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NON-GAUSSIAN MEASURES WITH GAUSSIAN STRUCTURE

BY

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Abstract. Examples of non-Gaussian multivariate distributions with Gaussian conditional moment structure, Gaussian marginals and normal projections are provided.

1. Introduction. Gaussian-like distributions are intriguing probability objects. Numerous constructions of such measures are given in the literature. They include among others:

– non-Gaussian n-dimensional measures with all Gaussian $(n-1)$-variate marginals – see, for example, Stoyanov [14];

– non-Gaussian measures with any finite number of normal projections – see Hamedani and Tata [8];

– non-Gaussian measures with normal conditional distributions – see, for example, Castillo and Galambos [5].

Further comments can be found, for example, in Hamedani [7] or Arnold et al. [1].

In this note we are interested in Gaussian conditional moment and marginal structure. Additionally, normal linear forms are considered. The examples we provide contribute towards better understanding of the miracle of the Gaussian measure in finite dimensions.

Consider a real square integrable random element $X = (X_t)_{t \in T}$. Assume (to keep everything in the simplest form without losing generality) that $\mathrm{E}(X_t) = 0$, $\mathrm{E}(X_t^2) = 1$, $t \in T$, and denote by Σ its correlation matrix. We say that X (or its distribution) has a *Gaussian conditional structure of order* s (belongs to $GCS_s(T)$) iff all conditional moments $\mathrm{E}(X_t^k \mid X_{t_1}, \ldots, X_{t_m})$, $t, t_1, \ldots, t_m \in T$, $m \geq 1$, $k = 1, \ldots, s$, are exactly like for the Gaussian distribution with the same correlation matrix Σ. The $GCS_2(T)$ measure has been intensively investigated during the last ten years. Main results are reviewed in Wesołowski [15].

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It is known that if $T=[0,\infty)$ (Plucińska [10]) or $T=\{0,1,\ldots\}$ (Bryc and Plucińska [3]), then under mild technical conditions $GCS_2(T)$ measure is Gaussian. However, in the two-dimensional case ($T=\{1,2\}$) a non-Gaussian example was given by Kwapień (reproduced in [3], where also infinite integrability of the GCS_2 families was proved) – it is a measure concentrated on four points $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ with weights $p/2$, $(1-p)/2$, $(1-p)/2$, $p/2$, respectively. No analogous construction in the n-variate case and no other non-Gaussian distribution belonging to $GCS_2(n)$, $n>2$, have been known up to now. Such examples are given in Section 2. Recall that $GCS_2(n)$ is a convex set (see Bryc [2]) – a nice property which is not utilized here.

In the celebrated Kagan et al. book [9] on characterization of probability distributions the following problem of identification of normality via Gaussian-like polynomial regressions and normal marginal is posed: Assume that Y is a normal r.v. and

$$\mathrm{E}(Y^k|X)=X^k+Q_{k-1}(X),\qquad k=1,2,\ldots,$$

where Q_{k-1} is a polynomial of degree not exceeding $k-1$. Is X normal? A counterexample with two-point distribution for X was given in Shanbhag [13].

Bryc and Szablowski [4] considered a symmetrized version of that problem involving the following conditions:

$$\mathrm{E}(X^k|Y)=P_k(Y),\qquad \mathrm{E}(Y^k|X)=\varrho^k X^k+Q_{k-1}(X),\qquad k=1,2,\ldots,$$

for normal $\mathcal{N}(0,1)$ random variables X and Y with $\varrho=\mathrm{E}(XY)$, where P_k and Q_k are polynomials of degree less than or equal to k. It appears that such assumptions ensure bivariate normality. This approach imposes a question of considering jointly GCS_s and Gaussian marginal structure. In Section 2 we give an example of a measure belonging to $GCS_s(n)$ for some arbitrary (but fixed) s which has all Gaussian $(n-1)$-variate marginals. Moreover, joint conditional moments up to some order are also like in the Gaussian case. It means that Theorem 3.1 of Bryc and Szablowski [4] cannot be essentially improved. Let us point out that the simplest case, i.e. non-Gaussian $GCS_1(2)$ distributions with normal marginals, was considered in Ruymgaart [12] – see also Feller [6, p. 99].

