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ON A NEW AFFINE INVARIANT AND CONSISTENT TEST FOR MULTIVARIATE NORMALITY

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Abstract. We propose a new test for multivariate normality based on the empirical characteristic function. We show that the test is affine invariant and consistent against every non-normal alternative. The test considered in this paper is also able to detect contiguous alternatives that converge to the normal distribution at the rate $n^{-1/2}$. The results of an extensive Monte Carlo study show that the test has power comparable with one of the best existing procedures.

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1. INTRODUCTION

Although there exist more than 50 procedures for testing multivariate normality (for a recent review see [7]), Henze and Wagner [8] noted that only BHEP test (introduced in the univariate case by Epps and Pulley [5] and extended to the multivariate one by Baringhaus and Henze [2]) shares all of the following desirable properties:

- affine invariance,
- consistency against each fixed non-normal alternative distribution,
- asymptotic power against contiguous alternatives of order $n^{-1/2}$,
- feasibility for any dimension and any sample size.

In this paper, we propose a new test having the foregoing properties.

Let $X_1, X_2, \ldots$ be a sequence of d-dimensional, independent, identically distributed random vectors with distribution P and characteristic function $C(t)$. By S_n we denote the empirical covariance matrix

$$S_n = \frac{1}{n}\sum_{j=1}^{n} (X_j - \bar{X}_n)(X_j - \bar{X}_n)^T,$$

where $\bar{X}_n$ stands for the sample mean, $\bar{X}_n = n^{-1}\sum_{j=1}^{n} X_j$. Let $S_n^{-1/2}$ denote a symmetric, positive definite square root of S_n^{-1} and define the scaled residuals by $Y_j = S_n^{-1/2}(X_j - \bar{X}_n)$. By $\tilde{C}_n(t)$ we denote the empirical characteristic function of the scaled residuals, i.e.

$$\tilde{C}_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(i\langle t, Y_j\rangle) = \exp(-i\langle S_n^{-1/2} t, \bar{X}_n\rangle)\, C_n(S_n^{-1/2} t),$$

where $C_n(t)$ is the empirical characteristic function of the sample $X_1, \ldots, X_n$, i.e.

$$C_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(i\langle t, X_j\rangle)$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\boldsymbol{R}^d$.

For testing a hypothesis that the sample comes from a non-degenerate d-dimensional normal distribution ($P \in \mathcal{N}_d$) we consider the following statistic:

$$T_n = T_n(X_1, \ldots, X_n) = \sqrt{n} \sup_{|t|<r} |W_n(t)|,$$

where

$$W_n(t) = \begin{cases} \dfrac{\tilde{C}_n(t) - \exp(-|t|^2/2)}{|t|}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

and $|\cdot|$ denotes the Euclidean norm in $\boldsymbol{R}^d$.

Note that the test statistic is defined only if S_n is non-singular. But, if P is the non-degenerate d-dimensional normal distribution, then S_n is non-singular with probability one.

The proposed statistic is a kind of distance between the empirical characteristic function of the scaled residuals and the theoretical characteristic function of the standard normal distribution. Two similar statistics have been considered before. The first, proposed by Csörgő [3], was as follows:

$$\sqrt{n} \sup_{|t|<r} \big| |\tilde{C}_n(t)|^2 - \exp(-|t|^2) \big|$$

and the statistic of the BHEP test was

$$n \int_{\boldsymbol{R}^d} \left| \tilde{C}_n(t) - \exp\left(-\frac{|t|^2}{2}\right) \right|^2 \phi_\beta(t)\, dt,$$

where

$$\phi_\beta(t) := (2\pi\beta^2)^{-d/2} \exp\left(-\frac{|t|^2}{2\beta^2}\right).$$

In this paper we consider the weighted supremum distance. Since the most important properties of a distribution are determined by the behaviour of its characteristic function in a neighbourhood of zero, we use the weight function $1/|t|$.

2. AFFINE INVARIANCE

As Szkutnik [9] pointed out, every full-rank affine transformation of the sample is equivalent to an orthogonal transformation of the scaled residuals. Therefore, to obtain the affine invariance of the test statistic, it is sufficient to ensure that the value of the statistic T_n does not change, when Y_j is replaced by HY_j for $j = 1, \ldots, n$, where H is an arbitrary orthogonal matrix. To this end,

$$\sup_{|t|<r} \frac{\left|n^{-1}\sum_{j=1}^{n} \exp(i\langle t, HY_j\rangle) - \exp(-|t|^2/2)\right|}{|t|}$$
$$= \sup_{|Hs|<r} \frac{\left|n^{-1}\sum_{j=1}^{n} \exp(i\langle s, Y_j\rangle) - \exp(-|Hs|^2/2)\right|}{|Hs|}$$
$$= \sup_{|s|<r} \frac{\left|n^{-1}\sum_{j=1}^{n} \exp(i\langle s, Y_j\rangle) - \exp(-|s|^2/2)\right|}{|s|},$$

which means that the test statistic is affine invariant.

