q)} = k^{-2}.$$

Since $\sum_k k^{-2}<\infty$, the Borel–Cantelli lemma implies that, for almost all k, Q contains a set in $\mathscr{S}_k$.

Let $\mathscr{J}_k$ be the family of sets from $\mathscr{I}_k$ rotated around 0 by the angle π/M. The same proof which works for $\mathscr{I}_k$ shows that, for almost all k, Q contains also a set in $\mathscr{J}_k$. It is now elementary to see that, with probability 1, Q contains at least M disjoint paths (except that they all start at 0) which connect 0 and some circle $\partial B(0, 2^{-k})$, $k < \infty$. Since M is an arbitrarily large number, we see that the order of ramification of 0 in Q is infinite a.s.

The same proof would apply if we killed X^j's at the hitting time of a circle $\partial B(0, 2^{-k})$ for some fixed $k > 1$. Note that if X is a Brownian motion and $t \in (0, 1)$ is fixed, then $\{X(t-s)-X(t), s \in [0, t]\}$ and $\{X(t+s)-X(t), s \in [0, 1-t]\}$ are independent Brownian motions starting from 0 and with probability 1 they both hit some circle $\partial B(0, 2^{-k})$. Thus for every fixed $t \in (0, 1)$ the order of ramification of $X(t)$ in $X([0, 1])$ is infinite a.s. Theorem 3 (i) now follows from the Fubini theorem. ∎

Proof of Theorem 4. Fix some $t \in (0, 1)$. Note that $Y = \{X(t+s)-X(t), s \geqslant 0\}$ and $Z = \{X(t-s)-X(t), s \in [0, t]\}$ are independent Brownian motions starting from 0. Theorem 2 applied to the first process shows that, with probability 1, for every $n \geqslant 1$ there is $s_n \in (t, t+1/n)$ such that $X[t, s_n]$ is disjoint from $X[s_n, 1]$ and $X(s_n) \neq X(s)$ for all $s \neq s_n$, $s \in [t, 1]$. Hence one can find $\varepsilon_n > 0$ such that

$$X[s_n-\varepsilon_n, s_n) \cap X(s_n, s_n+\varepsilon_n] = \emptyset.$$

Since Y and Z are independent and a Brownian motion with probability 1 does not hit a fixed point, it follows that $X[0, t]$ does not contain any point $X(s_n)$, $n \geqslant 1$, a.s. Thus with probability 1 there are cut points arbitrarily close to $X(t)$. The same is true simultaneously for all rational $t \in (0, 1)$ a.s. It remains to notice that the set of points $X(t)$ with rational $t \in (0, 1)$ is dense in $X[0, 1]$. ∎

Proof of Theorem 1'. Recall that X is a 2-dimensional Brownian motion with $X(0) = 0$. We will choose many constants depending on the particular path but we will suppress expressions "with probability 1."

We will construct a number of paths (we will call them (P)-paths) which are not, strictly speaking, Brownian paths, although they may be called "Brownian" as they have many typical properties of a Brownian trajectory. We start by defining a countable family $\mathscr{K}$ of C^1 lines. First, all line segments whose endpoints have rational coordinates are contained in $\mathscr{K}$. The family $\mathscr{K}$ also includes all arcs of circles such that the center of the circle and its radius are rational and the arc corresponds to an angle which is a rational multiple of π. Finally, $\mathscr{K}$ contains all C^1 Jordan curves which are finite unions of line segments and arcs described above. Note that for every finite union of balls M and every $\varepsilon > 0$ there exists an open set M_1 containing M such that $\partial M_1 \in \mathscr{K}$ and all points of ∂M_1 are within distance ε from M.

We will also need a family $\mathscr{A}$ of random times. A random time T *belongs to* $\mathscr{A}$ if it satisfies one of the following conditions:

(A1) There exists a rational t such that $T = t$ a.s.

(A2) There exist $K \in \mathscr{K}$ and rational t_1 such that $T = \inf\{t > t_1 : X(t) \in K\}$.

(A3) There exist $K \in \mathscr{K}$ and rational t_1 such that $T = \sup\{t < t_1 : X(t) \in K\}$.

(A4) There exist $S_1, S_2 \in \mathscr{A}$ such that $T = \min(S_1, S_2)$.

(A5) There exist $S_1, S_2 \in \mathscr{A}$ such that $T = \max(S_1, S_2)$.

(A6) There exist $K \in \mathscr{K}$ and $S \in \mathscr{A}$ such that $T = \inf\{t > S : X(t) \in K\}$.

(A7) There exist $K \in \mathscr{K}$ and $S \in \mathscr{A}$ such that $T = \sup\{t < S : X(t) \in K\}$.

(A8) There exist $S, S_1, S_2 \in \mathscr{A}$ such that $T = \inf\{t > S : X(t) \in X[S_1, S_2]\}$.

(A9) There exist $S, S_1, S_2 \in \mathscr{A}$ such that $T = \sup\{t < S : X(t) \in X[S_1, S_2]\}$.

Adelman and Dvoretzky [2] observed that the point where a Brownian motion hits the trace $Z[0, 1]$ of an independent Brownian motion Z is not a double point of Z with probability 1. This and a standard argument show that in (A8) there is only one $s \in [S_1, S_2]$ such that $X(s) = X(T)$. The fact that Brownian paths do not hit a point fixed in advance and the Markov property can then be used to show that there is no other $u \in [0, 1]$ besides s with $X(u) = X(T)$. A similar remark applies to (A1)–(A9).

We will say that a path $\{\Lambda(t), t \in [t_1, t_2]\}$ has *property* (P) if it satisfies the following. There exist a finite increasing sequence $\{s_j\}_{1 \leqslant j \leqslant k}$ such that $s_1 = t_1$, $s_k = t_2$, a (not necessarily increasing) sequence $\{u_j\}_{1 \leqslant j \leqslant k}$, and for every $j < k$ there is a $\Delta_j = 1$ or -1 such that $\Lambda(s_j + t) = X(u_j + \Delta_j t)$ for all $t < s_{j+1} - s_j$. In other words, Λ is assembled from a finite number of pieces of X. We also assume that the pieces of X that make up Λ are disjoint, i.e.,

$$X\big(u_j, u_j + \Delta_j(s_{j+1} - s_j)\big) \cap X\big(u_n, u_n + \Delta_n(s_{n+1} - s_n)\big) = \emptyset \qquad \text{if } j \neq n$$

(the formula needs an obvious modification when $\Delta_j = -1$ or $\Delta_n = -1$). Moreover, all times u_j and $u_j + \Delta_j(s_{j+1} - s_j)$ have to belong to $\mathscr{A}$. Finally, we require that if u_j or $u_j + \Delta_j(s_{j+1} - s_j)$ is equal to T defined by (A8) or (A9) and $s \in [S_1, S_2]$ is such that $X(s) = X(T)$, then there is $\delta > 0$ such that either $X[s - \delta, s]$ or $X[s, s + \delta]$ is not used in the construction of Λ.

Obviously, the original Brownian path $\{X(t), t \in [0, 1]\}$ is a (P)-path.

The following transformations may be used to obtain a (P)-path from another (P)-path although not every application of (T2) will guarantee that the pieces of X in the definition of a (P)-path are disjoint.

(T1) If $\{\Lambda(t), t \in [t_1, t_2]\}$ has property (P), $t_1 \leqslant T_1 < T_2 \leqslant t_2$, and $T_1, T_2 \in \mathscr{A}$, then $\{\Lambda(t), t \in [T_1, T_2]\}$ also has property (P).

(T2) Suppose that $\{\Lambda_1(t), t \in [t_1, t_2]\}$ and $\{\Lambda_2(t), t \in [t_3, t_4]\}$ are (P)-paths such that $\Lambda_1(t_2) = \Lambda_2(t_3)$. Then $\{\Lambda_3(t), t \in [t_1, t_2 + t_4 - t_3]\}$ defined by

$$\Lambda_3(t)=\begin{cases}\Lambda_1(t) & \text{if } t\in[t_1,t_2],\\ \Lambda_2(t-t_2+t_3) & \text{if } t\in[t_2,t_2+t_4-t_3],\end{cases}$$

may be (or may be not) a (P)-path.

(T3) If $\{\Lambda(t), t\in[t_1,t_2]\}$ has property (P) and $\Lambda_1(t)=\Lambda(t_2+t_1-t)$, then $\{\Lambda_1(t), t\in[t_1,t_2]\}$ is also a (P)-path.

We would like to point out one particular combination of these transformations which will play an important role in the proof. Suppose that $\{\Lambda(t), t\in[t_1,t_2]\}$ has property (P) and let $\mathscr{A}_\Lambda$ be defined relative to Λ in the same way that $\mathscr{A}$ has been defined relative to X. Let us assume that $T, S, S_1, S_2\in\mathscr{A}_\Lambda$, $S\leqslant S_1\leqslant S_2$, and these random times satisfy a condition analogous to (A8), i.e.,

$$T=\inf\{t>S\colon \Lambda(t)\in\Lambda[S_1,S_2]\}.$$

Let $S_3=\sup\{t\in[S_1,S_2]\colon \Lambda(t)=\Lambda(T)\}$. Now let us replace the piece $\Lambda[S,S_2]$ of Λ with $\Lambda[S,T]$ and $\Lambda[S_3,S_2]$. We obtain a new (P)-path Λ_1. The precise meaning of this operation should be quite obvious but we are going to spell out the definition of Λ_1 in this case. We will limit ourselves to the word description in future instances of this operation. We have

$$\Lambda_1(t)=\begin{cases}\Lambda(t) & \text{if } t\in[t_1,T],\\ \Lambda(S_3+t) & \text{if } t\in[T,T+(S_2-S_3)],\\ \Lambda(S_2+t) & \text{if } t\in[T+(S_2-S_3), T+(S_2-S_3)+(t_2-S_2)].\end{cases}$$

We will say that $\{\Lambda(t), t\in[t_1,t_2]\}$ *has a local double cut point* at x if there exist $s_1, u_1\in(t_1,t_2)$ and a small disc $B(x,r)$ with the following properties. Let

$$s_0=\inf\{t\colon \Lambda[t,s_1]\subset B(x,r)\},\qquad s_2=\sup\{t\colon \Lambda[s_1,t]\subset B(x,r)\},$$

$$u_0=\inf\{t\colon \Lambda[t,u_1]\subset B(x,r)\},\qquad u_2=\sup\{t\colon \Lambda[u_1,t]\subset B(x,r)\}.$$

Let D_1, D_2, D_3 and D_4 be the connected components of $B(x,r)\backslash(\Lambda[s_0,s_2]\cup\cup\Lambda[u_0,u_2])$ which touch the boundary of $B(x,r)$. The point x is called a *local double cut point* if $\Lambda(s_1)=\Lambda(u_1)$, $s_2<u_0$ or $u_2<s_0$ and at least two of D_j's contain x in their closure.