Another interesting question is: What happens if additionally $aX+bY$ is also normal? – see Theorem 3.3 in [4], which states that if all the conditional moments of X given Y and of Y given X are of polynomial type, marginals are normal and additionally $aX+bY$ is normal, then the joint distribution is Gaussian. Again due to the examples of non-Gaussian $GCS_s(n)$ measures with Gaussian marginals and normal projections, we provide in Section 3, it follows that this result cannot be improved by considering a finite number of regressions.

Also one cannot get rid of normality of a projection, while keeping all polynomial regressions. A simple example of a bivariate non-Gaussian measure with polynomial conditional moments and normal marginals is given in Section 4. The distribution is not of the type considered in the previous sections since it involves Gaussian mixtures.

2. Gaussian conditional moments and Gaussian marginals. In this section we give three examples of non-Gaussian measures with Gaussian marginal and conditional structure. Since the construction of the n-dimensional example is based essentially on modifying two- and three-dimensional case, which, in turn, makes use of the ideas developed in the univariate case, we present our examples in the order of increasing complexity, starting with a 'warm up' construction in one dimension.

Denote in the sequel an n-dimensional standardized Gaussian density function (i.e. with zero means, unit variances and a correlation matrix Σ) by $f_n(\cdot, \Sigma)$. Let further $s \in \boldsymbol{N}$ be a given positive natural number and let h_s be a function $h_s\colon \boldsymbol{R} \to \boldsymbol{R}$, bounded in $(-1, 1)$ and satisfying the condition

$$\int_{-1}^{1} x^k h_s(x)dx = 0 \quad \text{for } k = 0, 1, \ldots, s. \tag{2.1}$$

For example, one can take

$$h_s(x) = x(c_0 + c_1x^2 + c_2x^4 + \ldots + c_{s-1}x^{2s-2} - x^{2s}), \qquad x \in \boldsymbol{R}, \tag{2.2}$$

where $\{c_i\}_{i=0}^{s-1}$ is a sequence of constants satisfying the following system of linear equations:

$$\begin{array}{ccccccccccc}
\frac{c_0}{3} & + & \frac{c_1}{5} & + & \frac{c_2}{7} & + & \cdots & + & \frac{c_{s-1}}{2(s+1)-1} & = & \frac{1}{2(s+2)-1}, \\
\frac{c_0}{5} & + & \frac{c_1}{7} & + & \frac{c_2}{9} & + & \cdots & + & \frac{c_{s-1}}{2(s+2)-1} & = & \frac{1}{2(s+3)-1}, \\
\frac{c_0}{7} & + & \frac{c_1}{9} & + & \frac{c_2}{11} & + & \cdots & + & \frac{c_{s-1}}{2(s+3)-1} & = & \frac{1}{2(s+4)-1}, \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & = & \vdots \\
\frac{c_0}{2(s+1)-1} & + & \frac{c_1}{2(s+2)-1} & + & \frac{c_2}{2(s+3)-1} & + & \cdots & + & \frac{c_{s-1}}{2(s+s)-1} & = & \frac{1}{2(s+s+1)-1}.
\end{array}$$

The existence of the sequence $\{c_i\}_{i=0}^{s-1}$ follows from the fact that the $(s \times s)$-matrix with the entries

$$\left[\frac{1}{2(i+j)-1}\right]_{i,j=1,\ldots,s}$$

is a generalized Hilbert matrix, which is known to be of full rank.

In our first example we provide a formula for a univariate density function that is non-normal but has its first s moments exactly like a standard normal distribution. Moreover, it is related to the $\chi^2(1)$ distribution in exactly the same way as normal $\mathcal{N}(0, 1)$ distribution.

