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ON THE RESTRICTED CONVERGENCE OF THE INTERMEDIATE AND CENTRAL TERMS OF ORDER STATISTICS

BY

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Abstract. As the distribution function (d.f.) of the suitably normalized general intermediate (or central) term of order statistics converges on an interval $[c, d]$ to an arbitrary nondecreasing function, the continuation of this (weak) convergence on the whole real line to an intermediate (or central) value distribution is proved.

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1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables (r.v.'s) with common continuous d.f. $F(x) = P(X_n \leqslant x)$. Let $X_{r:n}$, $r = r_n$, represent the r_nth smallest order statistics. A rank sequence $\{r_n\}$ is called an *intermediate rank* and the sequence $\{X_{r:n}\}$, $r = r_n$, is called an *intermediate order statistic* if $r_n/n \to 0$ or 1.

The intermediate order statistics have many applications, e.g., in the theory of statistics, they can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to the available sample size. Pickands [17] has shown that intermediate order statistics of the form

$$X_{[t_1 r_n]:n}, X_{[t_2 r_n]:n}, \ldots, X_{[t_k r_n]:n}, \quad \text{where } 0 < t_1 < t_2 < \ldots < t_k,$$

can be used in constructing consistent estimators for the shape parameter α of the limiting extremal distribution in the parametric form $G_\alpha(x) = \exp\left(-(1-\alpha x)^{1/\alpha}\right)$. Many authors (e.g., Teugels [21] and Mason [16]) have also found estimators that are based, in part, on intermediate order statistics.

Chibisov [7] and Wu [22] both have shown that the normal and log-normal distributions are possible limiting distributions for the intermediate terms. Chibisov [7] considered the intermediate rank sequence $\{r_n\}$, which satisfied the limit relation (the Chibisov condition)

$$\lim_{n\to\infty}\left(\sqrt{r_{n+z_n}}-\sqrt{r_n}\right)=\frac{\alpha l\gamma}{2}, \quad l>0, \tag{1.1}$$

for any sequence $\{z_n\}$ of integer values for which $z_n/n^{1-\alpha/2}\to\gamma$ as $n\to\infty$ ($0<\alpha<1$ and γ is an arbitrary real number). Chibisov [7] showed that, whenever $\{r_n\}$ satisfies (1.1) and there exist sequences $a_n>0$ and b_n such that

$$\Phi_{n:n}(a_n x+b_n)=P(X_{r_n:n}\leqslant a_n x+b_n)$$

has a nondegenerate limit, the limiting distribution must be of the form $\mathcal{N}(W(x))$, where $\mathcal{N}(\cdot)$ is the standard normal distribution and $W(x)$ has one and only one of the following types (up to an affine transformation of x):

$$\text{Types I:}\quad W_1^{(\theta)}(x)=\begin{cases}-\infty, & x\leqslant 0,\\ \theta\ln x, & x>0;\end{cases} \tag{1.2}$$

$$\text{Types II:}\quad W_2^{(\theta)}(x)=\begin{cases}-\theta\ln|x|, & x\leqslant 0,\\ \infty, & x>0;\end{cases}$$

$$\text{Type III:}\quad W_3^{(\theta)}(x)=W_3(x)=x \quad \text{for all } x,$$

where θ is some positive constant which depends only on α, l and the type of the d.f. $F(x)$. Wu [22] generalized the Chibisov results for any nondecreasing intermediate rank sequence $\{r_n\}$ and proved that the only possibilities for $W(\cdot)$ are the same types as defined by (1.2). Some emendations and complements to Wu [22] are done by Barakat and Ramachandran [6]. The domains of attraction of the limiting forms $\mathcal{N}(W_i^{(\theta)}(x))$, $i=1,2,3$, have been obtained in Chibisov [7], Wu [22], Smirnov [20] and Balkema and de Haan [2], [3]. However, it could be stated that, for many intermediate terms among those under consideration, the fields of attraction of $\mathcal{N}(W_i^{(\theta)}(x))$, $i=1,2$, are empty.

