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SELFDECOMPOSABILITY PERPETUITY LAWS AND STOPPING TIMES

BY

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Abstract. The decomposability property of Lévy class L of probability distributions, on a Banach space, is extended to a family of stopping times associated with background driving Lévy processes (BDLP). As consequences, this allows us to show that all selfdecomposable measures are perpetuity laws and to get a representation of gamma distribution as an infinite product of independent uniform distributions.

In the probability theory, limit distributions (or probability measures) are often characterized by some convolution equations (factorization properties) rather than by Fourier transforms (the characteristic functionals). In fact, the latter usually follows the first one. Equations, in question, involve the multiplication by positive scalars c or an action of the corresponding dilation T_c on measures. In such a setting, it seems that there is no way for stopping times (or, in general, for the stochastic analysis) to come into the "picture". However, if one accepts the view that the primary objective, in the classical limit distributions theory, is to describe the limiting distributions (or random variables) by the tools of random integrals/functionals, then one can use the stopping times. In this paper we illustrate such a possibility in the case of selfdecomposability random variables (i.e. the Lévy class L) with values in a real separable Banach space. Also some applications of our approach to perpetuity laws are presented; cf. [2]–[4]. In fact, we show that all selfdecomposable distributions are perpetuity laws. Moreover, as a by-product we obtain a representation of gamma distributions in terms of products of independent uniform distributions.

1. Let E be a real separable Banach space. An E-valued random variable (r.v.) X, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is said to be *selfdecomposable* (or a *Lévy class L*) if for each $t > 0$ there exists an r.v. X_t indepen-

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dent of X such that

(1) $$X \stackrel{d}{=} X_t + e^{-t} X,$$

where $\stackrel{d}{=}$ means equality in distribution.

Remark 1. (i) The class of selfdecomposable r.v.'s (distributions) coincides with the class of limiting r.v.'s of the following infinitesimal triangular arrays:

$$a_n(Z_1 + Z_2 + \ldots + Z_n) + b_n,$$

where $a_n > 0$, $b_n \in E$, and $Z_1, Z_2, \ldots$ are independent E-valued r.v.'s; cf. for instance [8], Chapter 3.

(ii) In case of i.i.d. Z_n's one gets in the above scheme the class of all stable distributions.

(iii) In terms of probability distributions, equation (1) reads that for each $0 < c < 1$ there exists a probability measure μ_c such that

(2) $$\mu = \mu_c * T_c \mu,$$

where $*$ denotes the convolution of measures and $(T_c \mu)(\cdot) = \mu(c^{-1} \cdot)$. In other words, $T_c \mu$ is the image of a measure μ under the linear mapping T_c: $E \to E$ given by $T_c x = cx$, $x \in E$.

Let us also recall that a stochastic base is an increasing and right continuous family of σ-fields $\mathscr{F}_t \subset \mathscr{F}$ (i.e. $\mathscr{F}_s \subset \mathscr{F}_t$ for $s < t$ and $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$).

Furthermore, any mapping τ: $\Omega \to [0, \infty)$ such that

(3) $$\{\omega\colon \tau(\omega) \leqslant t\} \in \mathscr{F}_t, \quad t \geqslant 0,$$

is called a *stopping time.*

A family $Y(t)$, $t \geqslant 0$, of E-valued random variables is called a *Lévy process* provided

- $Y(0) = 0$ $\mathscr{P}$-a.s., $Y(t+s) - Y(s) \stackrel{d}{=} Y(t)$ for all $s, t \geqslant 0$;
- $Y(t_k) - Y(t_{k-1})$, $k = 1, 2, \ldots, n$, are independent for all $0 \leqslant t_0 < \ldots < t_n$, $n \geqslant 1$;
- $t \mapsto Y(t, \omega)$ are cadlag functions for $\mathscr{P}$-a.a. $\omega \in \Omega$.

Of course, for any $c > 0$ and a Lévy process Y, one has $Y_c(t) := Y(t+c) - Y(t)$, $t \geqslant 0$, is a new Lévy process with $Y_c \stackrel{d}{=} Y$ in the Skorohod space $D_E[0, \infty)$ of all cadlag functions. Moreover, Y_c is independent of the σ-field $\sigma\{Y(t)\colon 0 \leqslant t \leqslant c\} = \mathscr{F}_c^Y$. In fact, for any stopping time τ with respect to $\mathscr{F}_t^Y$ it follows that

(4) $$Y_\tau(t) := Y(t+\tau) - Y(\tau), \quad t \geqslant 0,$$

is a Lévy process such that $Y_\tau \stackrel{d}{=} Y$ and Y_τ is independent of the σ-field $\mathscr{F}_\tau$ defined by

(5) $$\mathscr{F}_\tau = \{A \in \mathscr{F}\colon A \cap [\tau \leqslant t] \in \mathscr{F}_t \text{ for each } t \geqslant 0\};$$

cf. for instance, [1], Theorem 32.5, to derive the above statements for any Lévy process.