EXAMPLE 2.1. *A non-normal random variable with normal moments and $\chi^2(1)$ distribution of its square.*

Let us consider a one-dimensional p.d.f. $g_{s,1}$ given by the formula

$$g_{s,1}(x) = \begin{cases} f_1(x, 1) + \alpha_1 h_s(x) & \text{for } |x| < 1, \\ f_1(x, 1) & \text{otherwise,} \end{cases}$$

where h_s is an odd function satisfying condition (2.1) (we may take for instance that given by formula (2.2)), and α_1 is a normalizing constant depending on h_s assuring that $g_{s,1}$ is non-negative.

Clearly, $g_{s,1}$ defines a density of a non-normal r.v. X with its first s moments being equal to those of standard normal distribution. Let us also note that since

$$g_{s,1}(x) + g_{s,1}(-x) = \sqrt{2/\pi} \exp\{-x^2/2\},$$

it follows from Roberts and Geisser [11] that X^2 has to be distributed according to the $\chi^2(1)$ law. ∎

In the next paragraph we give an example of a two-dimensional non-Gaussian measure with normal marginals belonging to $GCS_s(2)$. Let us point out that an example of a non-Gaussian measure belonging to $GCS_s(2)$ but without normal marginals was given by Kwapień (see Bryc and Plucińska [3]). On the other hand, non-Gaussian distributions with $GCS_1(2)$ structure and normal marginals were constructed in Ruymgaart [12] (here Gaussian mixtures are quite natural examples – see Example 4.1); see also Feller [6, p. 99].

EXAMPLE 2.2. *A non-Gaussian $GCS_s(2)$ measure with normal marginals.*

Let us define a function $h_{s,2}\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ by

$$h_{s,2}(\boldsymbol{x}) = h_s(x_1) h_s(x_2), \qquad \boldsymbol{x} = (x_1, x_2) \in \boldsymbol{R}^2,$$

where h_s is a function satisfying condition (2.1), and let us consider a two-dimensional p.d.f. given by

$$g_{s,2}(\boldsymbol{x}) = \begin{cases} f_2(\boldsymbol{x}, \Sigma) + \alpha_2 h_{s,2}(\boldsymbol{x}) & \text{for } |x_i| < 1,\ i = 1, 2, \\ f_2(\boldsymbol{x}, \Sigma) & \text{otherwise,} \end{cases}$$

$\boldsymbol{x} \in \boldsymbol{R}^2$. Here α_2 is a normalizing constant depending on $h_{s,2}$ and assuring that $g_{s,2}$ is non-negative. Let (X_1, X_2) be a random vector with density $g_{s,2}$. Then, by the definition of $h_{s,2}$, we easily see that

(i) X_1 and X_2 are both normal $\mathcal{N}(0, 1)$;

(ii) $(X_1, X_2) \in GCS_s(2)$.

See Fig. 1 for an example of such a density with $s = 2$. ∎

In the following paragraph we construct a three-dimensional non-Gaussian measure with Gaussian marginals and belonging to $GCS_s(3)$. This is a slight modification of the previous one but, as indicated in the Introduction, may be of independent interest, since no non-Gaussian $GCS_s(3)$ measure has been known until now.