In this paper an interesting stability property of the intermediate order statistics is proved. This property is a continuation of the convergence. More precisely, let $a_n>0$ and b_n be some suitable normalizing constants. Let $\Phi^*(x)$ be a monotone function which has more than two different values in a closed interval $[c,d]$. Finally, let $\Phi_{n:n}(a_n x+b_n)\to\Phi^*(x)$ as $n\to\infty$ for all $x\in[c,d]$. Then

$$\Phi_{n:n}(a_n x+b_n)\xrightarrow{w}\Phi(x)=\mathcal{N}(W_i^{(\theta)}(x)), \quad i\in\{1,2,3\},$$

where $\xrightarrow{w}$ stands for the weak convergence and $\Phi(x)=\Phi^*(x)$ for all $x\in[c,d]$. This property for maximum (minimum) and central order statistics has been discussed by many authors; for example, see Gnedenko [11], Gnedenko and Senussi-Bereksi [12], [13], Shokry [19], Galambos [9], [10] and Barakat [4], [5].

Recently, Barakat and Ramachandran [6] has extended the convergence of the distribution of the intermediate and central order statistics when the limit function $\Phi^*(x)$ is assumed to be continuous and strictly increasing on a compact interval (i.e., $\Phi^*(x)$ has an uncountable number of different values in a compact interval). The survey paper of Rossberg [18] stresses different points of the restricted convergence of order statistics and sums of independent r.v.'s. The main result of the present paper is Theorem 2.1 where the continuation property of a central irregular rank sequence, for which $\sqrt{n}(r_n/n-\lambda)\to\infty$ as $n\to\infty$, and of a general intermediate rank sequence is proved. The methods of proof of Theorem 2.1 are different from the methods given in Barakat and Ramachandran [6].

Throughout this paper, for any sequence of d.f.'s $\{\Phi_n\}$, we shall use the notation $\Phi_n(x)\xrightarrow{[c,d]}\Phi^*(x)$ as $n\to\infty$ to denote the restricted convergence of Φ_n on $[c, d]$. That is, $\Phi_n(x)$ converges to a monotone right continuous function $\Phi^*(x)$ for every continuity point $x\in[c, d]$ of $\Phi^*(x)$. If $\Phi_n(x)\xrightarrow{R}\Phi^*(x)$ as $n\to\infty$ and $\varrho(\Phi^*)=\Phi^*(\infty)-\Phi^*(-\infty)=1$, we write $\Phi_n(x)\xrightarrow{w}\Phi^*(x)$ (weak convergence). Moreover, for any two integers n_1 and n_2 we write $\Phi_{n_1:n_2}(x) = P(X_{r_{n_1}:n_2}\leqslant x)$.

We make an extensive use of the Khintchine convergence theorem (Feller [8], p. 246) and the following almost trivial result (see Leadbetter et al. [15]).

LEMMA 1.1. *Let $\{u_n, n\geqslant 1\}$ be a sequence of real numbers and $0\leqslant\tau\leqslant\infty$. Then*

$$\Phi_{n:n}(u_n)\to\mathcal{N}(\tau) \text{ as } n\to\infty \quad \text{if and only if} \quad \frac{nF(u_n)-r_n}{\sqrt{r_n}}\to\tau.$$

2. MAIN RESULT

THEOREM 2.1. *Let $F(x)$ be a d.f. for which there exist real constants $a_n>0$ and b_n such that*

$$(2.1)\qquad \Phi_{n:n}(a_n x+b_n)\xrightarrow{[c,d]}\Phi^*(x) \quad \text{as } n\to\infty,$$

where Φ^* is any nondecreasing (right continuous) function which has at least two growth points in the interval (c, d) and the rank sequence $\{r_n\}$ satisfies the condition

(A) $r_n\to\infty$, *where r_n is nondecreasing in n and*

$$r_n/n\to\lambda=0 \quad \text{as } n\to\infty$$

or the condition

(B) $r_n/n\to\lambda$ $(0<\lambda<1)$, *where r_n is nondecreasing in n and*

$$\sqrt{n}(r_n/n-\lambda)\to\infty \quad \text{as } n\to\infty.$$

Then $\Phi_{n:n}(a_n x+b_n) \xrightarrow{w} \Phi(x)$ *as* $n \to \infty$, *where* $\Phi^*(x) = \Phi(x)$ *for all* $x \in [c, d]$. *Moreover,* $\Phi(x) = \mathcal{N}(W(x))$, *where* $W(x)$ *takes one and only one of the types* (1.2). (*Transformations* $ax+b$, $a > 0$, *of the argument are permitted for each type.*)

Remark 2.1. Under the condition (A), Theorem 2.1 discusses the continuation of the convergence of the intermediate order statistics with general rank sequences, while under the condition (B) the theorem discusses the continuation of the convergence of the central order statistics with some irregular rank sequences. In view of the results of Wu [22] and Balkema and de Haan [2], [3] the only possibilities of the limit d.f.'s of these central order statistics are $\mathcal{N}(W_i^{(\theta)}(x))$, $i = 1, 2, 3$, where $W_i^{(\theta)}(x)$, $i = 1, 2, 3$, are defined in (1.2).

