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A LÉVY KHINTCHINE TYPE REPRESENTATION OF CONVOLUTION SEMIGROUPS ON COMMUTATIVE HYPERGROUPS

BY

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Abstract. We first study Lévy measures, Poisson and Gaussian convolution semigroups on commutative hypergroups. Then we present a Lévy–Khintchine type representation of a convolution semigroup $(\mu_t)_{t>0}$ with symmetric Lévy measure λ of the form $\mu_t = \gamma_t * e(t\lambda)$, $t \geqslant 0$, for some Poisson semigroup $(e(t\lambda))_{t>0}$ and some Gaussian semigroup $(\gamma_t)_{t>0}$.

1. Introduction. Limit theorems of random variables and distributions are of great interest in probability theory. The distributions occurring as limits are infinitely divisible and therefore can be embedded into convolution semigroups. For that reason, convolution semigroups play an important role and are thoroughly studied on various algebraic-topological structures.

On locally compact abelian groups, convolution semigroups are completely determind by negative definite functions, as described for instance in Berg and Forst [3]. In [9] Lasser proved a Lévy–Khintchine representation for negative definite functions defined on the dual of a commutative hypergroup under the assumption that this dual admits a dual convolution structure. It is also possible to give such representations for certain classes of negative definite functions which are defined on the hypergroup itself. This was shown by Bloom and Ressel in [5]. But these negative definite functions do not generally correspond to convolution semigroups.

In concrete situations Lévy–Khintchine type formulae are also available when the dual is not necessarily a hypergroup. Examples are given by Sturm–Liouville hypergroups (cf. [1] and [6]) and products of Sturm–Liouville hypergroups with $\boldsymbol{R}^n$ (see [13]).

The aim of the subsequent discussion is to decompose convolution semigroups on arbitrary commutative hypergroups into Gaussian and Poisson factors.

The present paper contains some sections of the author's doctor thesis. Parts of the results below have been announced without proof in [11].

For a locally compact space K we reserve the symbols $\mathcal{M}^1(K) \subset \mathcal{M}_+^b(K) \subset \mathcal{M}_+(K)$ for the sets of probability measures, bounded nonnegative and nonnegative measures on K, respectively. $C_c(K) \subset C_0(K) \subset C_b(K)$ denote the spaces of continuous functions on K which are compactly supported, vanish at infinity, and are bounded, respectively. $\mathcal{U}(x)$ stands for the neighborhood base of an element $x \in K$. The symbols τ_v-lim and τ_w-lim denote vague and weak limit relations of sequences or nets of measures, respectively.

A commutative hypergroup $(K, *)$ consists mainly of a locally compact set K and a bilinear, associative, commutative and weakly continuous convolution $*$ on $\mathcal{M}^b(K)$. Moreover, there exists a neutral element $e \in K$ (i.e. the point measure ε_e is the neutral element of the convolution $*$), and a continuous involution $x \mapsto x^-$ on K such that $e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ iff $x = y^-$ for arbitrary $x, y \in K$. If the convolution operation is fixed, the hypergroup is denoted by K instead of $(K, *)$.

For $A, B \subset K$ we use the abbreviation $A*B := \bigcup_{x \in A, y \in B} \{x\}*\{y\}$, where the symbol $\{x\}*\{y\}$ stands for $\operatorname{supp}(\varepsilon_x * \varepsilon_y)$. We set $f^-(x) := f(x^-)$ for $x \in K$, $\mu^-(A) := \mu(A^-)$ for $\mu \in \mathcal{M}(K)$, and Borel sets $A \subset K$; here A^- denotes the image of A under the involution mapping $^-$, whereas $\overline{A}$ is the closure of a set $A \subset K$.

The dual space $K^\wedge$ of a commutative hypergroup is defined by

$$K^\wedge := \{\chi \in C_b(K) \colon \chi \not\equiv 0,\ \chi(x*y^-) = \chi(x)\overline{\chi(y)}\ \forall x, y \in K\},$$

where $f(x*y^-) := \int f \, d\varepsilon_x * \varepsilon_{y^-}$ for admissible functions $f: K \to \mathbf{C}$. The dual space $K^\wedge$ furnished with the topology of uniform convergence on compact subsets of K becomes a locally compact space. The Fourier transformation $^\wedge\colon \mathcal{M}^b(K) \to C_b(K^\wedge)$ is defined by $\mu \mapsto \hat{\mu}$, where

$$\hat{\mu}(\chi) := \int \bar{\chi} \, d\mu \quad \text{for } \mu \in \mathcal{M}^b(K),\ \chi \in K^\wedge.$$

The inverse Fourier transform $\check{\mu}$ of a measure $\mu \in \mathcal{M}^b(K^\wedge)$ is given by

$$\check{\mu}(x) := \int_{K^\wedge} \chi(x)\, \mu(d\chi) \quad \text{for } x \in K.$$

Moreover, we denote the Haar measure and the Plancherel measure of K by ω_K and π_K, respectively, and define $\check{g} := (g\pi_K)^\vee$ for $g \in \mathcal{L}^1(K^\wedge, \pi_K)$. For $\mu \in \mathcal{M}_+^b(K)$, $f, h \in C_0(K)$, the functions $\mu*f, f*h \in C_0(K)$ are given by

$$\mu*f(x) := \int f(y^- * x)\, \mu(dy) \quad \text{and} \quad f*h(x) := \int f(y) h(x*y^-)\, \omega_K(dy)$$

for $x \in K$, respectively.

For details and further information about hypergroups we refer to Jewett [8] and to the monograph of Bloom and Heyer [4].

For the entire paper let K be a commutative hypergroup.