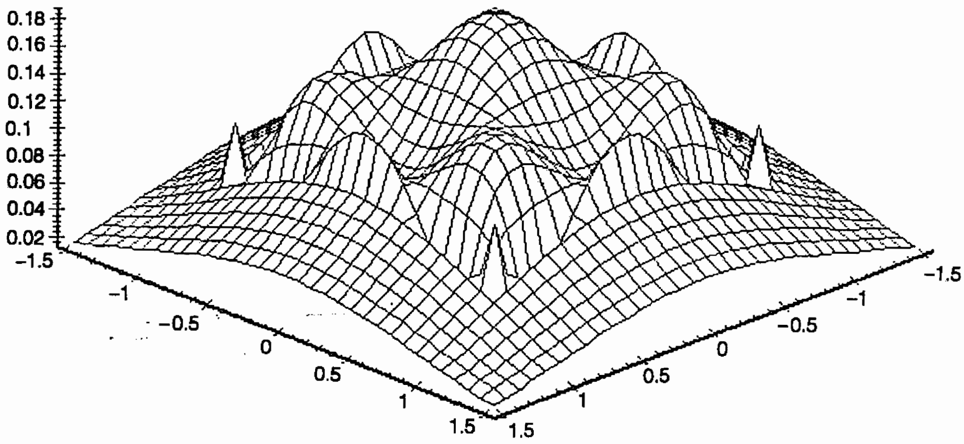

Fig. 1. The graph of the density

$$g_{2,2}(x_1, x_2) = \frac{1}{2\pi}\Big[\exp\{-(x_1^2+x_2^2)/2\} + \frac{8575}{192}e^{-5/7}\Big(\frac{3}{35}-\frac{6}{7}x_1^2+x_1^4\Big)\Big(\frac{3}{35}-\frac{6}{7}x_2^2+x_2^4\Big)I(|x_1|<1, |x_2|<1)\Big]$$

EXAMPLE 2.3. *A non-Gaussian $GCS_s(3)$ measure with Gaussian marginals.* As in the previous example let us first define a function $h_{s,3}$: $\boldsymbol{R}^3 \to \boldsymbol{R}$ by

$$h_{s,3}(\boldsymbol{x}) = h_s(x_1)h_s(x_2)h_s(x_3), \qquad \boldsymbol{x} = (x_1, x_2, x_3) \in \boldsymbol{R}^3,$$

where h_s satisfies condition (2.1). Now let us consider a three-dimensional p.d.f. given by

$$g_{s,3}(\boldsymbol{x}) = \begin{cases} f_3(\boldsymbol{x}, \Sigma) + \alpha_3 h_{s,3}(\boldsymbol{x}) & \text{for } |x_i| < 1,\ i = 1, 2, 3, \\ f_3(\boldsymbol{x}, \Sigma) & \text{otherwise,} \end{cases}$$

$\boldsymbol{x} = (x_1, x_2, x_3) \in \boldsymbol{R}^3$. Here, as in the previous cases, α_3 is a normalizing constant depending on $h_{s,3}$, assuring that $g_{s,3}$ is non-negative. Let (X_1, X_2, X_3) be a random vector with density $g_{s,3}$. Then

(i) (X_1, X_2), (X_1, X_3), (X_2, X_3) are all standardized Gaussian vectors;

(ii) $(X_1, X_2, X_3) \in GCS_s(3)$.

Observe that in this example also the conditional moments

$$\mathrm{E}(X_i^l X_j^m | X_k), \qquad \{i, j, k\} = \{1, 2, 3\},$$

where $l \leqslant s$ or $m \leqslant s$, are of the Gaussian form. ■

The generalization of Examples 2.2 and 2.3 to the case of n-dimensional measures, $n > 3$, is straightforward.

EXAMPLE 2.4. *An n-dimensional non-Gaussian measure with Gaussian $(n-1)$-variate marginals and multivariate Gaussian conditional moment structure.*

A further modification of Examples 2.2 and 2.3 gives us an example of n-dimensional non-Gaussian measure ($n \geqslant 3$) with Gaussian marginals and n-variate Gaussian conditional moment structure which imposes even more restrictions than that of $GCS_s(n)$. Following the scheme of the previous two constructions, let us define a function $h_{s,n}$: $\boldsymbol{R}^n \to \boldsymbol{R}$ by setting

$$(2.3) \qquad h_{s,n}(\boldsymbol{x}) = \prod_{i=1}^{n} h_s(x_i), \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in \boldsymbol{R}^n,$$

where h_s, as before, satisfies condition (2.1), and let us consider an n-dimensional p.d.f. given by

$$g_{s,n}(\boldsymbol{x}) = \begin{cases} f_n(\boldsymbol{x}, \Sigma) + \alpha_n h_{s,n}(\boldsymbol{x}) & \text{for } |x_i| < 1,\ i = 1, \ldots, n, \\ f_n(\boldsymbol{x}, \Sigma) & \text{otherwise,} \end{cases}$$