Every (P)-path $\{\Lambda(t), t\in[t_1,t_2]\}$ has the following properties:

(P1) Λ is continuous.

(P2) $\Lambda[t_1,t_2]\subset X[0,1]$.

(P3) Λ has no local double cut points.

Properties (P1) and (P2) are evident. (P3) follows from the fact that the Brownian path X has no local double cut points (Lemma 2.2) and from the fact that the points where the pieces of the X path are spliced (i.e., $X(u_j)$ and $X(u_j+\Delta_j(s_{j+1}-s_j))$ in the notation used in the definition of a (P)-path) are not double points. See the last condition in the definition of (P)-paths and the remarks preceding the definition.

For a continuous path $\{\Lambda(t),\ t\in[t_1, t_2]\}$ and sets $D\subset D^+$ let

$$
\begin{aligned}
&S_1^D(\Lambda) = \inf\{t > t_1\colon\ \Lambda(t)\in \overline{D}\},\\
&T_k^D(\Lambda) = \inf\{t > S_k^D(\Lambda)\colon\ \Lambda(t)\in \partial D^+\}, \qquad k\geqslant 1,\\
&S_k^D(\Lambda) = \inf\{t > T_{k-1}^D(\Lambda)\colon\ \Lambda(t)\in \overline{D}\}, \qquad k> 1,\\
&U_k^D(\Lambda) = \sup\{t < S_k^D(\Lambda)\colon\ \Lambda(t)\in \partial D^+\}, \qquad k\geqslant 1.
\end{aligned}
\tag{2.2}
$$

Suppose that $\{x_k\}_{k\geqslant 1}$ is a sequence of points of $\boldsymbol{R}^2$ and $\{\gamma_k\}_{k\geqslant 1}$, $\{\alpha_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ are positive sequences decreasing to 0. Let $C_k = B(x_k, \gamma_k)$, $C_k^+ = B(x_k, \gamma_k+\beta_k)$. Assume that:

(i) γ_k's, α_k's, β_k's and the coordinates of x_k's are rational.

(ii) $X(0)$ and $X(1)$ are outside the closure of C_k^+ for every k.

(iii) Every point in $\boldsymbol{R}^2\backslash\{X(0), X(1)\}$ is covered by infinitely many discs $B\big(x_k, (\gamma_k-\alpha_k)/2\big)$.

Other conditions will be imposed on these sequences later in the proof.

Find the smallest k_0 such that $X[0, 1]\cap C_{k_0}\neq\emptyset$. Sometimes we will suppress k_0 in the notation and write C and C^+ instead of C_{k_0} and $C_{k_0}^+$.

Note that $T_k^C(X)<\infty$ for only a finite number of k. Let V_k be a time t between $U_k^C(X)$ and $T_k^C(X)$ at which $|X(t)-x_0|$ takes its minimum value. Note that $|X(V_k)-x_0|<\gamma_{k_0}$ a.s. Then let

$$V_k^- = \sup\{t<V_k\colon\ |X(t)-x_0|\geqslant\gamma_{k_0}\},\qquad V_k^+ = \inf\{t>V_k\colon\ |X(t)-x_0|\geqslant\gamma_{k_0}\}.$$

The continuity of the Brownian path implies that for any given $\alpha_{k_0}>0$ we can choose sufficiently small $\beta_{k_0}>0$ so that $X[U_k^C(X), V_k^-]$ and $X[V_k^+, T_k^C(X)]$ lie in

$$L \overset{\mathrm{df}}{=} B(x_0, \gamma_{k_0}+\beta_{k_0})\backslash B(x_0, \gamma_{k_0}-\alpha_{k_0})\qquad\text{for all } k.$$

Fix some small $\varepsilon_1>0$, and find $\alpha_{k_0}, \beta_{k_0}>0$ such that the time Brownian motion X spends in L before time 1 is less than ε_1. The ϕ-measure of a piece of 2-dimensional Brownian path is equal to its time-length, where

$$\phi(u) = u^2\log(1/u)\log\log\log(1/u)$$

(see [18] and [16]). Hence the ϕ-measure of $X[0, 1]\cap L$ is less than ε_1.

Let $K_1 = X[0, 1]\backslash\bigcup_k X[U_k^C, T_k^C]$ and $K_2 = \bigcup_k X[V_k^-, V_k^+]$. Both sets K_1 and K_2 are closed and $K_1\cap K_2=\emptyset$. Let

$$R_1 = \inf\{t>U_1^C(X)\colon\ X(t)\in K_2\},$$

$$R_k = \begin{cases}\inf\{t>R_{k-1}\colon\ X(t)\in K_1\}, & k \text{ even},\\ \inf\{t>R_{k-1}\colon\ X(t)\in K_2\}, & k \text{ odd}.\end{cases}$$

Since K_1 and K_2 do not intersect and X is continuous, $R_k \leqslant V_1^-$ for only a finite number of k, say for $k \leqslant i_0$. Let $R_0 = U_1^C(X)$ and $R_{i_0+1} = V_1^-$. Now we apply Lemma 2.3 to $\{X(t), t \in [R_0, R_1]\}$ and sets K_1 and K_2. We replace this piece of the Brownian path with a new path $\{\Gamma_0(s), s \in [s_1^0, s_2^0]\}$ such that

$$X(R_0) = \Gamma_0(s_1^0), \quad X(R_1) = \Gamma_0(s_2^0), \quad \Gamma_0[s_1^0, s_2^0] \subset X[R_0, R_1]$$

and

$$(K_1 \cup \Gamma_0[s_1^0, s_0^0]) \cap (K_2 \cup \Gamma_0[s_0^0, s_2^0]) = \{\Gamma_0(s_0^0)\}$$

for some $s_0^0 \in (s_1^0, s_2^0)$.

The remarks before the definition of a (P)-path show that $\Gamma(s_0^0)$ is not a double point. Hence there is an open ball B_0 whose boundary belongs to $\mathscr{K}$ which contains $\Gamma(s_0^0)$ and satisfies $X[R_1, R_2] \cap B_0 = \emptyset$. Let

$$s_3^0 = \inf\{s \leqslant s_0^0 \colon \Gamma_0[s, s_0^0] \in B_0\}, \quad s_4^0 = \sup\{s \geqslant s_0^0 \colon \Gamma_0[s_0^0, s] \in B_0\},$$

$$K_1^0 = K_1 \cup \Gamma_0[s_1^0, s_3^0], \quad K_2^0 = K_2 \cup \Gamma_0[s_4^0, s_2^0].$$

Note that $s_3^0 < s_0^0 < s_4^0$ and $K_1^0 \cap K_2^0 = \emptyset$. Let us apply Lemma 2.3 again, this time to $\{X(t), t \in [R_1, R_2]\}$, K_1^0 playing the role of K_2 and K_2^0 playing the role of K_1. We will obtain $\{\Gamma_1(s), s \in [s_1^1, s_2^1]\}$ such that

$$X(R_1) = \Gamma_1(s_1^1), \quad X(R_2) = \Gamma_1(s_2^1), \quad \Gamma_1[s_1^1, s_2^1] \subset X[R_1, R_2]$$

and

$$(K_1 \cup \Gamma_0[s_1^0, s_0^0] \cup \Gamma_1[s_0^1, s_2^1]) \cap (K_2 \cup \Gamma_0[s_0^0, s_2^0] \cup \Gamma_1[s_1^1, s_0^1]) = \{\Gamma_0(s_0^0), \Gamma_1(s_0^1)\}$$

for some $s_0^0 \in (s_1^0, s_2^0)$ and $s_0^1 \in (s_1^1, s_2^1)$.

By repeating the same procedure we can construct continuous paths $\{\Gamma_k(s), s \in [s_1^k, s_2^k]\}$ for $0 \leqslant k \leqslant i_0$ such that

$$X(R_k) = \Gamma_k(s_1^k), \quad X(R_{k+1}) = \Gamma_k(s_2^k), \quad \Gamma_k[s_1^k, s_2^k] \subset X[R_k, R_{k+1}]$$

and

$$\Big(K_1 \cup \bigcup_{k \text{ even}} \Gamma_k[s_1^k, s_0^k] \cup \bigcup_{k \text{ odd}} \Gamma_k[s_0^k, s_2^k]\Big) \cap \Big(K_2 \cup \bigcup_{k \text{ even}} \Gamma_k[s_0^k, s_2^k] \cup \bigcup_{k \text{ odd}} \Gamma_k[s_1^k, s_0^k]\Big) = \bigcup_k \{\Gamma_k(s_0^k)\}$$

for some $s_0^k \in (s_1^k, s_2^k)$.

Next we can find continuous paths analogous to Γ_k's which will replace all pieces $X[U_k^C(X), V_k^-]$ and $X[V_k^+, T_k^C(X)]$ of the original Brownian path. Let $\{\Lambda_1(u), u \in [u_0, u_\infty]\}$ be the path obtained from $\{X(t), t \in [0, 1]\}$ by replacing all pieces $\{X(t), t \in [R_k, R_{k+1}]\}$ with the corresponding new paths $\{\Gamma_k(s), s \in [s_1^k, s_2^k]\}$. Let $\{u_j\}_{1 \leqslant j \leqslant j_0}$ be the family of all parameter values for the new path Λ_1 such that $\Lambda_1(u_j)$ is one of the points $\Gamma_k(s_0^k)$ (we recall that we include also the new paths and points corresponding to $X[U_k^C(X), V_k^-]$ and

$X[V_k^+, T_k^C(X)]$ for all $k \geqslant 1$). Let us suppose that u_j's are labeled so that $u_j \leqslant u_{j+1}$ for all j and let us rename u_∞ and call it u_{j_0+1}. As a result of our construction, there is a partition of integers between (and including) 0 and j_0 into two disjoint sets J_1 and J_2 such that

$$\bigcup_{j \in J_1} A_1[u_j, u_{j+1}] \cap \bigcup_{j \in J_2} A_1[u_j, u_{j+1}] = \bigcup_{1 \leqslant j \leqslant j_0} \{A_1(u_j)\}.$$

Moreover, for one of those sets, say for J_1, we have

$$\bigcup_{j \in J_1} A_1[u_j, u_{j+1}] \subset C^+ \quad \text{while} \quad \bigcup_{j \in J_2} A_1[u_j, u_{j+1}] \nsubseteq C^+.$$