2. Convolution semigroups. In this section we introduce the notion of convolution semigroups on commutative hypergroups.

2.1. DEFINITION. (a) We call a family $(\mu_t)_{t>0}$ in $\mathscr{M}^1(K)$ a *(continuous) convolution semigroup* if

$$\mu_s * \mu_t = \mu_{s+t} \text{ for all } s, t > 0 \quad \text{and} \quad \tau_w\text{-}\lim_{t\to 0} \mu_t = \varepsilon_e.$$

(b) For a convolution semigroup $(\mu_t)_{t>0}$ on $(K, *)$ the *(infinitesimal) generator* A is defined by

$$A\phi := \lim_{t\to 0} \frac{1}{t} (\mu_t * \phi - \phi)$$

for all

$$\phi \in D(A) := \left\{\psi \in C_0(K)\colon \lim_{t\to 0} \frac{1}{t} (\mu_t * \psi - \psi) \text{ exists and belongs to } C_0(K)\right\}.$$

Here the *limits* are understood as limits with respect to the topology of uniform convergence on $C_0(K)$.

The symbol 1_A denotes the characteristic function of the set $A \subset K$. Hence the function 1_K assigns to each $x \in K$ the value 1; it is therefore the constant character of the hypergroup.

2.2. DEFINITION. Let $\mathscr{M}_{ac}(K^\wedge) := \{c\varepsilon_{1_K} + g\pi_K \in \mathscr{M}_c(K^\wedge) : c \in C, g \in C_c(K^\wedge)\}$. We say that the function $\psi \in C(K^\wedge)$ is *strongly negative definite* if the following holds:

(1) For $\mu \in \mathscr{M}_{ac}(K^\wedge)$ with $\check{\mu} \geqslant 0$, $\check{\mu}(e) = 0$ we have $\int \psi \, d\mu \leqslant 0$.

(2) $\psi(1_K) \geqslant 0$ and $\psi(\overline{\chi}) = \overline{\psi(\chi)}$ for all $\chi \in K^\wedge$.

We denote the set of all strongly negative definite functions by $\mathrm{SN}(K^\wedge)$.

The following characterization of strongly negative definite functions is due to Voit [15], Theorem 3.7.

2.3. THEOREM. *If $(\mu_t)_{t>0}$ denotes a convolution semigroup on K, then there exists a uniquely determined function $\psi \in \mathrm{SN}(K^\wedge)$ with $\psi(1_K) = 0$ which satisfies $\hat{\mu}_t = e^{-t\psi}$ for all $t > 0$. Conversely, if ψ is a strongly negative definite function on $K^\wedge$ with $\psi(1_K) = 0$, then there exists a uniquely determined convolution semigroup $(\mu_t)_{t>0}$ on K such that $\check{\mu}_t|_{\mathrm{supp}(\pi_K)} = -e^{-t\psi}|_{\mathrm{supp}(\pi_K)}$ for all $t > 0$.*

We first study generators of convolution semigroups and their domains.

2.4. LEMMA. *Let A be the generator of a convolution semigroup $(\mu_t)_{t>0}$ on the hypergroup K. Then for every neighborhood U of $e \in K$ there exists a function $\phi \in D(A)$ which satisfies*

(2.1) $$\phi \geqslant 0, \quad \phi \not\equiv 0, \quad \textit{and} \quad \mathrm{supp}(\phi) \subset U.$$

Proof. Choose a neighborhood $V \in \mathscr{U}(e)$ such that $\overline{V} * \overline{V} \subset U$ (cf. Lemma 3.2 of [8]). Moreover, let $f \in C_c(K)$ with $f \geqslant 0$, $\mathrm{supp}(f) \subset V$ and f not identically zero. We regard the resolvent $\varrho_1 = \int_0^\infty e^{-t} \mu_t \, dt$ as the potential of the submarkovian semigroup $(e^{-t}\mu_t)_{t>0}$ and conclude from Theorem 6.4.16 in Bloom and Heyer [4] that ϱ_1 is perfect. This means in particular that there exists a measure $\sigma \in \mathcal{M}_+^b(K)$ with

(2.2) $$\sigma * \varrho_1 \leqslant \varrho_1, \quad \sigma * \varrho_1 \neq \varrho_1, \quad \text{and} \quad \sigma * \varrho_1 = \varrho_1 \quad \text{on } \complement V.$$

The function $\phi := (\varrho_1 - \sigma * \varrho_1) * f$ now has the properties (2.1). In fact, (2.2) ensures that $\phi \geqslant 0$ with $\phi \neq 0$ since $f \geqslant 0$ and $f \neq 0$. Moreover, $\operatorname{supp}(\varrho_1 - \sigma * \varrho_1) \subset \bar{V}$, and hence

$$\operatorname{supp}((\varrho_1 - \sigma * \varrho_1) * f) = \operatorname{supp}(\varrho_1 - \sigma * \varrho_1) * \operatorname{supp}(f) \subset \bar{V} * \bar{V} \subset U.$$

Finally, it follows from the general theory of resolvents of one-parameter contraction semigroups that $\phi = \varrho_1 * (f - \sigma * f) \in D(A)$; cf. Corollary IX.4.2 in Yosida [16]. ■

2.5. LEMMA. *Let A denote the generator of a convolution semigroup $(\mu_t)_{t>0}$ on K and U an open and relatively compact subset of K. Suppose that $\varepsilon > 0$ and $f \in C_c(K)$, $f \geqslant 0$, with $\operatorname{supp}(f) \subset U$. Then there exists a function $g \in D(A) \cap C_c(K)$ with $g \geqslant 0$ and $\operatorname{supp}(g) \subset U$ such that $\|f - g\|_\infty \leqslant \varepsilon$.*

Proof. Choose $V_1 \in \mathcal{U}(e)$ such that $V_1 * \operatorname{supp}(f) \subset U$ (see Lemma 3.2D in Jewett [8]). By Proposition 1.2.28 in Bloom and Heyer [4] the function f is α-uniformly continuous. This means that there exists an open set $V_2 \in \mathcal{U}(e)$ such that