For avoiding the complication in the proof of Theorem 2.1, we shall first prove it when the rank sequence satisfies the restrictive Chibisov condition (1.1) (the Chibisov condition implies the condition (A); see Wu [22]) and then extend the proof to the general case. Before giving details of the proof in Section 3, we first establish the following lemma.

LEMMA 2.1. *Let the rank sequence satisfy the Chibisov condition* (1.1). *Furthermore, let* (2.1) *be satisfied. Then* $\{\Phi_{n:n}(a_n x+b_n)\}_n$ *is the sequence of stochastically bounded d.f.'s* (for the definition see Feller [8]).

Proof. To prove this lemma, it suffices to show that, for any subsequence $\{n_k\}$ such that

$$\Phi_{n_k:n_k}(a_{n_k} x+b_{n_k}) \xrightarrow{R} \tilde{\Phi}(x) \quad \text{as } k \to \infty, \tag{2.2}$$

where $\tilde{\Phi}(x)$ is a nondecreasing right continuous function, we must have $\tilde{\Phi}(-\infty) = 0$ and $\tilde{\Phi}(\infty) = 1$. On the other hand, in view of Lemma 1.1, (2.1) and (2.2) are equivalent, respectively, to

$$W_{n:n}(a_n x+b_n) = \frac{nF(a_n x+b_n)-r_n}{\sqrt{r_n}} \xrightarrow{[c,d]} W^*(x) \quad \text{as } n \to \infty$$

and

$$W_{n_k:n_k}(a_{n_k} x+b_{n_k}) = \frac{n_k F(a_{n_k} x+b_{n_k})-r_{n_k}}{\sqrt{r_{n_k}}} \xrightarrow{R} \tilde{W}(x) \quad \text{as } k \to \infty,$$

where $\mathcal{N}(W^*(x)) = \Phi^*(x)$ for all $x \in [c, d]$, $\mathcal{N}(\tilde{W}(x)) = \tilde{\Phi}(x)$ for all x, $\tilde{W}(x) = W^*(x)$ for all $x \in [c, d]$, and

$$W_{n_1:n_2}(x) = \frac{n_2 F(x)-r_{n_1}}{\sqrt{r_{n_1}}}.$$

Hence, to prove this lemma it suffices to show that $\tilde{W}(-\infty) = -\infty$ and $\tilde{W}(\infty) = \infty$. Now, for any finite real number t, (2.1) may be written as

$$\Phi_{n(t):n(t)}(a_{n(t)} x+b_{n(t)}) \xrightarrow{[c,d]} \Phi^*(x) \quad \text{as } n \to \infty,$$

where $n(t) = n + [n^{1-\alpha/2} t]$ (letting [ ] denote integer part). The last limit relation, in view of Lemma 1.1, is equivalent to

$$W_{n(t):n(t)}(a_{n(t)} x + b_{n(t)}) \xrightarrow{[c,d]} W^*(x) \quad \text{as } n \to \infty.$$

On the other hand, we can write

$$W_{n(t):n(t)}(a_{n(t)} x + b_{n(t)}) = \frac{n(t)}{n}\sqrt{\frac{r_n}{r_{n(t)}}}\, W_{n:n}(a_{n(t)} x + b_{n(t)}) + \frac{n(t) r_n - n r_{n(t)}}{n\sqrt{r_{n(t)}}}.$$

It is clear that $\{z_n\} = \{n(t) - n\} = \{[n^{1-\alpha/2} t]\}$ are sequences of integer values for which

$$z_n/n^{1-\alpha/2} \to t \quad \text{as } n \to \infty.$$

Therefore, we can deduce that r_n satisfies the limit relation

$$\lim_{n\to\infty}(\sqrt{r_{n(t)}} - \sqrt{r_n}) = \lim_{n\to\infty}(\sqrt{r_{n+z_n}} - \sqrt{r_n}) = (\alpha l t)/2, \qquad l > 0.$$