Let

$$M = \bigcup_{\substack{j,k \in J_1 \\ j \neq k}} A_1[u_j, u_{j+1}] \cap A_1[u_k, u_{k+1}].$$

Note that M is a subset of the set of double points of X. Le Gall [16] has proved that the exact Hausdorff ϕ_2-measure for the double points of the 2-dimensional Brownian motion is given by

$$\phi_2(u) = u^2 \big(\log(1/u) \log\log\log(1/u)\big)^2.$$

Moreover, he has shown that for every fixed $\delta > 0$ the set of all points x such that $x = X(t_1) = X(t_2)$, $|t_1 - t_2| > \delta$ and $t_1, t_2 \in [0, 1]$ has a finite ϕ_2-measure. It is easy to see that there exists $\delta > 0$ such that for every pair of $j, k \in J_1$ the paths $A_1[u_j, u_{j+1}]$ and $A_1[u_k, u_{k+1}]$ are assembled from pieces of the Brownian path which correspond to subsets of the time axis whose distance is greater than some $\delta > 0$. Thus $\phi_2\text{-}m(M) < \infty$ a.s. Recall that

$$\phi(u) = u^2 \log(1/u) \log\log\log(1/u)$$

gives the exact Hausdorff measure for the whole Brownian path and note that $\phi_2(u)/\phi(u) \to \infty$ as $u \to 0$. It follows that every set whose ϕ_2-measure is finite must have zero ϕ-measure. This remark applies to M. Hence, it is possible to cover the set M with a family of open discs $\{B_j\}$ such that the ϕ-measure of $\bigcup_j B_j \cap X[0, 1]$ is arbitrarily small, say, less than $\varepsilon_2 > 0$. Since the set M is compact and B_j's are open, we can assume that the family $\{B_j\}$ is finite. It is easy to see that we can slightly enlarge the set $\bigcup_j B_j$ so that it becomes the union of a finite number of open disjoint sets A_j whose boundaries belong to $\mathscr{K}$ and such that

$$\phi\text{-}m\Big(\bigcup_j A_j \cap X[0, 1]\Big) < 2\varepsilon_2.$$

For every j find an open A_j^+ which contains the closure of A_j. We choose A_j^+'s so that they are disjoint, have boundaries in $\mathscr{K}$ and

$$\phi\text{-}m\Big(\bigcup_j A_j^+ \cap X[0, 1]\Big) < 3\varepsilon_2.$$

Let $A = \bigcup_j A_j$ and $A^+ = \bigcup_j A_j^+$. Recall the notation introduced in (2.2). Note that there is only a finite number of k with $T_k^A(\Lambda_1) < \infty$. By applying Lemma 2.5 successively to all A_j we construct for every $k \in J_1$ a continuous path $\Gamma_k^1(t)$ with endpoints $\Lambda_1(u_k)$ and $\Lambda_1(u_{k+1})$ whose range is contained in $\Lambda_1[u_k, u_{k+1}]$ and which contains all the points of $\Lambda_1[u_k, u_{k+1}]$ except possibly $\Lambda_1(u)$ for some $u \in (U_j^A(\Lambda_1), T_j^A(\Lambda_1)), j \geqslant 1$. Moreover, each Γ_k^1 has property (P) and

$$\Gamma_k^1[U_j^A(\Gamma_k^1), T_j^A(\Gamma_k^1)] \cap \Gamma_k^1[U_i^A(\Gamma_k^1), T_i^A(\Gamma_k^1)] = \emptyset \qquad \text{for all } i \neq j.$$

We construct a new path by replacing every $\Lambda_1[u_k, u_{k+1}]$ with Γ_k^1 before applying the next lemma.

Then Lemma 2.8 below shows that there is a continuous (P)-path, say, $\{X^1(t), t \in [t_1, t_2]\}$, with the same endpoints as Λ_1 which contains all the points of $\Lambda_1 \backslash A^+$ and such that

$$X^1[U_j^C(X^1), T_j^C(X^1)] \cap X^1[U_i^C(X^1), T_i^C(X^1)] = \emptyset \qquad \text{for } i \neq j.$$

Every path $X^1[U_j^C(X^1), T_j^C(X^1)]$ contains two points of the form $\Lambda_1(u_k)$ or $\Lambda_1(u_{k+1})$, $k \in J_1$. Let Φ_j denote the part of $X^1[U_j^C(X^1), T_j^C(X^1)]$ between these points. If we remove the endpoints of Φ_j, it becomes disconnected from the rest of the path. Every continuous self-avoiding path within $X^1[t_1, t_2]$ which has the same endpoints as X^1 and which contains at least 3 distinct points of Φ_j must pass through both endpoints of Φ_j. Let $\delta_1 = \min_j \phi\text{-}m(\Phi_j)$. It is easy to see that $\delta_1 > 0$. If a path inside $X^1[t_1, t_2]$ has ϕ-measure greater than $\phi\text{-}m(X^1[t_1, t_2]) - \delta_1/2$, then it must contain at least 3 points of every Φ_j and, therefore, it must pass through all endpoints of all Φ_j's.

Suppose that we have X^k. Find the smallest j such that C_j intersects X^k and construct X^{k+1} from X^k relative to C_j in the same way we constructed X^1 from X using C_{k_0}.

We will impose some more conditions on the parameters of our construction.

Suppose that X^k is parametrized by the time interval $[v_1^k, v_2^k]$. Let $\{y_1^k, y_2^k, \ldots, y_{m_k}^k\}$ be the family of all points on the path of X^k which are defined in the same way as $\Lambda_1(u_k)$ and $\Lambda_1(u_{k+1})$, $k \in J_1$, have been defined in the construction of X^1. We will argue that if we choose the parameters of our construction in a suitable way, then

$$\bigcup_{k \leqslant j} \{y_1^k, y_2^k, \ldots, y_{m_k}^k\} \subset X^j[v_1^j, v_2^j] \tag{2.3}$$

for every j. Let δ_k be the minimum of ϕ-measures of all pieces of X^k joining pairs of distinct points in $\bigcup_{k \leqslant j} \{y_1^k, y_2^k, \ldots, y_{m_k}^k\}$. Again, it is rather easy to see that $\delta_k > 0$. Then we choose $\gamma_k, \alpha_k, \beta_k$ and x_k so that the ϕ-measure of the intersection of X^{k-1} with $B(x_k, \gamma_k + \beta_k) \backslash B(x_k, \gamma_k - \alpha_k)$ is less than $\min(\varepsilon_1/2^k, \delta_{k-1}/4)$. See the construction of X^1 for the justification of this step. In the construction of X^k, we choose sets $A_{j,k}$ and $A_{j,k}^+$ analogous to A_j's and A_j^+'s sufficiently small so that

$$\phi\text{-}m\Big(\bigcup_j A_{j,k}^+ \cap X[0,1]\Big) < \min(3\varepsilon_2/2^k, \delta_{k-1}/4).$$

Hence the total ϕ-measure of all pieces removed in the k-th step is less than $\delta_{k-1}/2$ and, therefore, (2.3) follows just like in the case of X^1.

Choose any $\varepsilon > 0$. The pieces of the original Brownian path which are removed at the k-th stage are contained in the union of $B(x_k, \gamma_k+\beta_k)\backslash B(x_k, \gamma_k-\alpha_k)$ and $\bigcup_j A_{j,k}^+$. Hence the ϕ-measure of all pieces removed at all stages is less than $\sum_k \varepsilon_1/2^k + 3\varepsilon_2/2^k$ and we can assume that this number is smaller than ε by choosing some ε_1 and ε_2 smaller than $\varepsilon/100$.

Note that

$$X^{k+1}[v_1^{k+1}, v_2^{k+1}] \subset X^k[v_1^k, v_2^k]$$

for every k. Hence $U \overset{\text{df}}{=} \bigcap_k X^k[v_1^k, v_2^k]$ is a compact set whose ϕ-measure differs by no more than ε from that of the original Brownian path $X[0,1]$. It remains to show that U is a self-avoiding continuous path.

Recall the family $\{y_1^k, y_2^k, \ldots, y_{m_k}^k\}$. It follows from (2.3) that

$$\mathcal{Y} \overset{\text{df}}{=} \bigcup_k \{y_1^k, y_2^k, \ldots, y_{m_k}^k\} \subset U.$$

Since

$$\{y_1^k, y_2^k, \ldots, y_{m_k}^k\} \subset B(x_k, \gamma_k+\beta_k)\backslash B(x_k, \gamma_k-\alpha_k),$$

$\{\gamma_k\}_{k\geq 1}$, $\{\alpha_k\}_{k\geq 1}$ and $\{\beta_k\}_{k\geq 1}$ are decreasing to 0, and every point in the plane (except $X(0)$ and $X(1)$) is covered by infinitely many discs $B(x_k, (\gamma_k-\alpha_k)/2)$, the set $\mathcal{Y}$ is dense in U and, therefore, $\mathcal{Y}$ is infinite.

Let us (arbitrarily) order points of $\mathcal{Y}$, say $\mathcal{Y} = \{z_1, z_2, \ldots\}$. Recall that every path X^k visits all points of $\mathcal{Y}$. Observe that every point z_j is visited only once by X^k if k is sufficiently large. Find an infinite subsequence of $\{X^k\}$ such that the paths in this sequence visit z_1 and z_2 only once and in the same order. Then find a further subsequence so that the paths in this subsequence visit z_1, z_2 and z_3 only once and in the same order. By repeating the procedure and then using the diagonal method we can find a subsequence of $\{X^k\}$ such that all X^k in the subsequence with $k > \bar{k}(n)$ visit all points $z_1, z_2, \ldots, z_n$ in the same order. We will write $z_k \prec z_j$ if z_k is visited before z_j by the paths in the tail of our final subsequence.

Let us define a mapping F from $\mathcal{Y}$ onto a subset of $[0,1]$. We start with $F(z_1) = 1/2$. Suppose that we have defined $F(z_j)$ for all $j < k$. If $z_j \prec z_k$ for all $j < k$, then $F(z_k) \overset{\text{df}}{=} [1+\max_{j<k} F(z_j)]/2$. Similarly, if $z_k \prec z_j$ for all $j < k$, then $F(z_k) \overset{\text{df}}{=} \min_{j<k} F(z_j)/2$. If none of these conditions holds, find j_1 and j_2 so

that $z_{j_1} \prec z_k \prec z_{j_2}$ but no z_j, $j < k$, lies between z_{j_1} and z_{j_2}. In this case let $F(z_k) \overset{\mathrm{df}}{=} [F(z_{j_1}) + F(z_{j_2})]/2$. Note that $G \overset{\mathrm{df}}{=} F^{-1}$ is a univalent function. Let $\mathscr{I}$ be the smallest closed interval containing $F(\mathscr{Y})$. We will argue that $\mathscr{J} \overset{\mathrm{df}}{=} F(\mathscr{Y})$ is dense in $\mathscr{I}$ and that G can be extended to a continuous function on $\mathscr{I}$.