(2.3) $$|f(t) - f(s)| < \varepsilon$$

for all $s, t \in K$ satisfying $\varepsilon_s * \varepsilon_{t^-}(V_2) > 0$. For $x \in K$, $y \in V_2$ and $t \in \{x\} * \{y^-\} \subset \{x\} * V_2^-$, Lemma 4.1B of [8] yields $\{x\} * \{t^-\} \cap V_2 \neq \emptyset$. Hence (2.3) implies

(2.4) $$|f(x * y^-) - f(x)| \leqslant \int |f(t) - f(x)| \, \varepsilon_x * \varepsilon_{y^-}(dt) \leqslant \varepsilon.$$

By Lemma 2.4 there exists a function $\phi \in D(A)$ such that

(2.5) $$\phi \geqslant 0, \quad \operatorname{supp}(\phi) \subset V_1 \cap V_2, \quad \text{and} \quad \int \phi \, d\omega_K = 1.$$

Therefore, $\operatorname{supp}(\phi * f) = \operatorname{supp}(\phi) * \operatorname{supp}(f) \subset V_1 * \operatorname{supp}(f) \subset U$, and $\phi * f \geqslant 0$. Moreover, from (2.4) and (2.5) we infer for every $x \in K$ that

$$|\phi * f(x) - f(x)| \leqslant \int \phi(y) |f(x * y^-) - f(x)| \, \omega_K(dy) \leqslant \varepsilon.$$

Finally, convolution with a $C_c(K)$-function is a bounded linear operator on $C_b(K)$; hence $D(A) * C_c(K) \in D(A)$. ■

We say that K is a *Godement hygergroup* if the unit character $1_K \in K^\wedge$ is contained in the support of the Plancherel measure π_K of K. There are many examples known which lack this property; for instance Sturm–Liouville hypergroups with exponential growth (see Section 3.5 in Bloom and Heyer [4] for more detailed information).

2.6. LEMMA. *Let $(\mu_t)_{t>0}$ denote a convolution semigroup on K. Then we have*

(2.6) $$\limsup_{t \to 0} \frac{1}{t} \mu_t(\complement U) < \infty$$

for every open $U \in \mathscr{U}(e)$. *Furthermore, let* $\delta > 0$. *If the hypergroup K enjoys the Godement property, then we can find* $t_0 > 0$ *and a relatively compact open set* $U_0 \in \mathscr{U}(e)$ *such that*

$$\frac{1}{t}\,\mu_t(\complement U_0) < \delta \quad \textit{for all } 0 < t \leqslant t_0. \tag{2.7}$$

Proof. Let $\alpha \in K^\wedge$ denote the unique positive character in $\operatorname{supp}(\pi_K)$ (cf. Theorem 2.11 in [14]), and $\psi \in \mathrm{SN}(K^\wedge)$ the strongly negative definite function corresponding to $(\mu_t)_{t>0}$ according to Theorem 2.3. The continuous function ψ satisfies $\overline{\psi(\alpha)} = \psi(\bar{\alpha}) = \psi(\alpha) \geqslant 0$. Hence we may choose a neighborhood V of α in $K^\wedge$ such that

$$|\psi(\chi)| \leqslant \psi(\alpha) + \delta/2 \quad \text{for all } \chi \in V. \tag{2.8}$$

Since $V \cap \operatorname{supp}(\pi_K) \neq \emptyset$, there exists a function $\phi \in C_c(K^\wedge)$ having the following properties:

$$\phi \geqslant 0, \quad \operatorname{supp}(\phi) \subset V, \quad \phi(\chi) = \phi(\bar{\chi}) \text{ for all } \chi \in K^\wedge, \quad \int \phi\, d\pi_K = 1. \tag{2.9}$$

The inequality $\operatorname{Re}\psi \geqslant 0$ implies

$$\left|\frac{e^{-t\psi} - 1}{t}\right| \leqslant |\psi|,$$

and hence Lebesgue's theorem yields

$$\lim_{t\to 0} \frac{1}{t}\left(\int \check{\phi}\, d\mu_t - \check{\phi}(e)\right) = \lim_{t\to 0} \int \frac{e^{-t\psi} - 1}{t}\,\phi\, d\pi_K = -\int \psi\phi\, d\pi_K. \tag{2.10}$$

Moreover, the real-valued function ϕ satisfies $1 = \check{\phi}(e) = \|\check{\phi}\|_\infty$ by (2.9); hence the set $U_0 := \{x \in K\colon \check{\phi}(x) \geqslant \frac{1}{2}\}$ is an open neighborhood of e which is also relatively compact by Theorem 2.2.32 (vi) in Bloom and Heyer [4].

Given an open neighborhood U of e there is a function $g^- \in D(A)$ with $g(e) = 0$, $g \geqslant 1_{U_0 \cap \complement U}$. Here A denotes the generator of $(\mu_t)_{t \geqslant 0}$. From (2.8) to (2.10) we conclude that

$$\begin{aligned}
\limsup_{t\to 0} \frac{1}{t}\mu_t(\complement U) &\leqslant \lim_{t\to 0} \frac{1}{t}\left(\int g\, d\mu_t - g(e)\right) + \limsup_{t\to 0} \frac{1}{t}\,\mu_t(\complement U_0)\\
&\leqslant Ag^-(e) + \lim_{t\to 0} \frac{2}{t} \int (1 - \check{\phi})\, d\mu_t\\
&= Ag^-(e) + 2\int \psi\phi\, d\pi_K \leqslant Ag^-(e) + 2\psi(\alpha) + \delta.
\end{aligned}$$

If K is a Godement hypergroup (i.e. $1_K = \alpha \in \operatorname{supp}(\pi_K)$), we find with the help of (2.8) and (2.10) the following inequality:

$$\limsup_{t\to 0} \frac{1}{t}\,\mu_t(\complement U_0) \leqslant \lim_{t\to 0} \frac{2}{t} \int (1 - \check{\phi})\, d\mu_t = 2\int \psi\phi\, d\pi_K \leqslant 2\psi(1_K) + \delta = \delta,$$

since $1 = \check{\mu}_1(1_K) = e^{-\psi(1_K)}$ implies $\psi(1_K) = 0$. ∎

3. Lévy, Poisson and Gaussian measures. We introduce Lévy measures in two different ways and discuss the relation between them later. The following definition is suggested by the theory of probability measures on Banach spaces (see for instance Section 5.4 in Linde [10]). We say that $\lambda \in \mathcal{M}_+(K)$ is *symmetric* if $\lambda = \lambda^-$.