$\boldsymbol{x} = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$. Here, as before, α_n is a normalizing constant depending on $h_{s,n}$, assuring that $g_{s,n}$ is non-negative. Assume that an n-dimensional random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ has the density $g_{s,n}$. Then we can easily see that $(n-1)$-variate marginals of $\boldsymbol{X}$ are Gaussian. Moreover, using again condition (2.1) and definition (2.3), we conclude that the conditional moments

$$(2.4) \qquad \mathrm{E}(X_{i_1}^{s_1} X_{i_2}^{s_2} \ldots X_{i_k}^{s_k} | X_{j_1}, X_{j_2}, \ldots, X_{j_{n-k}}),$$

where $\{s_1, \ldots, s_k\}$ is a set of positive integers such that $s_l \leqslant s$ for some l, $1 \leqslant l \leqslant k$, and $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$, are equal to those of n-dimensional standardized Gaussian distribution. ■

3. Gaussian conditional moments, Gaussian marginals and normal projections. In this section we give some refinements of the examples from Section 2 by imposing an additional condition of normality of a finite number of linear forms. That question is settled by employing additionally an approach involving ch.f.'s. At first we consider a bivariate case and two linear forms.

EXAMPLE 3.1. *A non-Gaussian $GCS_s(2)$ measure with normal marginals and two normal projections.*

The function

$$g(\boldsymbol{x}) = f_2(\boldsymbol{x}, \Sigma) + \alpha \left[h_s^{(1)}(ax_1) h_s^{(2)}(bx_2) - h_s^{(2)}(ax_1) h_s^{(1)}(bx_2)\right] I\{a|x_1| < 1, b|x_2| < 1\},$$

$\boldsymbol{x} = (x_1, x_2) \in \boldsymbol{R}^2$, where $f_2(\cdot, \Sigma)$ is defined in Section 2, a, b are some positive constants, and $h_s^{(i)}$, $i = 1, 2$, are some odd functions bounded in the interval $(-1, 1)$, satisfying condition (2.1) for a given s, is a density function for some constant α.

It is easily seen that its ch.f. takes the form

$$\psi(t, u) = \phi_{\Sigma}(t, u) - \frac{2\alpha}{ab}[\phi_1(t/a)\phi_2(u/b) - \phi_2(t/a)\phi_1(u/b)]$$

for any real t, u, where ϕ_{Σ} is a ch.f. of the density $f_2(\cdot, \Sigma)$, and

$$\phi_i(t) = \int_0^1 h_s^{(i)}(x)\sin(tx)dx, \qquad i = 1, 2.$$

A random vector (X_1, X_2) with a density g (or a ch.f. ψ) has the following properties:

(i) X_1 and X_2 are normal $\mathcal{N}(0, 1)$;

(ii) $(X_1, X_2) \in GCS_s(2)$;

(iii) $aX_1 \pm bX_2$ are normal.

Only a comment on (iii) seems to be necessary: A ch.f. of $aX_1 + bX_2$ has the form

$$\mathrm{E}(\exp\{it(aX_1 + bX_2)\}) = \psi(at, bt) = \phi_{\Sigma}(at, bt), \qquad t \in \boldsymbol{R}.$$

Hence $aX_1 + bX_2$ is normal. Observe that ϕ_i, $i = 1, 2$, are odd. Consequently, $aX_1 - bX_2$ is also normal.