Suppose that there is a non-degenerate interval $(a, b) \subset \mathscr{I}$ such that $F(\mathscr{Y}) \cap (a, b) = \emptyset$. This easily implies that there exist z_j and z_k such that there is no z_m with $z_j \prec z_m \prec z_k$. In view of (iii), there are discs $B(x_n, \gamma_n + \beta_n)$ and $B(x_n, (\gamma_n - \alpha_n)/2)$ corresponding to the n-th stage of the construction such that z_k lies outside the bigger disc but z_j lies inside the smaller one. According to the construction, any continuous path within the range of X^n which passes through z_j and z_k must pass through $B(x_n, \gamma_n + \beta_n) \backslash B(x_n, \gamma_n - \alpha_n)$, and so it must pass through one of the points $\{y_1^n, y_2^n, \ldots, y_{m_n}^n\}$ after visiting z_j and before visiting z_k. This shows that there must exist z_m with $z_j \prec z_m \prec z_k$. We conclude that $F(\mathscr{Y})$ is dense in $\mathscr{I}$.

Next we prove that G can be continuously extended to $\mathscr{I}$. There are two possible reasons why this may fail. The first one is that there may be a discontinuity of the first kind at a point $s \in \mathscr{I}$. Suppose that

$$x_1 \overset{\mathrm{df}}{=} \lim_{\substack{t \to s- \\ t \in \mathscr{J}}} G(t) \neq \lim_{\substack{t \to s+ \\ t \in \mathscr{J}}} G(t) \overset{\mathrm{df}}{=} x_2. \tag{2.4}$$

We use an argument similar to the one in the previous paragraph. Find discs $B(x_n, \gamma_n + \beta_n)$ and $B(x_n, (\gamma_n - \alpha_n)/2)$ such that x_1 lies outside the bigger disc and x_2 lies inside the smaller one. In view of (2.4), there are t_1 and t_2 arbitrarily close to s and such that $G(t_1)$ and $G(t_2)$ are arbitrarily close to x_1 and x_2. A continuous path passing through $G(t_1)$ and $G(t_2)$ must also pass through $\{y_1^n, y_2^n, \ldots, y_{m_n}^n\}$ between its visits to $G(t_1)$ and $G(t_2)$. Hence we can find t_3 arbitrarily close to s with $G(t_3) \in \in \{y_1^n, y_2^n, \ldots, y_{m_n}^n\}$. We obtain a contradiction with (2.4) which shows that a discontinuity of the first kind is not possible.

Next we discuss the possibility of the discontinuity of the second kind. Suppose that G does not have a left or right limit at a point s of $\mathscr{I}$. Suppose, for example, that the left limit does not exist. Then we see using compactness that there are infinite increasing sequences $\{s_j\}$ and $\{t_j\}$ converging to s and such that $\lim_j G(s_j) = x_1$, $\lim_j G(t_j) = x_2 \neq x_1$ and $s_j < t_j < s_{j+1}$ for all j. Again, find discs $B(x_n, \gamma_n + \beta_n)$ and $B(x_n, (\gamma_n - \alpha_n)/2)$ such that x_1 lies outside the bigger disc and x_2 lies inside the smaller one. For large k, the path X^k has to pass alternatively through small neighborhoods of x_1 and x_2. Hence, it must also pass through $\{y_1^n, y_2^n, \ldots, y_{m_n}^n\}$. Since the path can visit the last set only m_n times, we obtain a contradiction with our assumption that the sequences $\{s_j\}$ and $\{t_j\}$ are infinite. This completes the proof of the claim that G can be continuously extended to $\mathscr{I}$.

Recall that $\mathscr{Y}$ is dense in U. Since G is continuous and its range contains $\mathscr{Y}$, it maps $\mathscr{I}$ onto U.

The last thing we have to prove is that $G(s) \neq G(t)$ if $s \neq t$. We will assume that this is not true and prove that this assumption leads to a contradiction. Assume that $G(s) = G(t) \stackrel{\text{df}}{=} x_1$ for some $s < t$. It follows from the definition of F that G is not constant on any interval. Hence, we must have $G(u) = x_2 \neq x_1$ for some $u \in (s, t)$. Suppose that the discs $B(x_n, \gamma_n + \beta_n)$ and $B(x_n, (\gamma_n - \alpha_n)/2)$ are such that x_1 lies outside the bigger disc and x_2 lies inside the smaller one. The inductive procedure which produced X^n from X^{n-1} insures that if a piece of a path X^k, $k > n$, starts at a point inside $B(x_n, (\gamma_n - \alpha_n)/2)$, leaves $B(x_n, \gamma_n + \beta_n)$, comes back to $B(x_n, (\gamma_n - \alpha_n)/2)$ and does not visit any point of $\{y_1^n, y_2^n, \ldots, y_{m_n}^n\}$ more than once cannot come within some positive distance, say, $\varrho > 0$, of its initial part inside $B(x_n, (\gamma_n - \alpha_n)/2)$. However, the definition of F and the assumption that x_1 is visited twice by G show that for every $r \in (0, \varrho/4)$ there are some points $x_3, x_4 \in B(x_1, r) \cap \mathscr{Y}$ such that for large k the paths X^k visit x_2 between the visits to x_3 and x_4. This is a contradiction. The proof is complete. ∎

Recall the definition of a local double cut point from the proof of Theorem 1′.

LEMMA 2.2. *Two-dimensional Brownian path* $\{X(t), t \in [0, 1]\}$ *does not have local double cut points a.s.*

Proof. Suppose that X has a local double cut point x. Then the definition says that there exist $s_1, u_1 \in (0, 1)$ and a small disc $B(x, r)$ with the following properties. Let

$$s_0 = \inf\{t\colon X[t, s_1] \subset B(x, r)\}, \qquad s_2 = \sup\{t\colon X[s_1, t] \subset B(x, r)\},$$

$$u_0 = \inf\{t\colon X[t, u_1] \subset B(x, r)\}, \qquad u_2 = \sup\{t\colon X[u_1, t] \subset B(x, r)\}.$$

Let D_1, D_2, D_3 and D_4 be the connected components of $B(x, r) \setminus (\Lambda[s_0, s_2] \cup \cup \Lambda[u_0, u_2])$ which touch the boundary of $B(x, r)$. Then $X(s_1) = X(u_1)$, $s_2 < u_0$ or $u_2 < s_0$ and at least two of D_j's contain x in their closure.

Suppose that $\bar{D}_1$ and $\bar{D}_2$ contain x, let the endpoints of $\bar{D}_1 \cap \partial B(x, r)$ be called y_1 and y_2 and let y_3 and y_4 have the analogous meaning for D_2. We can assume that y_1, y_2, y_3 and y_4 come in this order on the boundary (although they are not necessarily disjoint).

We will discuss three possible situations. First, suppose that $y_1 = X(s_0)$, $y_2 = X(s_2)$, $y_3 = X(u_0)$ and $y_4 = X(u_2)$. In this situation, the argument given in the proof of Theorem 1.5 (ii) of Burdzy and Lawler [6] applies and yields the lemma.

The next case is when $y_1 = X(s_0)$, $y_2 = X(u_0)$, $y_3 = X(u_2)$ and $y_4 = X(s_2)$. Suppose that $s_2 < u_0$. By conditioning on the value of $X(s_2)$ we may assume that $\{X(t), t \in [s_0, s_2]\}$ and $\{X(t), t \in [u_0, u_2]\}$ are independent. Since u_1 is the first hitting time of the first path by the second path, we may apply the strong Markov property at u_1. The path $\{X(t), t \in [u_1, u_2]\}$ contains infinitely many small closed loops around its starting point x and this shows that x cannot belong to the closure of $\bar{D}_1$ and $\bar{D}_2$.

The same argument applies when $y_1 = X(u_0)$, $y_2 = X(s_0)$, $y_3 = X(s_2)$ and $y_4 = X(u_0)$. Other situations can be handled in a similar manner. ∎

Lemma 2.3. *Suppose that $\{\Lambda(t), t \in [t_1, t_2]\}$ has property* (P) *and K_1 and K_2 are disjoint closed sets with $\Lambda(t_j) \in K_j$ for $j = 1, 2$. Assume that each K_j is the trace of a* (P)*-path. Suppose that for some t_0 we have $\Lambda[t_1, t_0] \cap K_2 = \emptyset$ and $\Lambda[t_0, t_2] \cap K_1 = \emptyset$. Then there exists a path $\{\Gamma(t), t \in [s_1, s_2]\}$ with property* (P) *which satisfies the following conditions*:

(i) $\Gamma(s_j) = \Lambda(t_j)$ *for* $j = 1, 2$,

(ii) $\Gamma[s_1, s_2] \subset \Lambda[t_1, t_2]$,

(iii) Γ *has a cut point* s_0 *such that*

$$(K_1 \cup \Gamma[s_1, s_0]) \cap (K_2 \cup \Gamma[s_0, s_2]) = \{\Gamma(s_0)\}.$$

Proof. Let $\tilde{K}_j = \{x: \operatorname{dist}(x, K_j) \leqslant \varepsilon\}$ for $j = 1, 2$. By compactness, we can choose $\varepsilon > 0$ so small that $\Lambda[t_1, t_0] \cap \tilde{K}_2 = \emptyset$ and $\Lambda[t_0, t_2] \cap \tilde{K}_1 = \emptyset$. Let

$$S = \inf\{t: \Lambda[t, t_2] \subset \tilde{K}_1^c\}, \qquad T = \inf\{t: \Lambda(t) \in \Lambda[S, t_2]\},$$

$$U = \sup\{t: \Lambda(t) = \Lambda(T)\}.$$

Let us define a continuous path Γ on an interval $[s_1, s_2] = [t_1, t_2 - U + T]$ by

$$\Gamma(t) = \begin{cases} \Lambda(t) & \text{if } t \in [t_1, T], \\ \Lambda(t+U-T) & \text{if } t \in [T, t_2 - U + T]. \end{cases}$$

Note that Γ has property (P), it has the same endpoints as Λ, its trace is a subset of $\Lambda[t_1, t_2]$ and it has a cut point $\Gamma(T)$. Observe that $\Gamma(T, s_2) \cap \cap \tilde{K}_1 \subset \{\Lambda(T)\} \subset \partial \tilde{K}_1$, and so $\Gamma(T, s_2) \cap K_1 = \emptyset$. It remains to show that $\Gamma(s_1, T) \cap K_2 = \emptyset$.