3.1. DEFINITION. A symmetric $\lambda \in \mathcal{M}_+(K)$ is said to be a *symmetric abstract Lévy measure* if the function

$$K^\wedge \ni \chi \mapsto \exp\left(\int (\operatorname{Re}\chi - 1)\, d\lambda\right) \tag{3.1}$$

is the Fourier transform of a probability measure $e(\lambda)$ on K. We call $e(\lambda)$ the (*generalized*) *Poisson measure related to* λ.

An arbitrary measure $\lambda \in \mathcal{M}_+(K)$ is an *abstract Lévy measure* if $\lambda + \lambda^-$ is a symmetric abstract Lévy measure.

3.2. Remarks. (1) We do not require the finiteness of the integral in (3.1). If the integral does not exist as a real number, the Fourier transform of the corresponding Poisson measure vanishes. If $\lambda \in \mathcal{M}_+(K)$ is a symmetric abstract Lévy measure such that $\int (1 - \operatorname{Re}\chi)\, d\lambda < \infty$ for all characters $\chi \in K^\wedge$, then it is clear that the related Poisson measures $(e(t\lambda))_{t>0}$ form a convolution semigroup on K.

(2) The exponential of a finite measure is defined by

$$\exp(\mu) := \sum_{k=0}^{\infty} \frac{1}{k!} \mu^{*k} \in \mathcal{M}_+^b(K) \qquad \text{for } \mu \in \mathcal{M}_+^b(K),$$

where μ^{*k} denotes the k-th convolution power of μ with respect to $*$, and $\mu^{*0} := \varepsilon_e$. Now, if $\mu \in \mathcal{M}_+^b(K)$ has no mass at the neutral element $e \in K$, then μ is an abstract Lévy measure, and

$$e(\mu) = e^{-\mu(K)} \exp(\mu). \tag{3.2}$$

Therefore, we take (3.2) as the definition of $e(\mu)$ for arbitrary measures $\mu \in \mathcal{M}_+^b(K)$, and call $e(\mu)$ again a *Poisson measure.*

In order to introduce Lévy measures which are related to convolution semigroups, we need the following lemma. Along with the abbreviation $K^\times := K \setminus \{e\}$ we set

$$C_c(K^\times) := \{f \in C_c(K)\colon\ e \notin \operatorname{supp}(f)\}.$$

For a Borel set $B \subset K$ and $\mu \in \mathcal{M}(K)$ let $\mu | B$ denote the measure $\mu(\cdot \cap B)$.

3.3. LEMMA. *Let $(\mu_t)_{t>0}$ be a convolution semigroup on K. Then there exists a uniquely determined measure $\lambda \in \mathcal{M}_+(K^\times)$ such that*

$$\lambda = \tau_v\text{-}\lim_{t \to 0} \frac{1}{t} \mu_t | K^\times.$$

Proof. Let A be the generator of $(\mu_t)_{t>0}$. We define

$$\lambda(\phi) = \int \phi \, d\lambda := \lim_{t\to 0} \frac{1}{t} \int \phi \, d\mu_t = A\phi^- (e)$$

for arbitrary $\phi^- \in D(A) \cap C_c(K^\times)$. The assertion is proved once we have shown that for every $f \in C_c(K^\times)$ the family $(t^{-1} \int f d\mu_t)_{t>0}$ is a Cauchy net for $t \to 0$. In fact, choose $U \in \mathscr{U}(e)$ with $\operatorname{supp}(f) \subset \complement U$ and let $\varepsilon > 0$. Lemma 2.5 ensures the existence of a function $\phi^- \in D(A) \cap C_c(K^\times)$ which satisfies $\operatorname{supp}(\phi) \subset \complement \bar{U}$ and $\|\phi - f\|_\infty < \varepsilon$. Hence, by (2.6), there exists $C > 0$ such that

$$\left|\frac{1}{s} \int f \, d\mu_s - \frac{1}{t} \int f \, d\mu_t\right| \leqslant \left|\frac{1}{s} \int \phi \, d\mu_s - \frac{1}{t} \int \phi \, d\mu_t\right| + \varepsilon \left(\frac{1}{s} \mu_s(\complement U) + \frac{1}{t} \mu_t(\complement U)\right) \leqslant C\varepsilon$$

for sufficiently small $s, t > 0$. Here the constant C may be chosen independently of s, t and ε. ■

We call $\lambda \in \mathscr{M}_+(K)$ the *Lévy measure* of the convolution semigroup $(\mu_t)_{t>0}$ if $\lambda(\{e\}) = 0$ and

$$\tau_v\text{-}\lim_{t\to 0} \frac{1}{t} \mu_t \,|\, K^\times = \lambda \,|\, K^\times. \tag{3.3}$$

3.4. LEMMA. *If λ_μ and λ_ν are the Lévy measures of the convolution semigroups $(\mu_t)_{t>0}$ and $(\nu_t)_{t>0}$, respectively, then $\lambda_\mu + \lambda_\nu$ is the Lévy measure of $(\mu_t * \nu_t)_{t>0}$.*

Proof. If A^- denotes the generator of $(\mu_t^-)_{t>0}$, we see for $f \in C_c(K^\times) \cap D(A^-)$ and $t > 0$ that

$$\frac{1}{t} \int f d\mu_t * \nu_t = \frac{1}{t} \int\int f(x*y) \, \mu_t(dx) \, \nu_t(dy)$$

$$= \int \left[\frac{1}{t}\left(\int f(x*y) \, \mu_t(dx) - f(y)\right) - A^- f(y)\right] \nu_t(dy) + \frac{1}{t} \int f d\nu_t + \int A^- f \, d\nu_t.$$