See Fig. 2 for an example of such a density with $a = b = 1$, $s = 2$. ∎

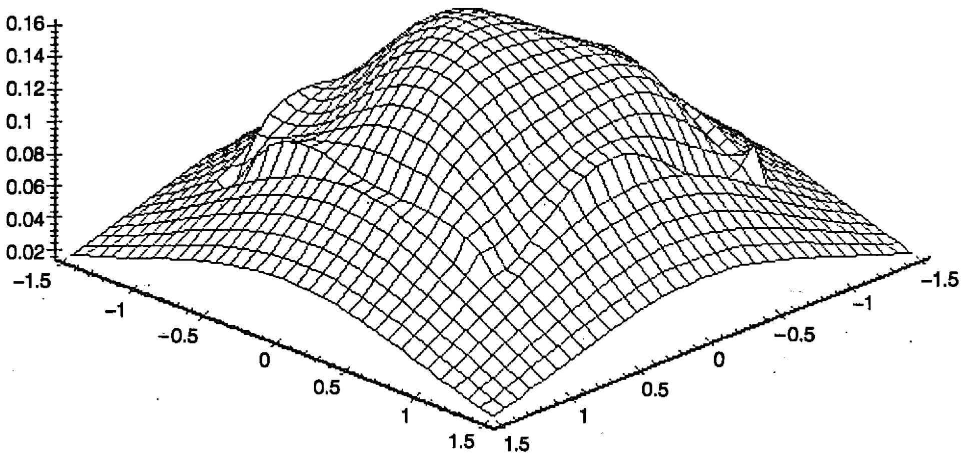

Fig. 2. The graph of the density

$$g(x_1, x_2) = \frac{1}{2\pi}\Big\{\exp\{-(x_1^2+x_2^2)/2\}$$
$$+\frac{105}{32}e^{-1}\Big[\Big(x_1 - \frac{5}{3}x_1^3\Big)\Big(\frac{3}{35} - \frac{6}{7}x_2^2 + x_2^4\Big) - \Big(\frac{3}{35} - \frac{6}{7}x_1^2 + x_1^4\Big)\Big(x_2 - \frac{5}{3}x_2^3\Big)\Big] I(|x_1| < 1, |x_2| < 1)\Big\}$$

Performing an extension similar to that which led us from Example 2.2 to Example 2.4 we arrive at the example of an n-dimensional non-Gaussian measure with even more unexpected property of linear forms.

EXAMPLE 3.2. *A non-Gaussian n-variate measure with Gaussian $(n-1)$-dimensional marginals, Gaussian conditional moments structure and uncountable number of normal projections.*

Let

$$A(t, u) = \phi_1(t)\phi_2(u) - \phi_2(t)\phi_1(u), \qquad t, u \in \mathbf{R},$$

where ϕ's are defined in Example 3.1. Take positive constants a, b and define an n-dimensional ch.f. by setting

$$\psi(\mathbf{t}) = \phi_{\Sigma}(\mathbf{t}) + CA(t_1/a, t_2/b) \prod_{j=3}^{n} \phi_j(t_j),$$

for any $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbf{R}^n$, where ϕ_{Σ} is a ch.f. of n-dimensional standardized Gaussian measure, $\phi_j(t) = \int_0^1 h_s(x)\sin(tx)dx$, $j = 3, \ldots, n$, for some odd function h_s, bounded in $(-1, 1)$ and satisfying condition (2.1) (see Example 3.1), and C is a constant. In other words, the p.d.f. associated with ψ takes the form

$$g_n(\mathbf{x}) = f(\mathbf{x}, \Sigma) + \alpha H(ax_1, bx_2) \prod_{j=3}^{n} h_s(x_j) I(|x_j| < 1)$$

for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, where

$$H(x, y) = [h_s^{(1)}(x)h_s^{(2)}(y) - h_s^{(2)}(x)h_s^{(1)}(y)] I(|x| < 1, |y| < 1), \qquad x, y \in \mathbf{R},$$

and α is some constant.

Now consider a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with a ch.f. ψ (or density g_n). It is easily seen that $\mathbf{X}$ is non-Gaussian and has the following properties:

(i) all $(n-1)$-dimensional marginals are Gaussian;

(ii) all conditional moments of the form (2.4) are exactly like in the Gaussian case;

(iii) all linear forms $aX_1 \pm bX_2 + c_3X_3 + \ldots + c_nX_n$, where $c_3, \ldots, c_n$ are any real numbers (a and b are fixed), are normal. ∎

However, an extension of Example 3.1 to any finite (fixed) number of normal linear forms in two dimensions is not immediate. To get them we need a result relating $GSC_2(2)$ structure to some properties of ch.f.'s.