Moreover, the following limit relations can easily be verified:

$$\frac{n(t)}{n}\sqrt{\frac{r_n}{r_{n(t)}}} \to 1$$

and

$$\frac{n(t) r_n - n r_{n(t)}}{n\sqrt{r_{n(t)}}} \to l(1-\alpha)t \quad \text{as } n \to \infty.$$

Hence we get

$$W_{n:n}(a_{n(t)} x + b_{n(t)}) \xrightarrow{[c,d]} W^*(x) - l(1-\alpha)t \quad \text{as } n \to \infty,$$

which, in view of Lemma 1.1, yields

$$\Phi_{n:n}(a_{n(t)} x + b_{n(t)}) \xrightarrow{[c,d]} \mathcal{N}\big(W^*(x) - l(1-\alpha)t\big) \quad \text{as } n \to \infty. \tag{2.3}$$

Therefore, applying a lemma due to Senussi-Bereksi (see Lemma 2.4 in Barakat [4]), by (2.1), (2.3) and (2.2) we deduce that there exist two real functions $\alpha(t) > 0$ and $\beta(t)$ such that $\mathcal{N}(W^*(x) - l(1-\alpha)t) = \mathcal{N}\big(\tilde{W}(\alpha(t)x + \beta(t))\big)$, i.e.,

$$W^*(x) = \tilde{W}\big(\alpha(t)x + \beta(t)\big) + l(1-\alpha)t \quad \text{for all } x \in [c, d]. \tag{2.4}$$

Now, if $\tilde{W}(-\infty) > -\infty$, then we have, by (2.4), $W^*(x) \geqslant \tilde{W}(-\infty) + l(1-\alpha)t$ for all arbitrary positive large values of t. Hence we deduce that $W^*(x) = \infty$ for all $x \in [c, d]$ (let $t \to \infty$), which contradicts our assumptions. On the other hand, if $\tilde{W}(\infty) < \infty$, then we have (in view of (2.4)) $W^*(x) \leqslant \tilde{W}(\infty) + l(1-\alpha)t$ for all arbitrary negative small values of t. Consequently, we get $W^*(x) = -\infty$, which again contradicts our assumptions. This completes the proof. ■

Remark 2.2. It may be possible to give a short proof of (2.4), using only the Khintchine's classical convergence of types theorem, by defining the following sequences of truncated d.f.'s:

$$F_{n:n}(x) = \frac{\Phi_{n:n}(x)-\mathscr{C}}{\mathscr{D}-\mathscr{C}}, \qquad \Phi_{n:n}^{-1}(\mathscr{C}) \leqslant x < \Phi_{n:n}^{-1}(\mathscr{D})$$

and

$$\tilde{F}(x) = \frac{\tilde{\Phi}(x)-\mathscr{C}}{\mathscr{D}-\mathscr{C}}, \qquad \tilde{\Phi}^{-1}(\mathscr{C}) \leqslant x < \tilde{\Phi}^{-1}(\mathscr{D}),$$

where $\mathscr{C} = \tilde{\Phi}(c) = \Phi^*(c) < \Phi^*(d) = \tilde{\Phi}(d) = \mathscr{D}$ and $F^{-1}(u) = \sup\{x\colon F(x) \leqslant u\}$ is the inverse of the cumulative d.f. F. Therefore, in view of (2.2) and the conditions of Theorem 2.1, it can be easily shown that

$$F_{n_k:n_k}(a_{n_k}x+b_{n_k}) \xrightarrow{w} \tilde{F}(x) \quad \text{as } k \to \infty$$

and

$$F_{n_k:n_k}(a_{n_k(t)}x+b_{n_k(t)}) \xrightarrow{w} F^*(x) \quad \text{as } k \to \infty,$$

where

$$F^*(x) = \frac{\Phi^*(x)-\mathscr{C}}{\mathscr{D}-\mathscr{C}}, \qquad \Phi^{*-1}(\mathscr{C}) \leqslant x < \Phi^{*-1}(\mathscr{D}), \qquad \Phi^*(x) = \mathscr{N}\big(\tilde{W}(x)-l(1-\alpha)t\big).$$

Clearly, both $\tilde{F}$ and F^* are nondegenerate. Hence, by virtue to the Khintchine's convergence of types theorem, we deduce that there exist two functions $\alpha(t) > 0$ and $\beta(t)$ such that $\tilde{F}\big(\alpha(t)x+\beta(t)\big) = F^*(x)$, which obviously leads to (2.4).