It follows from the definition of S that $\Lambda(S) \in \tilde{K}_1$, and so we must have $S \leqslant t_0$. Since $T \leqslant S$, we have $T \leqslant t_0$ and

$$\Gamma(s_1, T) \cap K_2 = \Lambda(t_1, T) \cap K_2 \subset \Lambda(t_1, t_0) \cap K_2 = \emptyset. \ ∎$$

Lemma 2.4. *Suppose that D is an open set homeomorphic to a disc and $\partial D \in \mathscr{K}$. Suppose that $\{\Gamma_1(s), s \in [s_1^1, s_2^1]\}$ and $\{\Gamma_2(s), s \in [s_1^2, s_2^2]\}$ are paths with property* (P) *which lie inside D except that their endpoints belong to ∂D. Moreover, let us suppose that their endpoints are distinct and they are arranged on the boundary in the following order: $\Gamma_1(s_1^1)$, $\Gamma_2(s_1^2)$, $\Gamma_1(s_2^1)$ and $\Gamma_2(s_2^2)$. Then there exist* (P)*-paths $\{\Psi_1(t), t \in [t_1^1, t_2^1]\}$ and $\{\Psi_2(t), t \in [t_1^2, t_2^2]\}$ such that $\Psi_1(t_1^1) = \Gamma_1(s_1^1)$, $\Psi_1(t_2^1) = \Gamma_2(s_1^2)$, $\Psi_2(t_1^2) = \Gamma_1(s_2^1)$, $\Psi_2(t_2^2) = \Gamma_2(s_2^2)$,*

$$\Psi_1[t_1^1, t_2^1] \cup \Psi_2[t_1^2, t_2^2] \subset \Gamma_1[s_1^1, s_2^1] \cup \Gamma_2[s_1^2, s_2^2]$$

and

$$\Psi_1[t_1^1, t_2^1] \cap \Psi_2[t_1^2, t_2^2] = \emptyset.$$

Proof. Let D_1 be the component of $D \backslash \Gamma_1[s_1^1, s_2^1] \cup \Gamma_2[s_1^2, s_2^2]$ whose boundary contains the piece of ∂D between $\Gamma_1(s_1^1)$ and $\Gamma_2(s_1^2)$ and which does not contain the other endpoints of Γ_j's. Likewise, let D_2 be the component of $D \backslash \Gamma_1[s_1^1, s_2^1] \cup \Gamma_2[s_1^2, s_2^2]$ which contains in its closure the piece of ∂D between $\Gamma_1(s_2^1)$ and $\Gamma_2(s_2^2)$ but no other endpoints.

The closures $\bar{D}_1$ and $\bar{D}_2$ are disjoint because (P)-paths have no local double cut points (see Lemma 2.2 and property (P3) in the proof of Theorem 1').

Now we can find open sets N_1, N_1^+, N_2 and N_2^+ whose boundaries belong to $\mathscr{K}$ and such that

$$\bar{D}_1 \subset N_1 \subset \bar{N}_1 \subset N_1^+, \quad \bar{D}_2 \subset N_2 \subset \bar{N}_2 \subset N_2^+, \quad \text{and} \quad \bar{N}_1^+ \cap \bar{N}_2^+ = \emptyset.$$

Recall the notation introduced in (2.2). Let $\{\Phi_k(u), u \in [u_1^k, u_2^k]\}_{1 \leqslant k \leqslant k_0}$ be the family of all paths of the form $\{\Gamma_1(s), s \in [S_j^{N_1}(\Gamma_1), T_j^{N_1}(\Gamma_1)]\}$ or $\{\Gamma_2(s), s \in [S_j^{N_1}(\Gamma_2), T_j^{N_1}(\Gamma_2)]\}$. The set $\bigcup_k \Phi_k[u_1^k, u_2^k]$ is connected and contains $\Gamma_1(s_1^1)$ and $\Gamma_2(s_1^2)$. Hence we may assume that Φ_k's are ordered in such a way that $\Gamma_1(s_1^1) \in \Phi_1[u_1^1, u_2^1]$, $\Gamma_2(s_1^2) \in \Phi_{k_1}[u_1^{k_1}, u_2^{k_1}]$ for some $k_1 \leqslant k_0$ and

$$\Phi_k[u_1^k, u_2^k] \cap \Phi_{k+1}[u_1^{k+1}, u_2^{k+1}] \neq \emptyset \qquad \text{for all } k < k_1.$$

Let $\tilde{v}_1^1 \in [u_1^1, u_2^1]$ be such that $\Phi_1(\tilde{v}_1^1) = \Gamma_1(s_1^1)$. If $\Phi_1[u_1^1, \tilde{v}_1^1] \cap \Phi_2[u_1^2, u_2^2] \neq \emptyset$, then let

$$v_2^1 = \sup\{v < \tilde{v}_1^1 \colon \Phi_1(v) \in \Phi_2[u_1^2, u_2^2]\}.$$

In this case we set $v_1^1 = \inf\{v > v_2^1 \colon \Phi_1(v) = \Gamma_1(s_1^1)\}$. The other possibility is that $\Phi_1[\tilde{v}_1^1, u_2^1] \cap \Phi_2[u_1^2, u_2^2] \neq \emptyset$. Then

$$v_2^1 = \inf\{v > \tilde{v}_1^1 \colon \Phi_1(v) \in \Phi_2[u_1^2, u_2^2]\} \text{ and } v_1^1 = \sup\{v < v_2^1 \colon \Phi_1(v) = \Gamma_1(s_1^1)\}.$$

Find $\tilde{v}_1^2 \in [u_1^2, u_2^2]$ such that $\Phi_2(\tilde{v}_1^2) = \Phi_1(v_2^1)$. Now we repeat the same procedure as in the previous paragraph to obtain v_1^2 and v_2^2. If $\Phi_2[u_1^2, \tilde{v}_1^2] \cap$ $\cap \Phi_3[u_1^3, u_2^3] \neq \emptyset$, then let

$$v_2^2 = \sup\{v < \tilde{v}_1^2 \colon \Phi_2(v) \in \Phi_3[u_1^3, u_2^3]\}.$$

In this case we set $v_1^2 = \inf\{v > v_2^2 \colon \Phi_2(v) = \Phi_1(v_2^1)\}$. The other possibility is that $\Phi_2[\tilde{v}_1^2, u_2^2] \cap \Phi_3[u_1^3, u_2^3] \neq \emptyset$. Then

$$v_2^2 = \inf\{v > \tilde{v}_1^2 \colon \Phi_2(v) \in \Phi_3[u_1^3, u_2^3]\} \text{ and } v_1^2 = \sup\{v < v_2^2 \colon \Phi_2(v) = \Phi_1(v_2^1)\}.$$

An inductive argument produces sequences of points $\{v_1^k\}$ and $\{v_2^k\}$ such that $\Phi_1(v_1^1) = \Gamma_1(s_1^1)$, $\Phi_{k_1}(v_1^{k_1}) = \Gamma_2(s_1^2)$ and $\Phi(v_2^k) = \Phi(v_1^{k+1})$ for all $1 \leqslant k \leqslant k_1 - 1$. The pieces $\{\Phi_k(v), v \in [v_1^k, v_2^k]\}$ can be spliced together to obtain a (P)-path with endpoints $\Gamma_1(s_1^1)$ and $\Gamma_2(s_1^2)$ which lies in N_1^+. An analogous construction gives a (P)-path in N_2^+ joining $\Gamma_1(s_2^1)$ and $\Gamma_2(s_2^2)$. The paths must be disjoint as N_1^+ and N_2^+ are disjoint. ■

Remark 2.1. It is clear from the proof of Lemma 2.4 that the new disjoint paths may be constructed in every case when the endpoints of Γ's alternate. We can also do it in the case when the endpoints do not alternate but the original paths intersect. It should be mentioned in connection with the last claim that an easy argument based on the strong Markov property shows that two pieces of a Brownian path cannot touch at a single point unless it is an endpoint of one of them (see the proof of Lemma 2.2).

DEFINITION 2.1. Suppose that Γ_1 and Γ_2 are continuous paths in an open simply connected set whose endpoints lie on its boundary. We will say that they *intersect for topological reasons* if the endpoints of Γ_1 alternate on the boundary with those of Γ_2.

Remark 2.2. Suppose that $\Gamma_1, \ldots, \Gamma_j$ are continuous paths in an open simply connected set A whose endpoints lie on its boundary. Moreover, assume that no pair of these paths must intersect for topological reasons. We will argue that a repeated application of Remark 2.1 will produce a sequence of disjoint paths $\Gamma_1^*, \ldots, \Gamma_j^*$ such that for each j the endpoints of Γ_j^* are the same as those of Γ_j.

Suppose we apply Remark 2.1 to a pair of paths Γ_n and Γ_m which do not have to intersect for topological reasons and we obtain a pair of disjoint paths Γ_n^1 and Γ_m^1 such that the endpoints of Γ_j^1 are the same as those of Γ_j for $j = n, m$. Then we will say that we applied *transformation* $\mathscr{D}$ to Γ_n and Γ_m.

The endpoints of a path Γ_k divide ∂A into two arcs. It is easy to see that there is a path Γ_k such that one of these arcs contains no endpoints of other Γ_m's. Assume without loss of generality that $k = 1$. Apply operation $\mathscr{D}$ to Γ_1 and Γ_2 to obtain Γ_1^1 and Γ_2^1. Then apply $\mathscr{D}$ to Γ_1^1 and Γ_3 to get Γ_1^2 and Γ_3^1. Proceed inductively – apply $\mathscr{D}$ to Γ_1^n and Γ_{n+2} for all $n \leqslant j-2$ to obtain Γ_1^{n+1} and Γ_{n+2}^1. An examination of the construction given in the proof of Lemma 2.4 shows that Γ_2^1 is disjoint not only from Γ_1^1 but also from Γ_1^n for every $n \geqslant 1$. More generally, Γ_i^1 is disjoint from Γ_1^m for all $m \geqslant i-1$. Let $\Gamma_1^* = \Gamma_1^{j-1}$. This curve is disjoint from $j-1$ curves $\Gamma_2^1, \Gamma_3^1, \ldots, \Gamma_j^1$. Note that these curves belong to a simply connected component A_1 of $A \backslash \Gamma_1^*$. Although A_1 is not a disc, Lemma 2.4 and Remark 2.1 can be applied as they have a topological nature. We have reduced the problem to that of $j-1$ curves, and hence we can achieve our goal in a finite number of steps.