LEMMA 1. *A bivariate standardized probability measure with correlation coefficient ϱ and a ch.f. τ belongs to $GCS_2(2)$ iff the following four identities hold:*

$$\left.\frac{\partial\tau(t, u)}{\partial t}\right|_{t=0} = \varrho\frac{d\tau(0, u)}{du}, \qquad \left.\frac{\partial\tau(t, u)}{\partial u}\right|_{u=0} = \varrho\frac{d\tau(t, 0)}{dt},$$

$$\left.\frac{\partial^2\tau(t,u)}{\partial t^2}\right|_{t=0} = (\varrho^2-1)\,\tau(0,u)+\varrho^2\frac{d^2\tau(0,u)}{du^2},$$

$$\left.\frac{\partial^2\tau(t,u)}{\partial u^2}\right|_{u=0} = (\varrho^2-1)\,\tau(t,0)+\varrho^2\frac{d^2\tau(t,0)}{dt^2}.$$

This is an immediate consequence of Lemma 1.1.3 of Kagan et al. [9].

EXAMPLE 3.3. *A non-Gaussian $GCS_2(2)$ measure with Gaussian marginals and an arbitrary (but fixed) number of normal projections.*

We use notation introduced in the preceding examples. Assume here that the functions h_s's, upon which ϕ_i's are built, satisfy (2.1) with $s = 2$. Observe that $A(t, 0) = A(0, u) = 0$ for any $t, u \in \boldsymbol{R}$. Now by Lemma 1 and Example 3.1 we conclude that

$$\text{(3.1)}\qquad \left.\frac{\partial^i A(t,u)}{\partial t^i}\right|_{t=0} = 0, \qquad \left.\frac{\partial^i A(t,u)}{\partial u^i}\right|_{u=0} = 0, \qquad i = 1, 2.$$

Consider now pairs of positive reals (a_i, b_i), $i = 1, \ldots, K$, where K is an arbitrary, but fixed number. Put additionally

$$A_i(t,u) = A(t/a_i, u/b_i), \qquad i = 1, \ldots, K,\ t, u \in \boldsymbol{R}.$$

LEMMA 2. *The function*

$$\Psi = \phi_\Sigma^K + c\prod_{i=1}^{K} A_i$$

is a bivariate ch.f. for some constant c.

Proof. Let

$$\Psi_m^{\pm} = \phi_\Sigma^m \pm c_m \prod_{i=1}^{m} A_i, \qquad m = 1, \ldots, K.$$

Apply mathematical induction to show that there exists c_m such that $\Psi_m^{\pm}$ are ch.f.'s, $m = 1, \ldots, K$. For $m = 1$ this follows from Example 3.1. Now assume that $\Psi_m^{\pm}$ for some $m = 1, \ldots, K-1$ are ch.f.'s and consider

$$\tilde{\Psi}_{m+1}^{\pm} = \phi_\Sigma \pm \tilde{c}_{m+1} A_{m+1}.$$

Observe that, again by Example 3.1, $\tilde{\Psi}_{m+1}^{\pm}$ are ch.f.'s. Since convex combinations of ch.f.'s are again ch.f.'s, the result follows from the formula

$$(\Psi_m^{+}\tilde{\Psi}_{m+1}^{\pm} + \Psi_m^{-}\tilde{\Psi}_{m+1}^{\mp})/2 = \phi_\Sigma^{m+1} \pm c_m \tilde{c}_{m+1} \prod_{i=1}^{m+1} A_i. \ \blacksquare$$

Consider a random vector (X_1, X_2) with a ch.f.

$$\psi(t,u) = \Psi\left(\frac{t}{\sqrt{K}}, \frac{u}{\sqrt{K}}\right), \qquad t, u \in \boldsymbol{R}.$$

Obviously, (X_1, X_2) is non-Gaussian and has the following properties:

(i) X_1 and X_2 are normal $\mathcal{N}(0, 1)$;

(ii) $(X_1, X_2) \in GCS_2(2)$;

(iii) $a_i X_1 + b_i X_2$ is normal, $i = 1, 2, \ldots, K$.