3. PROOF OF THEOREM 2.1

We now turn to the proof of Theorem 2.1 under the Chibisov condition. Since the proof is somewhat lengthy, we split it into several steps, some of which are of independent interest.

Step 1. *If there exist two real numbers $t' < t''$ such that $-\infty < t' < 0 < t'' < \infty$, $c \leqslant \alpha(t'')c+\beta(t'') < d$ and $c < \alpha(t')d+\beta(t') \leqslant d$, then $\Phi^*(c) = 0$ and $\Phi^*(d) = 1$.* (This means that the convergence in (2.1) will continue weakly to a nondegenerate d.f., and in this case the proof of the theorem will follow at once from the Chibisov results.)

Proof. If two such real numbers t' and t'' exist, then, by virtue of (2.4), we deduce that

$$W^*(c) \leqslant W^*\big(\alpha(t'')c+\beta(t'')\big) = \tilde{W}\big(\alpha(t'')c+\beta(t'')\big) = W^*(c)-l(1-\alpha)t'', \qquad t'' > 0,$$

$$W^*(d) \geqslant W^*\big(\alpha(t')d+\beta(t')\big) = \tilde{W}\big(\alpha(t')d+\beta(t')\big) = W^*(d)-l(1-\alpha)t', \qquad t' < 0,$$

which is impossible unless $W^*(c) = -\infty$ or ∞ and $W^*(d) = -\infty$ or ∞. But the cases $W^*(c) = \infty$ and $W^*(d) = -\infty$ are impossible because they lead to $\mathcal{N}(W^*(c)) = \Phi^*(c) = 1$ and $\mathcal{N}(W^*(d)) = \Phi^*(d) = 0$, respectively. Therefore, $W^*(c) = -\infty$ and $W^*(d) = \infty$, which completes the proof of Step 1. ∎

Step 2. *Under our assumptions (the assumptions of Lemma 2.1) it is impossible to find $0 < t'' < \infty$ such that $c < d \leqslant \alpha(t'')c+\beta(t'')$ and to find $-\infty < t' < 0$ such that $\alpha(t')d+\beta(t') \leqslant c < d$.*

Proof. If two such real numbers exist, then, in view of (2.4), we get the relations

$$\tilde{W}(\alpha(t'')c+\beta(t'')) \geqslant \tilde{W}(c) = W^*(c) = \tilde{W}(\alpha(t'')c+\beta(t''))+l(1-\alpha)t'', \qquad t'' > 0,$$

and

$$\tilde{W}(\alpha(t')d+\beta(t')) \leqslant \tilde{W}(d) = W^*(d) = \tilde{W}(\alpha(t')d+\beta(t'))+l(1-\alpha)t', \qquad t' < 0,$$

which are impossible unless $\tilde{W}(\alpha(t'')c+\beta(t'')) = -\infty$ or ∞ and $\tilde{W}(\alpha(t')d+\beta(t')) = -\infty$ or ∞. The value $-\infty$, in the preceding two cases, leads to $\Phi^*(d) = 0$ (in the first case we have $0 = \tilde{\Phi}(-\infty) = \tilde{\Phi}(\alpha(t'')c+\beta(t'')) \geqslant \tilde{\Phi}(d) = \Phi^*(d)$, i.e., $\Phi^*(d) = 0$, and in the second case we have $-\infty \leqslant \tilde{W}(d) = W^*(d) = -\infty+l(1-\alpha)t' = -\infty$, i.e., $\Phi^*(d) = 0$), which is impossible. The value ∞, in the above two cases, is also impossible because it leads to $\Phi^*(c) = 1$ (in the first case we get $\infty \geqslant \tilde{W}(c) = W^*(c) = \infty+l(1-\alpha)t'' = \infty$, i.e., $W^*(c) = \infty$, while in the second case we have $1 = \tilde{\Phi}(\infty) = \tilde{\Phi}(\alpha(t')d+\beta(t')) \leqslant \tilde{\Phi}(c) = \Phi^*(c)$). This proves Step 2. ∎