The first term on the right-hand side converges to zero if $t \to 0$. Moreover, by definition,

$$\int f d\lambda_\nu = \lim_{t\to 0} \frac{1}{t} \int f d\nu_t$$

and the continuity of the convolution semigroup $(\nu_t)_{t>0}$ yields

$$\lim_{t\to 0} \int A^- f d\nu_t = A^- f(e) = \int f d\lambda_\mu.$$

Hence

$$\lim_{t\to \infty} \frac{1}{t} \int f \, d\mu_t * \nu_t = \int f \, d(\lambda_\mu + \lambda_\nu) \tag{3.4}$$

whenever $f \in C_c(K^\times) \cap D(A^-)$. Now, equality (3.4) can be shown for arbitrary $f \in C_c(K^\times)$ by the same method as in the proof of Lemma 3.3 using the denseness of $C_c(K^\times) \cap D(A^-)$ in $C_c(K^\times)$. ■

3.5. EXAMPLE. Let $\lambda \in \mathcal{M}_+^b(K)$ be a finite Lévy measure. Standard arguments show that $T_\lambda - \lambda(K)\,\mathrm{id}_{C_0(K)}$ is the generator of the convolution semigroup $(e(t\lambda))_{t>0}$. Moreover, λ is the related Lévy measure. In fact, for $g \in C_c(K^\times)$ we have

$$\lim_{t\to 0} \frac{1}{t} \int g\, de(t\lambda) = \left[\left(T_\lambda - \lambda(K)\,\mathrm{id}_{C_0(K)}\right) g^-\right](e) = \int g\, d\lambda.$$

3.6. DEFINITION. A convolution semigroup $(\mu_t)_{t>0}$ on K is called *Gaussian* if

$$\lim_{t\to 0} \frac{1}{t}\, \mu_t(\complement U) = 0$$

for every open set U containing $e \in K$.

A measure $\mu \in \mathcal{M}^1(K)$ is said to be *Gaussian* if there exists a Gaussian convolution semigroup $(\mu_t)_{t>0}$ with $\mu_1 = \mu$.

We say that the generator A of a convolution semigroup is of *local type* if $\mathrm{supp}(A\phi) \subset \mathrm{supp}(\phi)$ for all $\phi \in D(A)$. A generator A is of local type iff $A\phi(e) = 0$ for all $\phi \in D(A)$ with $e \notin \mathrm{supp}(\phi)$.

The following characterization of Gaussian convolution semigroups is well known in the group setting (compare for instance Berg and Forst [3]). For commutative hypergroups admitting a dual convolution structure this characterization is due to Heyer [7].

3.7. THEOREM. *Let K be a commutative hypergroup and $(\mu_t)_{t>0}$ a convolution semigroup on K with Lévy measure λ and generator A. Then the following conditions are equivalent:*

(1) *$(\mu_t)_{t>0}$ is a Gaussian convolution semigroup,*

(2) *A is of local type, and*

(3) *$\lambda = 0$.*

Proof. For the equivalence (1)$\Leftrightarrow$(2) see Proposition 2.12 in Rentzsch and Voit [12]. The inclusion (1)$\Rightarrow$(3) is clear as

$$\lambda \,|\, K^\times = \tau_v\text{-}\lim_{t\to 0} \frac{1}{t} \mu_t \,|\, K^\times.$$

To prove (3)$\Rightarrow$(2) take a function $\phi \in D(A)$ which vanishes in the open neighborhood U of e. We have to show that $A\phi(e) = 0$. This is clear by (3.3) if $\mathrm{supp}(\phi)$ is compact. Otherwise, let $\varepsilon > 0$ and choose $g \in C_c(K^\times)$ such that $\mathrm{supp}(g) \subset \complement U$ and $\|\phi^- - g\|_\infty < \varepsilon$. Using (3.3) we conclude that

$$|A\phi(e)| = \left|\lim_{t\to 0} \frac{1}{t} \int [(\phi^- - g) + g]\, d\mu_t\right| \leqslant \varepsilon \limsup_{t\to 0} \frac{1}{t}\, \mu_t(\complement U) + \left|\int g\, d\lambda\right|.$$

By condition (3) and inequality (2.6), the right-hand side of the last inequality becomes arbitrarily small. ■

4. The Lévy-Khintchine formula. Before we state our main theorem we need some additional lemmata. For $\lambda \in \mathcal{M}_+(K)$ we call a Borel subset $B \subset K$ a *λ-continuity set* if $\lambda(\partial B) = 0$, where ∂B is the boundary of B.

4.1. LEMMA. *Let K be a Godement hypergroup, and $(\mu_t)_{t>0}$ a convolution semigroup on K with Lévy measure λ. Then*

$$\tau_w\text{-}\lim_{t\to 0} \frac{1}{t}\, \mu_t | \complement U = \lambda | \complement U \tag{4.1}$$

for every open λ-continuity set $U \in \mathscr{U}(e)$.

Proof. Let U be as above. By standard arguments it follows that

$$\tau_v\text{-}\lim_{t\to 0} \frac{1}{t}\, \mu_t | \complement U = \lambda | \complement U$$

(use, for instance, Satz 30.2 of Bauer [2]). Let $\varepsilon > 0$. By (2.7) we can choose $V \in \mathscr{U}(e)$ and $t_0 > 0$ such that

$$\frac{1}{t}\, \mu_t(\complement V) + \lambda(\complement V) < \varepsilon \quad \text{for all } t \leqslant t_0. \tag{4.2}$$