Remark 2.3. We will describe a splicing operation for a (P)-loop $\{\Lambda(u), u \in [u_1, u_2]\}$ (here $\Lambda(u_1) = \Lambda(u_2)$) and a (P)-path $\{\Gamma(u): u \in [s_1, s_2]\}$. Let us assume that these paths intersect. Let $t_1 = \inf\{u: \Gamma(u) \in \Lambda\}$ and $t_2 = \inf\{u: \Lambda(u) = \Gamma(t_1)\}$. Then we create a single continuous (P)-path by assembling the segments of Λ and Γ in the following order: $\Gamma[s_1, t_1]$, $\Lambda[t_2, u_2]$, $\Lambda[u_1, t_2]$ and $\Gamma[t_1, s_2]$. A similar remark applies to a pair of loops — in such a case we would obtain a single loop.

LEMMA 2.5. *Suppose that* $\{\Lambda(u), u \in [u_1, u_2]\}$ *is a* (P)*-path, A is a set homeomorphic to a disc, and A^+ is another set homeomorphic to a disc which contains the closure of A. Assume that the endpoints of Λ lie outside the closure of A^+. Then there exists a* (P)*-path Γ with the same endpoints as Λ, whose range is contained in $\Lambda[u_1, u_2]$ and which contains all the points of $\Lambda[u_1, u_2]$ except possibly $\Lambda(u)$ for some $u \in (U_j^A(\Lambda), T_j^A(\Lambda))$, $j \geqslant 1$. Moreover,*

$$\Gamma[U_j^A(\Gamma), T_j^A(\Gamma)] \cap \Gamma[U_i^A(\Gamma), T_i^A(\Gamma)] = \emptyset \quad \textit{for all } i \neq j.$$

Proof. We will modify the path Λ inductively. Let us start by replacing the two pieces $\Lambda[U_1^A(\Lambda), T_1^A(\Lambda)]$ and $\Lambda[U_2^A(\Lambda), T_2^A(\Lambda)]$ of Λ by two new disjoint paths $\{\Psi_1(t), t \in [t_1^1, t_2^1]\}$ and $\{\Psi_2(t), t \in [t_1^2, t_2^2]\}$. Suppose for a moment that $\Lambda(U_1^A(\Lambda))$, $\Lambda(U_2^A(\Lambda))$, $\Lambda(T_1^A(\Lambda))$, and $\Lambda(T_2^A(\Lambda))$ come in this order on the boundary of A^+. Then according to Lemma 2.4 we may choose Ψ_1 and Ψ_2 so that $\Psi_1(t_1^1) = \Lambda(U_1^A(\Lambda))$, $\Psi_1(t_2^1) = \Lambda(U_2^A(\Lambda))$, $\Psi_2(t_1^2) = \Lambda(T_1^A(\Lambda))$, $\Psi_2(t_2^2) = \Lambda(T_2^A(\Lambda))$, and $\Psi_1 \cup \Psi_2$ is a subset of $\Lambda[U_1^A(\Lambda), T_1^A(\Lambda)] \cup \Lambda[U_2^A(\Lambda), T_2^A(\Lambda)]$. Then we connect some pieces of Λ and Ψ_1 and Ψ_2 in the following order: $\Lambda[u_1, U_1^A(\Lambda)]$, $\Psi_1[t_1^1, t_2^1]$, time-reversed $\Lambda[T_1^A(\Lambda), U_2^A(\Lambda)]$, $\Psi_2[t_1^2, t_2^2]$, and $\Lambda[T_2^A(\Lambda), u_2]$. Let Λ_1 be the name of the resulting path. Note that the maximum k such that $S_k^A(\Lambda_1) < \infty$ is the same as that for $S_k^A(\Lambda) < \infty$. We also have

$$\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)] \cap \Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)] = \emptyset.$$

Some caution has to be exercised when choosing the endpoints of Ψ_1 and Ψ_2, for example, if we had chosen

$$\Psi_1(t_1^1) = \Lambda(U_1^A(\Lambda)), \quad \Psi_1(t_2^1) = \Lambda(T_2^A(\Lambda)),$$
$$\Psi_2(t_1^2) = \Lambda(T_1^A(\Lambda)), \quad \Psi_2(t_2^2) = \Lambda(U_2^A(\Lambda)),$$

then we could assemble these pieces together with some pieces of Λ to make a (not necessarily disjoint) curve and a closed loop — this is not what we want. The modifications of the argument needed in the case when $\Lambda(U_1^A(\Lambda))$, $\Lambda(U_2^A(\Lambda))$, $\Lambda(T_1^A(\Lambda))$, and $\Lambda(T_2^A(\Lambda))$ come in a different order on the boundary of A^+ are quite obvious (see Remark 2.1).

Next we will modify the path of Λ_1. At this point, we will consider only

$$\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)], \quad \Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)] \quad \text{and} \quad \Lambda_1[U_3^A(\Lambda_1), T_3^A(\Lambda_1)].$$

Recall that the first two pieces of the path do not intersect. If the third one intersects only one of the first two, we apply the same method as in the previous paragraph to replace the pair of intersecting pieces with new disjoint paths. The resulting curve Λ_2 will have the property that

$$\Lambda_2[U_1^A(\Lambda_2), T_1^A(\Lambda_2)], \quad \Lambda_2[U_2^A(\Lambda_2), T_2^A(\Lambda_2)] \quad \text{and} \quad \Lambda_2[U_3^A(\Lambda_2), T_3^A(\Lambda_2)]$$

are disjoint.

Suppose that $\Lambda_1[U_3^A(\Lambda_1), T_3^A(\Lambda_1)]$ intersects both $\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)]$ and $\Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)]$. First replace $\Lambda_1[U_3^A(\Lambda_1), T_3^A(\Lambda_1)]$ and $\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)]$ with disjoint Ψ_3 and Ψ_4 just like we did it in the first step. Let the new path be called $\hat{\Lambda}_2$. The key observation is that we may choose Ψ_3 and Ψ_4 so that one of them is disjoint from $\Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)]$. This is due to the fact that one of Ψ_3 or Ψ_4 must lie "on the opposite side" of $\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)]$ than $\Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)]$ because $\Lambda_1[U_1^A(\Lambda_1), T_1^A(\Lambda_1)]$ and $\Lambda_1[U_2^A(\Lambda_1), T_2^A(\Lambda_1)]$ are disjoint (see that proof of Lemma 2.4). For this to happen one has to make sure that the sets N_1^+ and N_2^+ in the proof of Lemma 2.4 are sufficiently small. We conclude that only two of the paths

$$\hat{\Lambda}_2[U_1^A(\hat{\Lambda}_2), T_1^A(\hat{\Lambda}_2)], \quad \hat{\Lambda}_2[U_2^A(\hat{\Lambda}_2), T_2^A(\hat{\Lambda}_2)] \quad \text{and} \quad \hat{\Lambda}_2[U_3^A(\hat{\Lambda}_2), T_3^A(\hat{\Lambda}_2)]$$

can intersect. We replace those two in the usual way with two new disjoint paths to obtain Λ_2 for which all three analogous pieces are disjoint. Again, the maximum k such that $S_k^A(\Lambda_2) < \infty$ is the same as that for $S_k^A(\Lambda) < \infty$.

We proceed by induction — we replace some or all curves $\Lambda_j[U_m^A(\Lambda_j), T_m^A(\Lambda_j)]$, $m \leqslant j+2$, in order to obtain a new path Λ_{j+1} for which all pieces $\Lambda_{j+1}[U_m^A(\Lambda_{j+1}), T_m^A(\Lambda_{j+1})]$, $m \leqslant j+3$, are disjoint. We obtain a curve with the desired properties in a finite number of steps. ■

The next two lemmas are concerned with smooth curves and have a combinatorial nature.

LEMMA 2.6. *Suppose that A is a set homeomorphic to a disc and $\Lambda_1, \ldots, \Lambda_k$ are disjoint smooth curves. Each curve intersects itself at a finite number of points and has endpoints on ∂A. Then we can find a function χ which takes values* 0 *or* 1 *and is constant on each component of $A \setminus \bigcup_j \Lambda_j$ and such that the components whose boundaries intersect along a non-degenerate piece of a curve Λ_j have different values.*

Proof. First consider the case when we have only one curve Λ. Suppose that $\{\Lambda(u), u \in [u_1, u_2]\}$ is a continuous parametrization of Λ. By using the Riemann mapping, we may assume that A is the upper half-plane and the endpoints of Λ are $\Lambda(u_1) = 0$ and $\Lambda(u_2) = 1$. For each $x \in A \setminus \Lambda$ choose a continuous version of the function $u \to \arg(x - \Lambda(u))$ and let

$$l(x) = \arg(x - \Lambda(u_2)) - \arg(x - \Lambda(u_1)).$$

It is easy to see that $l(x)$ is a continuous function of x within each component of $A \setminus \Lambda$ and takes values in $(2k\pi, \pi + 2k\pi)$ for some integer k. It has a jump of size 2π on the boundary between components. It will suffice to let $\chi(x) = 0$ if k is even and 1 otherwise.

Now suppose that the lemma is true for $k-1$ curves. Find a curve Λ_j among the curves $\{\Lambda_1, \ldots, \Lambda_k\}$ such that all the remaining curves lie in one component of $A \backslash \Lambda_j$, say D_1. Let D_2 be the component of $A \backslash \bigcup_{m \neq j} \Lambda_m$ which contains Λ_j. By the induction assumption, we can find χ_1 and χ_2 corresponding to the components of $A \backslash \Lambda_j$ and $A \backslash \bigcup_{m \neq j} \Lambda_m$. By flipping the values of 0 and 1 if necessary we can assume that χ_j is equal to 0 on D_j for $j = 1, 2$. Then let $\chi(x) = \chi_1(x) + \chi_2(x)$. The lemma follows by induction. ∎

Before we state the next lemma, let us give a name to a certain operation on smooth curves. Suppose that two smooth curves intersect at a finite number of points and let x be one of them. Let B be a small disc centered at x which does not contain any other points of intersection. Let us remove the parts of the curves which lie within B and add two disjoint line segments within B connecting the points at which the curves intersect ∂B. This gives us two new curves – each new curve contains a piece of each of the original curves and a new line segment within B. Note that there are two ways to perform this operation – each one produces a different pair of curves. We will refer in what follows to this operation as "reconnecting the curves at x."