Step 3. Combining Steps 1 and 2, we deduce, immediately, the following

(i) *If there exists a real number $0 < t'' < \infty$ such that $c \leqslant \alpha(t'')c+\beta(t'')$, then $\Phi^*(c) = 0$, which implies the continuation of the convergence in* (2.1) *to the left (i.e., to $-\infty$).*

(ii) *If there exists a real number $-\infty < t' < 0$ such that $\alpha(t')d+\beta(t') \leqslant d$, then $\Phi^*(d) = 1$, which implies the continuation of the convergence in* (2.1) *to the right (i.e., to ∞).*

(iii) *If there exist two such numbers, defined above, then the convergence in* (2.1) *will continue weakly, for all x, to a nondegenerate d.f. which coincides with $\Phi^*(x)$ on $[c, d]$.*

Moreover, in this case, the theorem will follow immediately from the Chibisov results.

Step 4. *Under our conditions there exist at least two growth points x_1 and x_2 in (c, d). Let us assume that $x_1 < x_2$. Assume further that $d < \alpha(t)d+\beta(t)$ for all $t < 0$. Then there exists $t' < 0$ such that*

$$\alpha(t')c+\beta(t') < x_1 < d < \alpha(t')d+\beta(t'). \tag{3.1}$$

Proof. Since the limit $\tilde{\Phi}(x)$ in (2.2) is a nondegenerate d.f., it follows from the Chibisov results that it must be of the form $\mathcal{N}(\tilde{W}(x))$. Moreover, $\tilde{W}(x)$ must be one of the types (1.2). Hence, using Khintchine's convergence theorem it is easy to prove that

$$\tilde{W}(x) = \tilde{W}(\alpha(t)x+\beta(t))+l(1-\alpha)t \quad \text{for all } t \text{ and } x.$$

Now, a quick check shows that $\beta(t)$ is either zero or $-l(1-\alpha)t$ and $\alpha(t) = e^{f(t)}$, where $f(t) = -l(1-\alpha)t/\theta$ for the first type of (1.2), $f(t) = l(1-\alpha)t/\theta$ for the second type of (1.2) and $f(t) = 0$ for the third type of (1.2). Hence we deduce that $\alpha(t)$ and $\beta(t)$ are continuous and monotonic functions of t. Let us define a new continuous function $g_c(t) = \alpha(t)c+\beta(t)$. Clearly, $g_c(0) = c$. Hence there exists $\delta > 0$ such that

$$|g_c(t')-g_c(0)| = |\alpha(t')c+\beta(t')-c| < x_1-c \quad \text{whenever } -\delta < t' < 0.$$

This implies that for all $-\delta < t < 0$ (there are infinitely many t such that $-\delta < t < 0$) we have $\alpha(t)c+\beta(t) < x_1$, which completes the proof of Step 4. ∎

Step 5. *Assume that for all $t < 0$ we have $d < \alpha(t)d+\beta(t)$. Then the convergence in* (2.1) *will continue weakly, for all x, to the right* (*i.e., for all $x > d$*).

Proof. Let $\mathcal{T}$ be the set of all $t < 0$, which satisfies the condition (3.1). Henceforth, we shall consider only the values $t \in \mathcal{T}$. Furthermore, let us consider the following three cases:

(I) there exists $t \in \mathcal{T}$ such that $\alpha(t) < 1$;

(II) there exists $t \in \mathcal{T}$ such that $\alpha(t) = 1$;

(III) for all $t \in \mathcal{T}$ we have $\alpha(t) > 1$.

First let us consider the case (I). Clearly, we have $d < \beta(t)/(1-\alpha(t))$. Now, if we show that the convergence of the sequence $\{\Phi_{n:n}(a_n x+b_n)\}$ will continue at the point $D = \beta(t)/(1-\alpha(t))$, then, in view of Step 3 (ii), the convergence will continue, for all x, to the right (since $D = \alpha(t)D+\beta(t)$). Indeed, by (2.3) and (2.4), we have

$$\Phi_{n:n}(a_{n(t)}x+b_{n(t)}) \xrightarrow{[c,d]} \tilde{\Phi}(\alpha(t)x+\beta(t)) \quad \text{as } n\to\infty. \tag{3.2}$$