For some $\phi \in C_c(K^\times)$ with $1 \geqslant \phi \geqslant 1_{V \cap \complement U}$ and an arbitrary function $f \in C_b(K)$ we have

$$\begin{aligned}\left|\frac{1}{t}\int_{\complement U} f\, d\mu_t - \int_{\complement U} f\, d\lambda\right| &= \left|\frac{1}{t}\int_{\complement U} (f(1-\phi)+f\phi)\, d\mu_t - \int_{\complement U} (f(1-\phi)+f\phi)\, d\lambda\right| \\ &\leqslant \left|\frac{1}{t}\int_{\complement U} f\phi\, d\mu_t - \int_{\complement U} f\phi\, d\lambda\right| + \|f\|_\infty \left(\frac{1}{t}\, \mu_t(\complement V) + \lambda(\complement V)\right).\end{aligned}$$

Hence $\tau_v\text{-}\lim_{t\to 0} t^{-1} \mu_t | \complement U = \lambda | \complement U$ together with (4.2) yields the assertion. ∎

4.2. LEMMA. *For each convolution semigroup $(\mu_t)_{t>0}$ on K with Lévy measure λ we have*

$$\tau_w\text{-}\lim_{t\to 0} e\left(\frac{s}{t}\mu_t\right) = \mu_s \quad \textit{for all } s > 0. \tag{4.3}$$

Moreover, if K is a Godement hypergroup, then

$$\tau_w\text{-}\lim_{t\to 0} e\left(\frac{s}{t}\mu_t | \complement U\right) = e(s\lambda | \complement U) \quad \textit{for all } s > 0 \tag{4.4}$$

and each open λ-continuity set $U \in \mathscr{U}(e)$.

Proof. Writing ψ for the strongly negative definite function associated with $(\mu_t)_{t>0}$ according to Theorem 2.3 we conclude for every $\chi \in K^\wedge$ and $s > 0$ that

$$\lim_{t\to 0} e\left(\frac{s}{t}\, \mu_t\right)^\wedge (\chi) = \exp\left(s \lim_{t\to 0} \frac{e^{-t\psi(\chi)} - 1}{t}\right) = e^{-s\psi(\chi)} = \hat{\mu}_s(\chi).$$

Hence (4.3) follows from the Lévy continuity theorem (cf. Theorem 4.2.11 in [4]) by noting that $\mu_s \in \mathscr{M}^1(K)$.

Assertion (4.4) follows easily from Lemma 4.1 by using again the Lévy continuity theorem. ∎

Unfortunately, we do not know whether the second statement of Lemma 4.1 remains true without the additional Godement property. But, if a hypergroup does not enjoy the Godement property, it is possible to modify the convolution so that the unit character belongs to the support of the Plancherel measure of the modified hypergroup. If $\alpha \in K^\wedge$ denotes a positive character of the hypergroup $(K, *)$, the modified convolution $\circ$ on K is defined by

$$\varepsilon_x \circ \varepsilon_y := \frac{1}{\alpha(x)\alpha(y)}\, \alpha \cdot (\varepsilon_x * \varepsilon_y) \tag{4.5}$$

for $x, y \in K$. For details we refer to the original paper by Voit [15] or to [4].

To stress the underlying convolution in the construction of Poisson measures we write for $\lambda \in \mathcal{M}_+^b(K)$

$$e_*(\lambda) := e^{-\lambda(K)} \sum_{k \geq 0} \frac{1}{k!} \lambda^{*k} \quad \text{and} \quad e_\circ(\lambda) := e^{-\lambda(K)} \sum_{k \geq 0} \frac{1}{k!} \lambda^{\circ k}$$

referring to the respective convolutions on K.

4.3. Lemma. *Let $\mu, \nu \in \mathcal{M}^1(K)$, $\lambda \in \mathcal{M}_+^b(K)$, and $\alpha \in K^\wedge$ a positive character be such that*

$$m(\alpha\mu) := e_\circ(\alpha\lambda) \circ \nu, \tag{4.6}$$

where $m := \left(\int \alpha \, d\mu\right)^{-1} = \hat{\mu}(\alpha)^{-1}$. Then

$$s := \int \frac{1}{\alpha}\, de_\circ(\alpha\lambda) = \left(e_*(\lambda)^\wedge(\alpha)\right)^{-1} < \infty, \qquad g := \int \frac{1}{\alpha}\, d\nu < \infty,$$

and

$$\mu = e_*(\lambda) * \gamma, \tag{4.7}$$

where $\gamma := g^{-1}(\alpha^{-1}\nu) \in \mathcal{M}^1(K)$.

Proof. Definition (4.5) of the modified convolution in K implies

$$\int \frac{1}{\alpha}\, d\varepsilon_x \circ \varepsilon_y = \frac{1}{\alpha(x)\alpha(y)} \quad \text{for } x, y \in K.$$

Hence, by the Fubini–Tonelli theorem,

$$m = \int \frac{1}{\alpha}\, d\big(m(\alpha\mu)\big) = \int \frac{1}{\alpha}\, de_\circ(\alpha\lambda) \int \frac{1}{\alpha}\, d\nu = sg.$$

Thus $s, g < \infty$. Writing

$$\sigma := \frac{1}{s}\left(\frac{1}{\alpha}\, e_\circ(\alpha\lambda)\right) \in \mathcal{M}^1(K)$$

we obtain from (4.6)

$$m(\alpha\mu) = s(\alpha\sigma) \circ g(\alpha\gamma) = sg\big(\alpha(\sigma * \gamma)\big).$$

This yields that $\mu = \sigma * \gamma$, since α is positive. Therefore, we have to show that $\sigma = e_*(\lambda)$ and $s^{-1} = e_*(\lambda)^\wedge(\alpha) = \exp\left(\int(\alpha-1)\,d\lambda\right)$. Using $(\alpha\lambda)^{\circ k} = \alpha\lambda^{*k}$ $(k = 0, 1, 2, \ldots)$ we conclude for characters $\chi \in K^\wedge$ that

$$\hat{\sigma}(\chi) = \frac{1}{s}\left(\frac{1}{\alpha}e_\circ(\alpha\lambda)\right)^\wedge(\chi)$$

$$= \frac{1}{s}\exp\left(-\int\alpha\,d\lambda\right)\sum_{k\geq 0}\frac{1}{k!}\int(\overline{\chi}/\alpha)\,d(\alpha\lambda^{*k}) = \frac{1}{s}\exp\left(\int(\overline{\chi}-\alpha)\,d\lambda\right).$$

