PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 16, Fasc. 1 (1996), pp. 139–142

A PROBABILISTIC PROPERTY OF THE SPACE l_2^m

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Abstract. It is shown that for every sequence (x_k) of elements of l_2^m the following two properties are equivalent:

(a) $(x_k/k) \in l_2(l_2^m)$.

(b) $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale, where $S_n = \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k$, (ε_k) being a sequence of independent Rademacher r.v. and $\mathscr{F}_n$ denoting the σ-field generated by $(\varepsilon_1, \ldots, \varepsilon_n)$.

In a recent paper [3], we studied a new geometrical property of Banach spaces – property related with Kolmogorov's (non i.i.d.) strong law of large numbers. The purpose of the present note is to show that the Euclidean space l_2^m has this geometrical property.

In the sequel, $(B, \| \ \|)$ will be a real separable Banach space. We will denote by (ε_k) a sequence of independent Rademacher r.v.; for every n, $\mathscr{F}_n$ will be the σ-field generated by $(\varepsilon_1, \ldots, \varepsilon_n)$. With every sequence $x = (x_n)$ of elements of B we will associate the random sums

$$S_n = S_n(x) = \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k.$$

The announced geometrical property is defined as follows:

DEFINITION 1. We say that $(B, \| \ \|)$ has the *Kolmogorov quasimartingale property* (in short, Kqm-property) if and only if, for every sequence $x = (x_n)$ of elements of B, the following two properties are equivalent:

(i) $(\|S_n(x)/n\|^2, \mathscr{F}_n)$ is a quasimartingale.

(ii) $(x_k/k) \in l_2(B)$.

Remark. In this very special case, the fact that $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale simply reduces to

$$\sum_{n \geqslant 1} \mathrm{E} \left| \mathrm{E}\left(\left\| \frac{S_{n+1}}{n+1} \right\|^2 | \mathscr{F}_n \right) - \left\| \frac{S_n}{n} \right\|^2 \right| < +\infty \tag{1}$$

(see [1] or [2] for the general definition of a quasimartingale and the main properties of such a sequence of r.v.).

Among the properties proved in [3], let us mention that

(a) the real line has the Kqm-property;

(b) if $(B, \| \ \|)$ has the Kqm-property, then B is isomorphic to a Hilbert space;

(c) an infinite-dimensional Hilbert space does not have the Kqm-property.

In the conclusion of [3], the following question is asked: "Does a finite-dimensional Euclidean space have the Kqm-property?" The purpose of the present note is to answer this question. The result is as follows:

THEOREM 2. *The space l_2^m has the* Kqm-*property.*

Proof. A straightforward computation shows that in the space l_2^m property (ii) in Definition 1 implies property (i). To check that the converse implication also holds we recall first an exponential lower bound. That inequality is due independently to Ledoux and Talagrand ([4], Lemma 4.9) and Montgomery-Smith [5]. We give the statement under Montgomery-Smith's form:

PROPOSITION 3. *There exists a constant $C > 1$ such that for every $y \in l_2$ and all $t > 0$ we have*

$$P\Big(\sum_{k\geqslant 1} \varepsilon_k y_k > \frac{t}{C}\sqrt{\sum_{k\geqslant [t^2]+1} (y_k^*)^2}\Big) \geqslant \frac{1}{C}\exp(-Ct^2),$$

where $[\ ]$ denotes the integer part of a real number and (y_k^*) is the non-increasing rearrangement of the sequence $(|y_k|)$.

Let us define $\mu = [4Cm+1]$ and $M = \mu^2$.

Let now (x_n) be a sequence of elements of l_2^m such that $(\|S_n/n\|^2, \mathscr{F}_n)$ is a quasimartingale. By arguing as in the scalar case (see [3], Lemma 1.6), we obtain easily the following technical lemma:

LEMMA 4. *For every integer n we denote by $z_1(n), \ldots, z_n(n)$ the non-increasing rearrangement of the sequence $(\|x_1\|, \ldots, \|x_n\|)$. Then for every $n \geqslant M$ we have*

$$\frac{2n+1}{n^2(n+1)^2}\sum_{1\leqslant k\leqslant M} z_k^2(n) \leqslant u_n,$$

where u_n is the general term of a convergent series.