Put $y = \alpha(t)x+\beta(t)$; we get

$$\Phi_{n:n}\left(\frac{a_{n(t)}}{\alpha(t)}y+b_{n(t)}-\frac{a_{n(t)}\beta(t)}{\alpha(t)}\right) \xrightarrow{[c_1,d_1]} \tilde{\Phi}(y) \quad \text{as } n\to\infty, \tag{3.3}$$

where $c_1 = \alpha(t)c+\beta(t)$ and $d_1 = \alpha(t)d+\beta(t)$. In view of the facts that $\tilde{\Phi}(x) = \Phi^*(x)$ for all $x \in [c, d]\cap[c_1, d_1]$ and $\Phi^*(x)$ has more than two different values in the interval $[c, d]\cap[c_1, d_1]$, we can apply Lemma 2.3 of Barakat [4] (this lemma is originally due to Senussi-Bereksi [12]) in (2.1) and (3.3) to get

$$\frac{a_n\alpha(t)}{a_{n(t)}} \to 1 \quad \text{and} \quad \frac{b_n-b_{n(t)}+a_{n(t)}\beta(t)/\alpha(t)}{a_n} \to 0 \quad \text{as } n\to\infty.$$

By another appeal to Lemma 2.3 of Barakat [4], $a_{n(t)}/\alpha(t)$ and $b_{n(t)}-a_{n(t)}\beta(t)/\alpha(t)$ in (3.3) may be changed to a_n and b_n, respectively. Hence we get

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c_1,d_1]} \tilde{\Phi}(y) \quad \text{as } n\to\infty,$$

which, in view of (2.1), leads obviously to

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c,d_1]} \tilde{\Phi}(y) \quad \text{as } n\to\infty.$$

Repeating this argument N times, we get the relation

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c,d_N]} \tilde{\Phi}(y) \quad \text{as } n\to\infty,$$

where

$$\begin{aligned} d_N &= \alpha^N(t)\,d+\beta(t)\,[1+\alpha(t)+\ldots+\alpha^{N-1}(t)] \\ &= \alpha^N(t)\,d+\frac{1-\alpha^N(t)}{1-\alpha(t)}\beta(t)\to D \quad \text{as } N\to\infty. \end{aligned}$$

Therefore, due to the continuity of the function $\tilde{W}(y)$ for all y, the proof of Step 5 follows in this case.

Let us consider now the case (II). In this case, we can easily show that $\beta(t)$ for all $t\in\mathcal{T}$ is strictly positive. Hence, if we put $y=x+\beta(t)$ in (3.2), we get

$$\Phi_{n:n}\big(a_{n(t)}y+b_{n(t)}-a_{n(t)}\beta(t)\big) \xrightarrow{[c_1',d_1']} \tilde{\Phi}(y) \quad \text{as } n\to\infty, \tag{3.4}$$

where $c_1'=c+\beta(t)$ and $d_1'=d+\beta(t)$. An application of Lemma 2.3 of Barakat [4] thus yields (using (2.1) and (3.4))

$$\frac{a_{n(t)}}{a_n}\to 1 \quad \text{and} \quad \frac{b_n-b_{n(t)}+a_{n(t)}\beta(t)}{a_n}\to 0 \quad \text{as } n\to\infty.$$

(Note that $\tilde{\Phi}(x)=\Phi^*(x)$ for all $x\in[c,d]\cap[c_1',d_1']$ and $\Phi^*(x)$ has more than two different values in the interval $[c,d]\cap[c_1',d_1']$.) By another appeal to Lemma 2.3 of Barakat [4], $a_{n(t)}$ and $b_{n(t)}-a_{n(t)}\beta(t)$ in (3.4) may be changed to a_n and b_n, respectively. Hence, we get

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c_1',d_1']} \tilde{\Phi}(y) \quad \text{as } n\to\infty,$$

which leads obviously to

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c,d_1']} \tilde{\Phi}(y) \quad \text{as } n\to\infty.$$

Using the last procedure N times, we deduce that

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c,d_N']} \tilde{\Phi}(y) \quad \text{as } n\to\infty,$$

where $d_N'=d+N\beta(t)$. Therefore, when $N\to\infty$, the proof in this case follows immediately in view of the continuity of $\tilde{W}(y)$ for all y and the fact that $\beta(t)>0$.