Thus $s = s\hat{\sigma}(1_K) = \exp\left(\int(1-\alpha)\,d\lambda\right)$. The Fourier transformation is a one-to-one mapping, and therefore $\sigma = e_*(\lambda)$. ∎

4.4. Theorem. *Let K be a second countable commutative hypergroup and let $(\mu_t)_{t>0}$ be a convolution semigroup on K with Lévy measure λ. Assume that λ is symmetric. Then λ is a σ-finite abstract symmetric Lévy measure with*

(4.8) $$\int(1-\operatorname{Re}\chi)\,d\lambda < \infty \quad \textit{for all } \chi\in K^\wedge.$$

Moreover, there exists a Gaussian semigroup $(\gamma_t)_{t>0}$ such that

(4.9) $$\mu_t = \gamma_t * e(t\lambda) \quad \textit{for every } t>0.$$

Proof. (1) Let $\alpha\in K^\wedge$ denote the uniquely determined positive character in the support of the Plancherel measure of K (cf. Theorem 2.11 in [15] or Theorem 2.3.19 in [4]). If we modify the convolution $*$ with respect to α according to (4.5), then $(K, \circ)$ is a Godement hypergroup (see Theorem 2.3.6 in [4]). Writing $m_t := \left(\int\alpha\,d\mu_t\right)^{-1} = \hat{\mu}_t(\alpha)^{-1}$ and applying (4.5), it is easy to show that $\left(m_t(\alpha\mu_t)\right)_{t>0}$ is a convolution semigroup on K with respect to the convolution $\circ$. The Lévy measure of $\left(m_t(\alpha\mu_t)\right)_{t>0}$ is given by the limit relation

$$\tau_v\text{-}\lim_{t\to 0}\frac{1}{t}m_t(\alpha\mu_t)|K^\times = (\alpha\lambda)|K^\times.$$

Moreover, $\alpha\lambda\in\mathcal{M}_+(K)$ is symmetric. Since K is second countable, we may choose open, symmetric λ-continuity sets $U_n\in\mathcal{U}(e)$, $n\in N$, with $U_n\downarrow\{e\}$. From (4.4) it follows that

(4.10) $$\tau_w\text{-}\lim_{t\to 0} e_\circ\left(\frac{s}{t}m_t(\alpha\mu_t)\,|\,\complement U_n\right) = e_\circ\left(s(\alpha\lambda_n)\right) \quad \text{for all } s>0,\ n\in N,$$

where λ_n denotes the finite measure $\lambda|\complement U_n$.

(2) If we consider the representation

(4.11) $$e_\circ\left(\frac{s}{t}m_t(\alpha\mu_t)\,|\,\complement U_n\right)\circ e_\circ\left(\frac{s}{t}m_t(\alpha\mu_t)|\,U_n\right) = e_\circ\left(\frac{s}{t}m_t(\alpha\mu_t)\right) \quad \text{for } s, t>0,$$

we know from (4.3) that the right-hand side of (4.11) converges to $m_s(\alpha\mu_s)$ for $t\to 0$. Therefore, Theorem 5.1.4 in [4] together with Prohorov's theorem and (4.10) shows that the sets

(4.12) $$\left\{e_\circ\left(\frac{s}{t}m_t(\alpha\mu_t)|U_n\right):\ t>0\right\}, \quad s>0,\ n\in N,$$

are relatively compact in the weak topology. Taking limits $t \to 0$ on both sides of equation (4.11), we see by the continuity of the convolution together with (4.3) and (4.10) that

$$e_{\circ}(s\alpha\lambda_n) \circ \nu_{sn} = m_s \alpha\mu_s \quad \text{for } n \in N,\ s > 0, \tag{4.13}$$

where $\nu_{sn} \in \mathcal{M}^1(K)$ denotes an accumulation point of the set (4.12). But (4.13) refers exactly to the situation (4.6) in Lemma 4.3. Therefore, by (4.7) there exist probability measures $\gamma_{sn} \in \mathcal{M}^1(K)$ such that

$$e_*(s\lambda_n) * \gamma_{sn} = \mu_s \quad \text{for all } n \in N,\ s > 0. \tag{4.14}$$

In the remaining part of the proof we do not need the setting of modified hypergroups and use the symbol $e(\cdot)$ instead of $e_*(\cdot)$ for the corresponding Poisson measures.

(3) Since the measures λ_n, $n \in N$, are symmetric, the monotone convergence theorem shows for every $\chi \in K^\wedge$ that

$$\lim_{n \to \infty} \int (1-\overline{\chi})\, d\lambda_n = \lim_{n \to \infty} \int_{\complement U_n} (1-\operatorname{Re}\chi)\, d\lambda = \int (1-\operatorname{Re}\chi)\, d\lambda. \tag{4.15}$$

With the notation $e^{-\infty} := 0$ this yields

$$\lim_{n \to \infty} e(s\lambda_n)^\wedge(\chi) = \exp\left(s \int (\operatorname{Re}\chi - 1)\, d\lambda\right) \quad \text{for } s > 0. \tag{4.16}$$

We use the Lévy continuity theorem (cf. Theorem 4.2.11 in [4]) to conclude that $s\lambda$ is a symmetric abstract Lévy measure, and the sequence $(e(s\lambda_n))_{n \in N}$ converges weakly to the generalized Poisson measure $e(s\lambda)$, $s > 0$. Again by (4.14), (4.16) and Theorem 5.1.4 in [4] we see that the set $\{\gamma_{sn}\colon n \in N\}$ is relatively compact in the weak topology. For any accumulation point $\gamma_s \in \mathcal{M}^1(K)$ of this set we then obtain the representation (4.9) by applying (4.14) and (4.16).