Lemma 2.7. *Suppose that $A_1, A_2, \ldots, A_{j_0}$ are open disjoint discs whose closures lie inside a disc C. Suppose that smooth curves $\Phi_1, \Phi_2, \ldots, \Phi_{k_0}$ lie inside C and have their endpoints on ∂C. Assume that for every j and k the intersection of Φ_k and A_j is either empty or is a finite number of line segments. Assume that the endpoints of these line segments are all distinct and that for each k the line segments comprising $\bigcup_j A_j \cap \Phi_k$ do not intersect. Suppose that $\Phi_1, \Phi_2, \ldots, \Phi_{k_0-1}$ are pairwise disjoint and for all $k < k_0$ the intersection of Φ_k and Φ_{k_0} lies within $\bigcup_j A_j$. Then we can reconnect Φ_k's at the intersection points within $\bigcup_j A_j$ so that we obtain new disjoint curves $\Theta_1, \Theta_2, \ldots, \Theta_{k_0}$ and possibly some closed loops $\Pi_1, \ldots, \Pi_{j_0}$. The set of all endpoints of Θ_j's is the same as the set of all endpoints of Φ_k's. If $J \subset \{1, 2, \ldots, j_0\}$ and $\bigcup_{j \in J} \Pi_j$ is connected, then this set can intersect at most one curve Θ_k.*

Proof. Find a function χ corresponding to the components of $\tilde{A} = C \backslash \bigcup_{k<k_0} \Phi_k$ according to Lemma 2.6.

The intersection points of Φ_k's divide these curves into a finite number of pieces, say, J_k's. Let $I_1, \ldots, I_{m_0}$ be the consecutive pieces J_k of Φ_{k_0}. Imagine that initially each Φ_k is painted with color k. We will repaint Φ_k's and we start with Φ_{k_0}. The first piece, i.e., I_1, will retain its color k_0. We will put colors into categories of cool or warm colors as we proceed with our coloring scheme. The color k_0 will be called *cool* if I_1 belongs to the component of $\tilde{A}$ with χ-value 0. Otherwise we will call it *warm.*

As we move along Φ_{k_0} we will encounter intersections with lines that have been already crossed by Φ_{k_0} and also new lines. Suppose that I_k is the first piece of Φ_{k_0} touching Φ_j. Then I_{k+1} is painted with color j. At the same time we record the color of I_k and call it $\mathscr{C}(j)$. The color j will be called *cool* if I_{k+1} belongs to the component of $\tilde{A}$ with χ-value 0. Otherwise we will call j a *warm* color.

If I_k ends at a line Φ_m which has been already visited in the past, we repaint I_{k+1} depending on the color of I_k. If the color of I_k is different from m, we give color m to I_{k+1}; otherwise I_{k+1} gets color $\mathscr{C}(m)$.

We will show that for all j and k, the colors $\mathscr{C}(j)$ and j belong to different categories of cool and warm colors and the color of I_k is cool iff it is in a component where $\chi = 0$. It is easy to see that this is true for $k = 1, 2$, for the color j of I_2 and the corresponding $\mathscr{C}(j)$. Suppose that this is true for all $k < k_1$ and all colors j (and corresponding $\mathscr{C}(j)$) which have been used to paint all I_k's with $k < k_1$. Now observe that I_{k_1-1} and I_{k_1} lie in components with different values of χ. It follows from our coloring scheme that their colors must belong to different categories of cool and warm colors, no matter whether we are crossing an old line or a new line. Hence, the induction assumption implies that I_{k_1} is cool iff it is in a component where $\chi = 0$. If we are crossing a new line Φ_{j_0}, a moment thought confirms that the induction assumption and the coloring scheme imply that $\mathscr{C}(j_0)$ and j_0 belong to different categories of cool and warm colors.

For every $k < k_0$, we repaint Φ_k by alternating colors of its adjacent components J_m between k and $\mathscr{C}(k)$. We do it so that at least one of the endpieces of Φ_k is painted with color k.

Next we will show that all J_m's that meet at the intersections on the path Φ_k have one of two colors: k or $\mathscr{C}(k)$. This is obviously true for J_m's that lie on Φ_k. This is also true for all J_m's which lie on Φ_{k_0} and originate at Φ_k, no matter where they have their second endpoints (the sense of direction is the same as for the sequence of I_j's). It remains to consider I_j's which originate at some Φ_n with $n \neq k$ and end at a point of Φ_k. The first such I_j has color $\mathscr{C}(k)$, by definition.

Let D be a component of $\tilde{A}$. Suppose that the first I_j which lies in D has color m. The path Φ_{k_0} may leave D by crossing some line Φ_i for the first time. In such a case $\mathscr{C}(i) = m$. The first time (if there is such a time) the curve Φ_{k_0} returns to D, it must enter D by crossing the same line Φ_i. The color of the I_j that lies within D after that crossing can be only i or $\mathscr{C}(i) = m$. Since only one of these two colors is cool and the path is in the same component of $\tilde{A}$ which already contains a path colored m, we can conclude that I_j has color m. A similar argument applies to multiple crossings of the same line Φ_i. We conclude that all I_j's in the same component of $\tilde{A}$ must have the same color. Since I_j leaving a path Φ_k may have only one of two colors: k or $\mathscr{C}(k)$, so must the I_j's which come to Φ_k from a different Φ_n. This shows that all J_m's that meet at the intersections on the path Φ_k have one of two colors: k or $\mathscr{C}(k)$.

We have shown that the paths J_m meeting at an intersection point can have only two colors and the construction clearly shows that the colors come in two adjacent pairs as we go around the point. Hence we can apply the reconnecting procedure described before the lemma at every intersection point.

We always join J_k's which have the same color. The result will be a number of continuous paths $\Theta_1, \Theta_2, \ldots, \Theta_{k_0}$ and possibly some closed loops $\Pi_1, \Pi_2, \ldots, \Pi_{j_0}$, none of which intersect within A_k's, $k > 0$. Each of these paths is painted with only one color. It is easy to see that the set of all endpoints of Θ_j's is the same as the set of all endpoints of Φ_k's.

Recall that we have already shown that all I_j's in the same component of $\tilde{A}$ must have the same color. Fix some k and consider the components of $\hat{A}_k = C\backslash\bigcup_{j\neq k}\Phi_j$. Then using Lemma 2.6 and the same argument as that with the I_j's we can prove that J_m's which lie on Φ_k and belong to the same component of $\hat{A}_k$ must have the same color.

Suppose that two J_m's intersect outside $\bigcup_j A_j$ and, therefore, they both belong to the same Φ_j. It follows that they lie in the same component of $\tilde{A}$ and the same component of every $\hat{A}_k$, $k \neq j$, and so these two J_m's must have the same color. Note that all Θ_j's must have different colors because we have painted (at least) k_0 initial pieces of Φ_k's with different colors. Hence Θ_j's are pairwise disjoint. A loop Π_j can intersect other loops only if they are of the same color and all loops of the same color can intersect only one Θ_k – the one with the same color. This proves the last assertion of the lemma. ∎

LEMMA 2.8. *Suppose that C is an open disc whose closure is contained in an open disc C^+. Assume that for every $j \leqslant j_0$ we have an open simply connected set A_j whose closure lies within an open simply connected set A_j^+. The sets A_j^+ are assumed to be pairwise disjoint and their closures are contained in C. Let $A = \bigcup_j A_j$ and $A^+ = \bigcup_j A_j^+$. Suppose that $\{\Lambda(u), u\in[u_1, u_2]\}$ is a* (P)*-path with endpoints outside C^+. Let Λ_k be the piece of Λ between $U_k^C(\Lambda)$ and $T_k^C(\Lambda)$. Assume that all the intersections of Λ_k and Λ_j for $j \neq k$ lie within A. Moreover, assume that*

$$\Lambda_k[U_j^A(\Lambda_k), T_j^A(\Lambda_k)] \cap \Lambda_k[U_i^A(\Lambda_k), T_i^A(\Lambda_k)] = \emptyset \quad \textit{for all } k \textit{ and } i \neq j.$$

Then there is a (P)*-path Γ with the same endpoints as Λ which contains all the points of $\Lambda\backslash A^+$ and such that*

$$\Gamma[U_j^C(\Gamma), T_j^C(\Gamma)] \cap \Gamma[U_i^C(\Gamma), T_i^C(\Gamma)] = \emptyset \quad \textit{for } i \neq j.$$

Proof. First we replace the pairs of paths $\Lambda_k[U_j^{A_i}(\Lambda_k), T_j^{A_i}(\Lambda_k)]$ and $\Lambda_m[U_n^{A_i}(\Lambda_m), T_n^{A_i}(\Lambda_m)]$ which do not have to intersect for topological reasons with pairs of disjoint paths as in Lemma 2.4 (see Remark 2.1). This will change the path Λ only within A. From now on we will assume that if $\Lambda_k[U_j^{A_i}(\Lambda_k), T_j^{A_i}(\Lambda_k)]$ and $\Lambda_m[U_n^{A_i}(\Lambda_m), T_n^{A_i}(\Lambda_m)]$ intersect, then they have to do so for topological reasons.

We would like to apply Lemma 2.7 to the curve Λ but this curve is not smooth. Hence we will approximate it with a smooth curve Φ as follows. We choose Φ so that

(i) Φ has the same endpoints as Λ;

(ii) $\Phi(U_k^C(\Phi)) = \Lambda(U_k^C(\Lambda))$ and $\Phi(T_k^C(\Phi)) = \Lambda(T_k^C(\Lambda))$ for all k such that $T_k^C(\Lambda) < \infty$; let Φ_k be the piece of Φ between $U_k^C(\Phi)$ and $T_k^C(\Phi)$;

(iii) $\Phi_k(U_j^A(\Phi_k)) = \Lambda_k(U_j^A(\Lambda_k))$ and $\Phi_k(T_j^A(\Phi_k)) = \Lambda_k(T_j^A(\Lambda_k))$ for all j and k;

(iv) intersections of Φ_k and Φ_j for $j \neq k$ lie within A;

(v) $\Phi_k[U_j^A(\Phi_k), T_j^A(\Phi_k)] \cap \Phi_k[U_i^A(\Phi_k), T_i^A(\Phi_k)] = \emptyset$ for all k and $i \neq j$;

(vi) if Λ_k and Λ_j are disjoint, then so are Φ_k and Φ_j;

(vii) if $\Lambda_k[U_j^{A_i}(\Lambda_k), T_j^{A_i}(\Lambda_k)]$ and $\Lambda_m[U_n^{A_i}(\Lambda_m), T_n^{A_i}(\Lambda_m)]$ are disjoint, then so are $\Phi_k[U_j^{A_i}(\Phi_k), T_j^{A_i}(\Phi_k)]$ and $\Phi_m[U_n^{A_i}(\Phi_m), T_n^{A_i}(\Phi_m)]$;

(viii) $\Phi_k[U_j^{A_i}(\Phi_k), T_j^{A_i}(\Phi_k)]$ and $\Phi_m[U_n^{A_i}(\Phi_m), T_n^{A_i}(\Phi_m)]$ have at most one intersection point.