Here is the main result which extends (1) for some stopping times.

THEOREM 1. *Suppose X is a selfdecomposable E-valued r.v. Then there exists a stochastic base $(\mathcal{F}_t)_{t\geq 0}$ such that for each $\mathcal{F}_t$-stopping time τ there are independent E-valued r.v.'s X_τ and X' satisfying*

(i) $X \overset{d}{=} X_\tau + e^{-\tau} X'$;

(ii) *X' is independent of X and $X' \overset{d}{=} X$;*

(iii) *the vector $(X_\tau, e^{-\tau})$ is independent of X'.*

Proof. From [5] or [8], p. 124, we conclude that the r.v. X is selfdecomposable (i.e. (1) holds) if and only if there exists a unique, in distribution, Lévy process Y such that $E[\log(1+\|Y(1)\|)] < \infty$ and

$$X \overset{d}{=} \int_{(0,\infty)} e^{-s} dY(s). \tag{6}$$

(We refer to Y as the **b**ackground **d**riving **L**évy **p**rocess of X; in short, Y is a BDLP for X, cf. [6]). Taking $\mathcal{F}_t = \sigma(Y(s)\colon s \leq t)$ and defining

$$Z(t) := \int_{(0,t]} e^{-s} dY(s) \equiv e^{-t} Y(t) + \int_{(0,t]} Y(s-) e^{-s} ds, \tag{7}$$

we see that $Z(\tau(\omega))$ is $\mathcal{F}_\tau$-measurable for a stopping time τ; cf. [9], pp. 18–20. Finally, using (4) and (7) we get

$$\begin{aligned} X &\overset{d}{=} \int_{(0,\tau]} e^{-s} dY(s) + \int_{(\tau,\infty)} e^{-s} dY(s) \\ &= Z(\tau) + e^{-\tau} \int_{(0,\infty)} e^{-s} dY_\tau(s) = X_\tau + e^{-\tau} X', \end{aligned} \tag{8}$$

with $X_\tau = Z(\tau)$ independent of $X' = \int_{(0,\infty)} e^{-s} dY(s) \overset{d}{=} X$ because Y_τ is independent of $\mathcal{F}_\tau$. This completes the proof of Theorem 1. ∎

Remark 2. The integral in (6) is defined as a limit of $Z(t)$, given by (7), as $t \to \infty$. Existence of the limit (in probability, a.s. or in distribution) is equivalent to the condition $E[\log(1+\|Y(1)\|)] < \infty$; cf. [5] or [8], p. 122.

COROLLARY 1. *Let Y be a BDLP of a selfdecomposable r.v. X and let $\mathcal{F}_t = \sigma(Y(s)\colon s \leq t)$, $t \geq 0$, be the stochastic base given by Y. Then for each $\mathcal{F}_t$-stopping time τ there exists an r.v. X_τ independent of X such that*

$$X \overset{d}{=} X_\tau + e^{-\tau} X. \tag{9}$$

The equality in distribution in (9) can be strengthened as follows:

COROLLARY 2. *Let Y be an E-valued Lévy process such that*

$$E[\log(1+\|Y(1)\|)] < \infty$$

and let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration given by Y. Then for any $\mathcal{F}_t$-stopping time τ one has

$$\int_{(0,\infty)} e^{-s} dY(s,\omega) = \int_{(0,\tau(\omega)]} e^{-s} dY(s,\omega) + e^{-\tau(\omega)} \int_{(0,\infty)} e^{-s} dY(s+\tau(\omega),\omega) \tag{10}$$

for $\mathcal{P}$-a.a. $\omega \in \Omega$.

Proof. This is a consequence of Remark 2 and the equality in formula (8). ■

It is well known that any Lévy process Y can be written as a sum of two independent Lévy processes Y^c and Y^d, i.e., $Y = Y^c + Y^d$, where Y^c is purely continuous (Gaussian) while Y^d is a purely discontinuous (cadlag) process. Furthermore, for a Borel subset A separated from zero (i.e., $A \subset \{x \in E: \|x\| \geqslant \varepsilon\}$ for some $\varepsilon > 0$) we define

$$Y^d(t; A) := \sum_{0<s\leqslant t} \Delta Y^d(s)\, 1_A(\Delta Y^d(s)),$$

where the jumps $\Delta Y^d(s) := Y^d(s) - Y^d(s-)$ are in the set A, which is a Lévy process independent of the process $Y^d(t) - Y^d(t; A)$. All the above allows us to conclude the following:

COROLLARY 3. (i) *Let Y^d be a purely discontinuous Lévy process with finite logarithmic moment and let*

$$\tau_0 = \inf\{t > 0: \; Y^d(t) \neq 0\}$$

be the stopping time of the first non-zero value. Then

$$\int_{(0,\infty)} e^{-s} dY^d(s) = e^{-\tau_0} Y^d(\tau_0) + e^{-\tau_0} \int_{(0,\infty)} e^{-s} dY^d(s+\tau_0) \quad \mathscr{P}\text{-a.s.} \tag{11}$$

(ii) *Let $\tau_A = \inf\{t > 0: \; Y^d(t; A) \neq 0\}$ be the stopping time of the first jump whose value is in A. Then*

$$\int_{(0,\infty)} e^{-s} dY^d(s; A) = e^{-\tau_A} Y^d(\tau_A; A) + e^{-\tau_A} \int_{(0,\infty)} e^{-s} dY^d(s+\tau_A; A) \quad \mathscr{P}\text{-a.s.} \tag{12}$$

Proof. Apply the above stopping times in equation (10). ■

Remark 3. (a) Random integrals appearing in (12) are i.i.d. and selfdecomposable, and so are integrals in (11) and the outmost integrals in (10).

(b) If $\tau_1 = \tau_A$ and τ_k, $k \geqslant 1$, are the consecutive random times of the jumps of the process $Y^d(t; A)$ with $\tau_k \uparrow +\infty$ a.e., then one gets the factorization

$$\int_{(0,\infty)} e^{-s} dY^d(t; A) = \sum_{k=1}^{\infty} e^{-\tau_k} \Delta Y(\tau_k; A) \quad \text{a.e.,} \tag{13}$$

where the jumps $\Delta Y(\tau_k; A)$ are independent of $\tau_k - \tau_{k-1}$ for $k \geqslant 1$.

2. In this section we consider only *real* valued random variables. Let $(A, B), (A_1, B_1), (A_2, B_2), \ldots$ be a sequence of i.i.d. random vectors in $\boldsymbol{R}^2$ which define the stochastic difference equation

$$Z_{n+1} = A_n Z_n + B_n, \qquad n \geqslant 1. \tag{14}$$

Equation (14) appears in modelling many real situations including economics, finanse or insurance; cf. for instance [3], [4] and the references therein. One may look at (14) as an iteration of the affine random mapping $x \mapsto Ax+B$. So, starting with Z_0 and $(A_0, B_0) = (A, B)$, we get

$$Z_{n+1} = A_n A_{n-1} \ldots A_0 Z_0 + \sum_{k=0}^{n} B_k A_{k+1} A_{k+2} \ldots A_n.$$

Putting $Z_0 = 0$ and assuming that (Z_n) converges to Z, we obtain

$$Z \stackrel{d}{=} \sum_{k=1}^{\infty} B_k \prod_{l=1}^{k-1} A_l.$$

In insurance mathematics, distributions of Z are called *perpetuities*. Note that by (14) perpetuities are the solutions to

$$Z \stackrel{d}{=} AZ+B, \tag{15}$$

i.e., Z is a *distributional fixed-point* of the random affine mapping $x \mapsto Ax+B$, $x \in \mathbf{R}$.

What triplets A, B, X satisfy (15) with (A, B) independent of X? Or are there independent r.v.'s A, C, Z such that

$$Z \stackrel{d}{=} A(Z+C)? \tag{16}$$

It seems that there are not too many explicit examples of (15) or (16); cf. [2], p. 288. Results from the previous section can now be phrased as follows:

COROLLARY 4. (i) *All selfdecomposable distributions are perpetuities, i.e. they satisfy* (15) *with non-trivial* $0 \leqslant A \leqslant 1$ *a.s.*

(ii) *All selfdecomposable distributions whose BDLP Y have a non-zero purely discontinuous part have convolution factors that satisfy the equation* (16).

Let $\gamma_{\alpha,\lambda}$ denote a gamma r.v. with parameters $\alpha > 0$, $\lambda > 0$, i.e., it has the probability density

$$f_{\alpha,\lambda} = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} 1_{(0,\infty)}(x).$$

It is known (cf. [6] and [7]) that $\gamma_{\alpha,\lambda}$ is selfdecomposable and its BDLP is given by $\lambda^{-1} Y_0(\alpha t)$, where

$$Y_0(t) = \sum_{j=1}^{N(t)} \gamma_{1,1}^{(j)}, \tag{17}$$

$\gamma_{1,1}^{(1)}, \gamma_{1,1}^{(2)}, \ldots$ are i.i.d. copies of $\gamma_{1,1}$, and $N(t)$ is a standard Poisson process, i.e., it has stationary independent increments, $N(0) = 0$ a.e. and for $t > s > 0$

$$\mathcal{P}[N(t)-N(s) = k] = e^{-(t-s)} \frac{(t-s)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

If $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ are the consecutive random times (arrival times) of the jumps of N, then $\tau_n - \tau_{n-1} \stackrel{d}{=} \gamma_{1,1}$ for $n \geqslant 1$ are independent and $\tau_n \stackrel{d}{=} \gamma_{n,1}$ for $n \geqslant 1$.