For characters $\chi \in K^\wedge$ we have $\hat{\mu}_1(\chi) \neq 0$. Hence $e(\lambda)^\wedge(\chi) \neq 0$, and this shows (4.8). On the other hand, (4.8) and (4.9) imply that $(e(t\lambda))_{t>0}$ and $(\gamma_t)_{t>0}$ are convolution semigroups.

(4) It remains to prove that $(\gamma_t)_{t>0}$ is Gaussian. We show this together with the fact that λ is the Lévy measure of $(e(t\lambda))_{t>0}$.

First we argue that $\lambda | U_n$ is a symmetric abstract Lévy measure. In fact, for $n \in N$ the function

$$K^\wedge \ni \chi \mapsto \int_{U_n} (1-\operatorname{Re}\chi)\, d\lambda =: \psi_n(\chi)$$

is continuous since it is the difference of two (continuous) negative definite functions. Moreover, it is easily verified that ψ_n is strongly negative definite. Hence $(e(t\lambda | U_n))_{t>0}$ is a convolution semigroup for every $n \in N$.

We denote the Lévy measures of the convolution semigroups

$$(e(t\lambda | U_n))_{t>0}, \quad (e(t\lambda_n))_{t>0}, \quad (e(t\lambda))_{t>0}, \quad (\gamma_t)_{t>0}$$

by

$$\lambda_{e(\lambda|U_n)}, \quad \lambda_{e(\lambda_n)}, \quad \lambda_{e(\lambda)}, \quad \lambda_\gamma,$$

respectively. Because of $e(t\lambda|U_n)*e(t\lambda_n) = e(t\lambda)$ and (4.9), Lemma 3.4 yields $\lambda_{e(\lambda|U_n)}+\lambda_{e(\lambda_n)} = \lambda_{e(\lambda)}$ and $\lambda_{e(\lambda)}+\lambda_\gamma = \lambda$. Therefore, for $f\in C_c(K^\times)$, $f \geqslant 0$, we have

$$\int f\,d\lambda_{e(\lambda_n)} \leqslant \int f\,d\lambda_{e(\lambda|U_n)}+\int f\,d\lambda_{e(\lambda_n)} = \int f\,d\lambda_{e(\lambda)} \leqslant \int f\,d\lambda_{e(\lambda)}+\int f\,d\lambda_\gamma = \int f\,d\lambda.$$

But $\lambda_{e(\lambda_n)} = \lambda_n = \lambda|\complement U_n$ (cf. Example 3.5), and hence $\int f\,d\lambda_n = \int f\,d\lambda$ if n is sufficiently large. Thus $\lambda_{e(\lambda)} = \lambda$ and $\lambda_\gamma = 0$. Therefore $(\gamma_t)_{t>0}$ is necessarily Gaussian by Theorem 3.7. ■

4.5. Remark. Consider the case where the Lévy measure λ of the convolution semigroup $(\mu_t)_{t>0}$ on K is not symmetric, but satisfies

$$\int |1-\chi|\,d\lambda < \infty \quad \text{for all } \chi\in K^\wedge. \tag{4.17}$$

Then (4.9) is also available for some Gaussian semigroup $(\gamma_t)_{t>0}$. To show this, one can argue as in the proof of Theorem 4.4. Together with obvious changes we have, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int (1-\overline{\chi})\,d\lambda_n = \lim_{n\to\infty} \int_{\complement U_n} (1-\overline{\chi})\,d\lambda = \int (1-\overline{\chi})\,d\lambda$$

instead of (4.15).

4.6. Theorem. *Let $\lambda\in\mathcal{M}_+(K)$ be an abstract Lévy measure on K satisfying* (4.17). *Then λ is the Lévy measure of $(e(t\lambda))_{t>0}$.*

We omit the proof since it is similar to part (4) of that of Theorem 4.4. ■

4.7. Theorem. *Let $\lambda_1, \lambda_2\in\mathcal{M}_+(K)$ be σ-finite symmetric abstract Lévy measures satisfying* (4.8) *and let $\gamma^{(1)}, \gamma^{(2)}$ be Gaussian measures on K. Then the equality*

$$\gamma^{(1)}*e(\lambda_1) = \gamma^{(2)}*e(\lambda_2) \tag{4.18}$$

implies $\lambda_1 = \lambda_2$ and $\gamma^{(1)} = \gamma^{(2)}$.

Proof. Choose Gaussian semigroups $(\gamma_t^{(i)})_{t>0}$ such that $\gamma^{(i)} = \gamma_1^{(i)}$ $(i = 1, 2)$. First we show that

$$\gamma_t^{(1)}*(\gamma_t^{(1)})^-*e(2t\lambda_1) = \gamma_t^{(2)}*(\gamma_t^{(2)})^-*e(2t\lambda_2) \tag{4.19}$$

for every $t > 0$. In fact, if ψ_i denotes the strongly negative definite function corresponding to $(\gamma_t^{(i)})_{t>0}$ (according to Theorem 2.3), for each $\chi\in K^\wedge$ we obtain

$$(\gamma_t^{(i)}*(\gamma_t^{(i)})^-*e(2t\lambda_i))^\wedge(\chi) = \exp\left[-t\left(\psi_i(\chi)+\psi_i(\overline{\chi})+2\int(1-\mathrm{Re}\,\chi)\,d\lambda_i\right)\right],$$

where the exponent on the right-hand side is real. Therefore (4.19) holds for all $t > 0$, since (4.18) implies (4.19) for the particular value $t = 1$. Theorem 4.6 shows that $2\lambda_i$ is the Lévy measure of the convolution semigroup $(\gamma_t^{(i)}*(\gamma_t^{(i)})^-*e(2t\lambda_i))_{t>0}$. Lévy measures are uniquely determined by the convolution semigroups. Hence $\lambda_1 = \lambda_2$ by (4.19). Finally, the Fourier transform $e(\lambda_1)^\wedge = e(\lambda_2)^\wedge$ is positive, which yields $\gamma^{(1)} = \gamma^{(2)}$. ■

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