Finally, let us consider the case (III). Clearly, in this case $d+\beta(t)/(\alpha(t)-1)$ is strictly positive. Using the same procedure which was applied in the case (I), we can prove the relation

$$\Phi_{n:n}(a_n y+b_n) \xrightarrow{[c,d_N]} \tilde{\Phi}(y) \quad \text{as } n\to\infty,$$

where

$$d_N = \alpha^N(t)\,d+\frac{1-\alpha^N(t)}{1-\alpha(t)}\beta(t) = \left[d+\frac{\beta(t)}{\alpha(t)-1}\right]\alpha^N(t)-\frac{\beta(t)}{\alpha(t)-1}.$$

Since $d+\beta(t)/(\alpha(t)-1)>0$, it follows that $d_N\to\infty$ as $N\to\infty$, which means that the convergence in (2.1) will continue to $\tilde{\Phi}(x)$ for all $x>d$ (to the right). This completes the proof. ∎

Step 6. *Let $\alpha(t)c+\beta(t)<c$ for all $t>0$. Then there exists $t''>0$ such that*

$$\alpha(t'')c+\beta(t'')<c<x_2<\alpha(t'')d+\beta(t'').$$

Proof. The proof of this step is exactly the same as the proof of Step 4 (with obvious modifications). ∎

Step 7. *Let $\alpha(t)c+\beta(t)<c$ for all $t>0$. Then the convergence in* (2.1) *will continue weakly, for all x, to the left* (*i.e.*, *to* $-\infty$).

Proof. The proof of this step is exactly the same as the proof of Step 5 (with obvious modifications). ∎

Proof of Theorem 2.1 under the Chibisov condition. The proof of Theorem 2.1 in this case follows immediately by combining Lemma 2.1 and all Steps 1–7. ∎

Proof of Theorem 2.1 under the general conditions (A) and (B). The key of the extended proof consists in that, under the condition (A) or (B) and for any given positive real number t, there are two subsequences $\{n'\}$ and $\{n''\}$ of the natural numbers such that $n''>n'$ for all n, $n''/n'\to\mu_t^2$ as $n\to\infty$ $(1\leqslant\mu_t<\infty)$, and

$$n'\frac{r_{n'}/n'-r_{n''}/n''}{\sqrt{r_{n'}}}\to t\sqrt{1-\lambda} \quad \text{as } n\to\infty$$

(see Wu [22]). Moreover,

$$\lim_{n\to\infty}\frac{r_{n''}}{r_{n'}} = \lim_{n\to\infty}\frac{n''}{n'} = \mu_t^2.$$

Therefore, repeating the first part of the proof of Lemma 2.1, by changing the roles of the sequences $\{n\}$ and $\{n(t)\}$ to $\{n'\}$ and $\{n''\}$, respectively, we deduce that there exist two real functions $\alpha_1(t)>0$ and $\beta_1(t)$ such that

$$\mu_t W^*(t) = \tilde{W}(\alpha_1(t)x+\beta_1(t))-t\mu_t \quad \text{for all } x\in[c,d]. \tag{3.5}$$

On the other hand, repeating again the first part of the proof of Lemma 2.1, by changing the roles of the sequences $\{n\}$ and $\{n(t)\}$ to $\{n''\}$ and $\{n'\}$, respectively, we deduce that there exist two real functions $\alpha_2(t) > 0$ and $\beta_2(t)$ such that

$$\mu_t^{-1} W^*(t) = \tilde{W}(\alpha_2(t)x+\beta_2(t))+t\mu_t \quad \text{for all } x\in[c, d]. \tag{3.6}$$

Now, by applying the relation (3.5) and using the same argument as in Lemma 2.1, we can easily prove that $\tilde{W}(-\infty) = -\infty$, while by applying the relation (3.6) and using the same argument, we can easily prove that $\tilde{W}(\infty) = \infty$. Furthermore, we can see that the remaining part of the extended proof is exactly the same as Steps 1–7 (with only the obvious modifications). By changing the roles of $\alpha(t')$, $\beta(t')$, $\alpha(t'')$ and $\beta(t'')$ to $\alpha_1(t')$, $\beta_1(t')$, $\alpha_2(t'')$ and $\beta_2(t'')$, respectively, and using the relations (3.5) (the functions $\alpha_1(\cdot)$ and $\beta_1(\cdot)$) and (3.6) (the functions $\alpha_2(\cdot)$ and $\beta_2(\cdot)$), respectively, we can prove the left and right continuation of the convergence. This completes the proof. ∎

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