PROPOSITION 1. *For a gamma r.v. $\gamma_{\alpha,\lambda}$ one has:*

(i) $$\gamma_{\alpha,\lambda} \stackrel{d}{=} e^{-\gamma_{\alpha,1}}(\gamma_{1,\lambda}+\gamma_{\alpha,\lambda}) \stackrel{d}{=} U^{1/\alpha}\gamma_{\alpha+1,\lambda},$$

where U is uniformly distributed on $[0, 1]$, independent of $\gamma_{\alpha+1,\lambda}$ and the three middle r.v.'s are also independent.

(ii) $$\gamma_{\alpha,\lambda} \stackrel{d}{=} \sum_{n=1}^{\infty} U_1^{1/\alpha} U_2^{1/\alpha} \ldots U_n^{1/\alpha} \gamma_{1,\lambda}^{(n)},$$

where $\gamma_{1,\lambda}^{(1)}, \gamma_{1,\lambda}^{(2)}, \ldots$ are i.i.d. copies of $\gamma_{1,\lambda}$, $U_1, U_2, \ldots$ are i.i.d. copies of U and both sequences are also independent.

Proof. (i) Since $\gamma_{\alpha,\lambda}$ has a BDLP $Y(t) = \lambda^{-1} Y_0(\alpha t)$, by Corollary 3 (i) and Remark 3 (a) we have

$$\gamma_{\alpha,\lambda} \stackrel{d}{=} e^{-\gamma_{\alpha,\lambda}}\gamma_{1,\lambda}^{(1)}+e^{-\gamma_{\alpha,1}} \int_{(0,\infty)} e^{-s}\,dY(s+\tau_1) = U^{1/\alpha}(\gamma_{1,\lambda}^{(1)}+\tilde{\gamma}_{\alpha,\lambda}) \stackrel{d}{=} U^{1/\alpha}\gamma_{1+\alpha,\lambda}.$$

(Equality of the two outmost terms in (i) can be also easily checked by comparing the corresponding characteristic functions.)

(ii) Repeating the middle equality in (i) and using the facts that $\tau_n \uparrow +\infty$ a.s., and $Y(\cdot)$ is independent of $Y(\cdot+\tau_1+\tau_2+ \ldots +\tau_k)-Y(\tau_1+\tau_2+ \ldots +\tau_k)$, we arrive at (ii). ∎

3. The method of random integral representation is also applicable to operator-selfdecomposable distributions; cf. [8], Chapter 3, or [5]. Recall that a Banach space E-valued r.v. X is Q-selfdecomposable if for each $t > 0$ there exists an r.v. X_t independent of X such that

(18) $$X \stackrel{d}{=} X_t+e^{-tQ}X;$$

Q is a bounded linear operator on E and e^{-tQ} is the operator given by a power series.

THEOREM 2. *Suppose that X is a Q-decomposable E-valued r.v. and $e^{-tQ} \to 0$ as $t \to \infty$, in the norm topology. Then there is a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for each stopping time τ there exist independent E-valued r.v.'s X_τ and X' satisfying*

(i) $X \stackrel{d}{=} X_\tau+e^{-\tau Q}X'$;

(ii) *X' is an independent copy of X;*

(iii) *the random vector $(X_\tau, e^{-\tau Q})$ is independent of X'.*

Proof. From [5] or [8], Chapter III, it follows that X is Q-selfdecomposable if and only if

(19) $$X \stackrel{d}{=} \int_{(0,\infty)} e^{-tQ}\,dY(t)$$

for a uniquely defined Lévy process Y such that $E[\log(1+\|Y(1)\|)] < \infty$. Consequently, (19) allows us to proceed as in the proof of Theorem 1. ∎

Remark 4. (a) Corollaries from Section 1 have their "operator" counterparts.

(b) For a given E-valued r.v. B and a random bounded linear operator A on a Banach space E, consider the affine random mapping $x \mapsto Ax+B$. The question of finding all distributional fix-points, i.e., all E-valued r.v.'s X such that

$$X \stackrel{d}{=} AX+B, \tag{20}$$

seems to be more difficult as the composition of operators is not commutative. However, random integrals of the form

$$\int_{(a,b]} f(t)\,dY(r(t)), \tag{21}$$

where Y is a Lévy process, $r(t)$ is a change of time, f is a process or deterministic function, *might provide a tool* of constructing X satisfying the equation (20) or its variants (like (16)). The present paper illustrates this approach in the case of the selfdecomposable distributions and their random integral representations.

Added in proof. Corollary 1 is also true when the stopping time τ is replaced by a non-negative random variable T independent of the BDLP Y.

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