Now, denote by $(x_k^1, \ldots, x_k^m)$ the coordinates of x_k and consider the following $m+1$ sets of positive integers:

$$\forall j = 1, \ldots, m,\ H_j = \left\{n\colon \frac{\mu}{C}\left(\sqrt{\frac{2n+1}{n^2(n+1)^2}\sum_{1\leqslant k\leqslant n}(x_k^j)^2} - \sqrt{u_n}\right) \geqslant \sqrt{2}\frac{\|x_{n+1}\|}{n+1}\right\}$$

and

$$H_0 = N^* - \bigcup_{1 \leqslant j \leqslant m} H_j.$$

For all n belonging to H_j we get

$$(2) \quad P\left(\frac{\sqrt{2n+1}}{n(n+1)} \Big\| \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k \Big\| > \sqrt{2}\, \frac{\|x_{n+1}\|}{n+1}\right)$$

$$\geqslant P\left(\frac{\sqrt{2n+1}}{n(n+1)} \Big| \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k^j \Big| > \sqrt{2}\, \frac{\|x_{n+1}\|}{n+1}\right)$$

$$\geqslant P\left(\frac{\sqrt{2n+1}}{n(n+1)} \Big| \sum_{1 \leqslant k \leqslant n} \varepsilon_k x_k^j \Big| > \frac{\mu}{C} \sqrt{\frac{2n+1}{n^2(n+1)^2} \sum_{k \geqslant [\mu^2]+1} (x_k^j)^{*2}}\right) \geqslant \frac{1}{C} \exp(-C\mu^2),$$

where in the last step Proposition 3 has been used. From the quasimartingale property (1) we obtain, by the definition of the norm on l_2^m,

$$(3) \qquad \sum_{n \geqslant 1} \mathrm{E} \left| \frac{(2n+1)\,\|S_n\|^2}{n^2(n+1)^2} - \frac{\|x_{n+1}\|^2}{(n+1)^2} \right| < +\infty,$$

and so, by the application of (2),

$$\sum_{n \in H_0^c} \frac{\|x_{n+1}\|^2}{(n+1)^2} < +\infty.$$

Consider that time an integer $n \in H_0$. By the choice of μ we have

$$\frac{2n+1}{n^2(n+1)^2} \sum_{1 \leqslant k \leqslant n} \|x_k\|^2 = \frac{2n+1}{n^2(n+1)^2} \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant m} (x_k^j)^2$$

$$\leqslant 2m \left(\frac{2C^2}{\mu^2} \frac{\|x_{n+1}\|^2}{(n+1)^2} + u_n\right) \leqslant 2mu_n + \frac{\|x_{n+1}\|^2}{4(n+1)^2}.$$

Denote by A the set of elements of H_0 such that

$$u_n \leqslant \frac{\|x_{n+1}\|^2}{8m(n+1)^2},$$

and by B the complement of A in H_0. By Jensen's inequality, it follows easily from (3) that

$$\sum_{n \in A} \frac{\|x_{n+1}\|^2}{(n+1)^2} < +\infty.$$

Remembering finally that u_n is the general term of a convergent series, we get

$$\sum_{n\in B} \frac{\|x_{n+1}\|^2}{(n+1)^2} < +\infty,$$

and this completes the proof of the implication (i) $\Rightarrow$ (ii).

From Theorem 2 and the fact that an infinite-dimensional Hilbert space does not have the Kqm-property we deduce easily the following:

COROLLARY 5. *A real separable Banach space* $(B, \|\ \|)$ *which is isometrically isomorphic to a Hilbert space has the* Kqm-*property if and only if B is finite dimensional.*

REFERENCES

[1] C. Dellacherie et P. A. Meyer, *Probabilités et potentiel*, chap. V–VIII, *Théorie des martingales*, Hermann, Paris 1980.

[2] L. Egghe, *Stopping time techniques for analysts and probabilists*, Cambridge University Press, Cambridge 1984.

[3] B. Heinkel, *On the Kolmogorov quasimartingale property*, this fascicle, pp. 113–126.

[4] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer, Berlin 1991.

[5] S. J. Montgomery-Smith, *The distribution of Rademacher sums*, Proc. Amer. Math. Soc. 109 (1990), pp. 517–522.

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Received on 20.7.1993