It is easy to choose Φ so that it satisfies (i)–(iii). In order to have (iv)–(vii), it will suffice to construct Φ so that for a sufficiently small $\delta > 0$ and all u we have $|\Phi(u) - \Lambda(u)| < \delta$. Condition (viii) is easily satisfied if A_i^+'s are discs – it is enough to modify Φ's so that all pieces $\Phi_k[U_j^{A_i}(\Phi_k), T_j^{A_i}(\Phi_k)]$ are line segments Since (viii) is a topological assumption, it can be satisfied for any family of simply connected open sets A_i^+.

Let us rename paths $\Lambda_1[U_j^{A_1}(\Lambda_1), T_j^{A_1}(\Lambda_1)]$ as $I_j = \{I_j(u), u \in [u_1^j, u_2^j]\}$ and let $J_k = \{J_k(u), u \in [s_1^k, s_2^k]\}$ be the new names for $\Lambda_2[U_k^{A_1}(\Lambda_2), T_k^{A_1}(\Lambda_2)]$. Recall that I_j's are disjoint and the same is true for J_k's. If I_j and J_k intersect, then they have to do so for topological reasons. We will say that I_j *hits* J_k *before* J_m if

$$\inf\{u\colon I_j(u) \in J_k\} < \inf\{u\colon I_j(u) \in J_m\}.$$

Suppose that I_j and J_k intersect and that I_n is the first curve hit by J_k after hitting I_j. Let T be the infimum of u such that $J_k(u) \in I_n$. Then we let $t_k^j = \inf\{u\colon I_j(u) \in J_k[s_1^k, T)\}$ and $x_k^j = I_j(t_k^j)$. Let $r_k^j = \inf\{u\colon J_k(u) = x_k^j\}$. Note that $r_k^j < T$. Since $J_k[s_1^k, T)$ does not intersect I_n, we see that $x_k^j = J_k(r_k^j)$ and $x_k^n = J_k(r_k^n)$ for some $r_k^j < r_k^n$. More generally, if we move along J_k from $J_k(s_1^k)$ to $J_k(s_2^k)$ and we hit I_j before hitting I_m, then $x_k^j = J_k(r_k^j)$ and $x_k^m = J_k(r_k^m)$ for some $r_k^j < r_k^m$. Now suppose that I_j hits J_k before hitting J_m. Let D be the component of $A_1 \backslash I_n$ which contains $J_k[s_1^k, T)$. Note that I_j and $J_k[s_1^k, T)$ have to intersect in D for topological reasons. Moreover, I_j has to hit $J_k[s_1^k, T)$ before hitting J_m in D. We conclude that if we move along I_j from $I_j(u_1^j)$ to $I_j(u_2^j)$ and we hit J_k before hitting J_m, then $x_k^j = I_j(t_k^j)$ and $x_m^j = I_j(t_m^j)$ for some $t_k^j < t_m^j$.

Let I_j^Φ and J_k^Φ be defined relative to Φ_1 and Φ_2 in the same way as I_j and J_k have been defined relative to Λ_1 and Λ_2. Use these curves to define points y_k^j analogous to x_k^j's.

The main point of the construction of x_k^j's is that they come on the paths of I_j's and J_k's in the same order as y_k^j's on the paths of I_j^Φ's and J_k^Φ's. Let $\overline{y_{k_1}^{j_1} y_{k_2}^{j_2}}$ denote a piece of one of I_j^Φ's or one of J_k^Φ's between points $y_{k_1}^{j_1}$ and

$y^{j_2}_{k_2}$ – we will tacitly assume that either $j_1 = j_2$ or $k_1 = k_2$. Suppose that a path made of arcs $\overline{y^{j_1}_{k_1} y^{j_2}_{k_2}}, \overline{y^{j_2}_{k_2} y^{j_3}_{k_3}}, \ldots, \overline{y^{j_{m-1}}_{k_{m-1}} y^{j_m}_{k_m}}$ intersects another path made of $\overline{y^{i_1}_{n_1} y^{i_2}_{n_2}}, \overline{y^{i_2}_{n_2} y^{i_3}_{n_3}}, \ldots, \overline{y^{i_{r-1}}_{n_{r-1}} y^{i_r}_{n_r}}$ only at some y^j_k's. Then the paths made of pieces of I_j's and J_k's:

$$\overline{x^{j_1}_{k_1} x^{j_2}_{k_2}}, \overline{x^{j_2}_{k_2} x^{j_3}_{k_3}}, \ldots, \overline{x^{j_{m-1}}_{k_{m-1}} x^{j_m}_{k_m}} \quad \text{and} \quad \overline{x^{i_1}_{n_1} x^{i_2}_{n_2}}, \overline{x^{i_2}_{n_2} x^{i_3}_{n_3}}, \ldots, \overline{x^{i_{r-1}}_{n_{r-1}} x^{i_r}_{n_r}}$$

intersect only at some x^j_k's.

We will apply Lemma 2.7 to Φ_1 and Φ_2. The sets A^+_j in the statement of Lemma 2.7 will play the role of A_j's in Lemma 2.6. Although A^+_j's are not discs, they are simply connected open sets and we can apply Lemma 2.6 as it deals only with topological and combinatorial properties.

According to Lemma 2.7, Φ_1 and Φ_2 may be reconnected within A so that we obtain two paths Θ_1 and Θ_2 with the same endpoints as Φ_1 and Φ_2 (although the endpoints of Θ_j need not be the same as those of Φ_j) and possibly a finite number of loops Π_j. Every curve $\Theta_1[U^{A_1}_j(\Theta_1), T^{A_1}_j(\Theta_1)]$ contains a number of arcs of the form $\overline{y^{j_1}_{k_1} y^{j_2}_{k_2}}, \overline{y^{j_2}_{k_2} y^{j_3}_{k_3}}, \ldots, \overline{y^{j_{m-1}}_{k_{m-1}} y^{j_m}_{k_m}}$ except that very small pieces of these arcs close to y^j_k's were replaced by new connections. Consider the corresponding curves Ω^1_j consisting of paths $\overline{x^{j_1}_{k_1} x^{j_2}_{k_2}}, \overline{x^{j_2}_{k_2} x^{j_3}_{k_3}}, \ldots, \overline{x^{j_{m-1}}_{k_{m-1}} x^{j_m}_{k_m}}$. Note that Ω^1_j are (P)-paths in A^+_1 with the same endpoints as $\Theta_1[U^{A_1}_j(\Theta_1), T^{A_1}_j(\Theta_1)]$. Define Ω^2_j's in a similar way relative to Θ_2. The proof of Lemma 2.7 shows that not only Θ_1 and Θ_2 are disjoint but the paths $\Theta_1[U^{A_1}_j(\Theta_1), T^{A_1}_j(\Theta_1)]$ are also disjoint for different j (the same is true for Θ_2). It follows that no pair of paths from the family of Ω^1_j's and Ω^2_j's have to intersect for topological reasons. In view of Remark 2.2 we can replace Ω^1_j's and Ω^2_j's with $\hat{\Omega}^1_j$'s and $\hat{\Omega}^2_j$'s which are disjoint (the endpoints of $\hat{\Omega}^k_j$ are the same as those of Ω^k_j). If we perform the operation $\mathscr{D}$ discussed in Remark 2.2, $\hat{\Omega}^1_j$'s and $\hat{\Omega}^2_j$'s will not necessarily pass through the points x^i_n as the old paths. However, we will record the fact that an original path Ω^k_j passed through x^i_n by writing $x^i_n \rhd \hat{\Omega}^k_j$.

We have constructed disjoint (P)-paths $\hat{\Omega}^1_j$ and $\hat{\Omega}^2_j$ which connect the points $\Lambda_1(U^{A_1}_j(\Lambda_1))$, $\Lambda_1(T^{A_1}_j(\Lambda_1))$, $\Lambda_2(U^{A_1}_j(\Lambda_2))$, and $\Lambda_2(T^{A_1}_j(\Lambda_2))$ within A^+_1. Moreover, these endpoints are connected by the new paths in the same way as by Θ's. Next we repeat the same operation in every A^+_j.

Let us replace all $\Lambda_1[U^{A_i}_j(\Lambda_1), T^{A_i}_j(\Lambda_1)]$ and $\Lambda_2[U^{A_i}_j(\Lambda_2), T^{A_i}_j(\Lambda_2)]$ with $\hat{\Omega}^1_j$'s and $\hat{\Omega}^2_j$'s and the analogous paths in A^+_j's for $j > 1$. The original curve Λ will be replaced by a new curve $\tilde{\Lambda}$ and possibly a number of (P)-loops Σ_j corresponding to Π_j's.

If some loops Σ_j intersect, then we combine them into a single loop using Remark 2.3. If any of the new loops intersects $\tilde{\Lambda}$, we again apply Remark 2.3 to get rid of this loop and incorporate it into $\tilde{\Lambda}$. We may be still left with some loops that do not intersect any other loop or $\tilde{\Lambda}$. Note that such loops must contain an $\hat{\Omega}^k_j$ (or an analogous path in some A^+_n) such that there exist x^i_n

and $\hat{\Omega}^r_m$ with the following properties: $x^i_n \rhd \hat{\Omega}^k_j$, $x^i_n \rhd \hat{\Omega}^r_m$ and $\hat{\Omega}^r_m$ belongs either to another loop or to $\tilde{\Lambda}$. By switching the original reconnection at the point x^i_n we will get rid of one loop. A repeated application of the procedure will get rid of all loops.

Let Λ^* denote the path obtained from $\tilde{\Lambda}$ by attaching to it all loops. Our construction and Lemma 2.7 imply that

$$\Lambda^*[U_1^c(\Lambda^*), T_1^c(\Lambda^*)] \cap \Lambda^*[U_2^c(\Lambda^*), T_2^c(\Lambda^*)] = \emptyset.$$

Next we approximate Λ^* with a smooth line Φ just like in the case of Λ. Then we apply Lemma 2.7 to Φ_1, Φ_2 and Φ_3 in order to construct a new curve Λ^{**} with the property that

$$\Lambda^{**}[U_i^c(\Lambda^{**}), T_i^c(\Lambda^{**})] \cap \Lambda^{**}[U_j^c(\Lambda^{**}), T_j^c(\Lambda^{**})] = \emptyset \quad \text{for } i \neq j,\ i, j = 1, 2, 3.$$

We inductively repeat the procedure each time applying Lemma 2.7 to a larger number of Φ_j's. ∎

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Received on 8.3.1994