Now (iii) holds by means of the same argument as in Example 3.1, (i) follows from the fact that $A = 0$ if only one of its arguments is zero, and (ii) follows from Lemma 1 and (3.1). ■

The last example of this section extends the idea of Example 3.3 to any finite dimension and a more restrictive Gaussian structure. It summarizes all the ideas developed in Section 3.

EXAMPLE 3.4. *A non-Gaussian n-variate measure with $(n-1)$-variate Gaussian marginals, Gaussian conditional moments structure and normal projections.*

As in Example 3.3 it can be proved that

$$\Psi(t) = \phi_\Sigma(t) + c \prod_{i=1}^{K} B_i(t), \qquad t \in \mathbf{R}^n,$$

where

$$B_i(t) = A_i(t_1, t_2) \prod_{j=3}^{n} \phi_j(t_j), \qquad t \in \mathbf{R}^n, \ i = 1, \ldots, K$$

(A_i's are defined in Example 3.3), is a ch.f. for some constant c. Then a random vector $X = (X_1, \ldots, X_n)$ with the ch.f. Ψ has the following properties:

(i) all its $(n-1)$-dimensional marginals are Gaussian;

(ii) all its conditional moments (2.4) are exactly of the Gaussian form;

(iii) all linear forms $a_i X_1 + b_i X_2 + c_3 X_3 + \ldots + c_n X_n$, $i = 1, \ldots, K$, where $c_3, \ldots, c_n$ are any real numbers, are normal.

Properties (i) and (iii) are immediate, and (ii) follows from an analogue of Lemma 1 for higher conditional moments — it involves higher order derivatives of the ch.f.; see, for example, formula (13) in Bryc and Szablowski [4]. ■

4. Polynomial regressions and normal marginals. Here we are interested in bivariate measures for which all the conditional moments are of polynomial type and marginals are normal. A symmetrized version of the Kagan–Linnik–Rao problem is discussed (Example 4.1).

In the preceding section we presented an example of a bivariate non-Gaussian measure with normal marginals, Gaussian-like (polynomial) conditional moments up to some arbitrary (but fixed order) and normal projections. Consequently, Theorem 3.3 of Bryc and Szablowski [4] — see Section 1 — cannot be improved by considering only a finite number of conditional moments. On the other hand, in Section 2 we gave an example of a non-Gaussian measure which yields that the same kind of refinement for Theorem 3.1 of that paper (see again Section 1) is impossible.

Both the above-mentioned results rely on assumptions of normality of marginals and polynomial regressions. These assumptions are completed by some conditions, seemingly of technical nature – a special form of leading polynomial coefficients or normality of a linear form. However, the Gaussian mixture example, given beneath, proves that these additional assumptions are really important.

EXAMPLE 4.1. *A bivariate non-Gaussian measure with normal marginals and all polynomial regressions.*

Let F be a c.d.f. of a distribution with the support $(-1, 1)$. Denote by f_r the standardized bivariate Gaussian density with a correlation coefficient r. Let (X, Y) be a Gaussian mixture with the density

$$g = \int_{-1}^{1} f_r \, dF(r).$$

Then it can be easily checked that the marginals are normal $\mathcal{N}(0, 1)$ and the regressions take the forms

$$\mathrm{E}(Y^k|X) = \varrho_k X^k + Q_{k-1}(X), \quad \mathrm{E}(X^k|Y) = \varrho_k Y^k + Q_{k-1}(Y),$$

where Q_{k-1} is a polynomial of order less than or equal to $k-1$ and

$$\varrho_k = \int_{-1}^{1} r^k dF(r), \quad k = 1, 2, \ldots$$

Obviously, the identity $\varrho_k = \varrho^k$, $k = 1, 2, \ldots$, where $\varrho \in (-1, 1)$, implies bivariate normality (since then F is degenerate). ∎

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