3. THE ASYMPTOTIC BEHAVIOUR OF THE TEST STATISTIC UNDER NORMALITY

Let $r > 0$ be fixed. By $C(B_r)$ we denote the space consisting of all complex-valued continuous functions with domain B_r, where B_r denotes the closed ball in $\boldsymbol{R}^d$ with center in 0 and radius r, endowed with the supremum norm

$$\|f\|_{C(B_r)} = \sup_{t\in B_r} |f(t)|.$$

Define the process Z_n by the formula

$$Z_n(t) = \sqrt{n}\left(\tilde{C}_n(t) - \exp(-|t|^2/2)\right)$$
$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\cos\langle t, Y_j\rangle - \exp(-|t|^2/2) + i\sin\langle t, Y_j\rangle\right).$$

Under the null hypothesis, Z_n converges weakly to a certain Gaussian process. Due to the affine invariance of T_n, it is sufficient to consider the standardized normal distribution of the sample.

THEOREM 3.1. *Let $X_1, X_2, \ldots$ be a sequence of d-dimensional independent, identically distributed random vectors with distribution $N_d(0, I_d)$. Then there exists a centered, complex-valued Gaussian process Z in $C(B_r)$, with the covariance* *kernel*

$$K(s, t) = \exp\left(-\frac{|t+s|^2}{2}\right) - \left(1 - \langle s, t\rangle + \frac{\langle s, t\rangle^2}{2}\right)\exp\left(-\frac{|s|^2+|t|^2}{2}\right)$$

for all $t, s \in B_r$ *such that*

$$Z_n \xrightarrow{d} Z \quad \text{in } C(B_r),$$

where $\xrightarrow{d}$ *denotes weak convergence.*

Since the proof of this theorem is analogous to the proof 2.1 in [8], we give only its main steps (the process considered in [8] is of the form $\Re Z_n + \Im Z_n$, where $\Re Z_n$ and $\Im Z_n$ stand for the real and imaginary part of the complex-valued process Z_n defined above).

Sketch of the proof. Define the auxiliary processes Z_n^* and $\tilde{Z}_n$ as follows:

$$Z_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\left(\cos\langle t, X_j\rangle - \exp\left(-\frac{|t|^2}{2}\right)\right) + \left(\frac{1}{2}\langle t, X_j\rangle^2 - \frac{|t|^2}{2}\right)\exp\left(-\frac{|t|^2}{2}\right)\right)$$

$$+ \frac{i}{\sqrt{n}} \sum_{j=1}^{n} \left(\sin\langle t, X_j\rangle + \langle t, X_j\rangle \exp\left(-\frac{|t|^2}{2}\right)\right)$$

and

$$\tilde{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\left(\cos\langle t, X_j\rangle - \exp\left(-\frac{|t|^2}{2}\right)\right) - \langle t, \Delta_j\rangle \sin\langle t, X_j\rangle\right)$$

$$+ \frac{i}{\sqrt{n}} \sum_{j=1}^{n} \left(\sin\langle t, X_j\rangle + \langle t, \Delta_j\rangle \cos\langle t, X_j\rangle\right),$$

where $\Delta_j = (S_n^{-1/2} - I_d)X_j - S_n^{-1/2}\bar{X}_n$ for $j = 1, \ldots, n$. By straightforward calculations it is easy to prove that

$$E(Z_n^*(t)) \equiv 0 \quad \text{and} \quad E(Z_n^*(s) Z_n^*(t)) = K(s, t).$$

In a way analogous to that in [8], one can obtain

$$\|Z_n - \tilde{Z}_n\|_{C(B_r)} \xrightarrow{P} 0, \quad \|Z_n^* - \tilde{Z}_n\|_{C(B_r)} \xrightarrow{P} 0$$

as well as the existence of a complex-valued Gaussian process $Z \in C(B_r)$ such that $E(Z(t)) \equiv 0$, $E(Z(s)Z(t)) = K(s, t)$ and

$$Z_n^* \xrightarrow{d} Z \quad \text{in } C(B_r). \blacksquare$$

Before formulating the next theorem we introduce the following notation:

$$\frac{f(\cdot)}{|\cdot|}(t) = \begin{cases} f(t)/|t|, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

THEOREM 3.2. *Under the conditions of Theorem* 3.1,

$$\frac{Z_n(\cdot)}{|\cdot|} \xrightarrow{d} \frac{Z(\cdot)}{|\cdot|} \quad \text{in } C(B_r).$$

Proof. In order to prove the theorem, we show that

$$\frac{Z_n^*(\cdot)}{|\cdot|} \xrightarrow{d} \frac{Z(\cdot)}{|\cdot|} \quad \text{in } C(B_r), \tag{1}$$

$$\left\| \frac{\tilde{Z}_n(\cdot)}{|\cdot|} - \frac{Z_n(\cdot)}{|\cdot|} \right\|_{C(B_r)} \xrightarrow{P} 0, \tag{2}$$

and

$$\left\| \frac{\tilde{Z}_n(\cdot)}{|\cdot|} - \frac{Z_n^*(\cdot)}{|\cdot|} \right\|_{C(B_r)} \xrightarrow{P} 0. \tag{3}$$

To show (2), one can notice that (cf. [8])

$$|\tilde{Z}_n(t) - Z_n(t)| \leqslant |t|^2 o_P(1).$$

To obtain (3), one can estimate

$$\left| \frac{\tilde{Z}_n(t)}{|t|} - \frac{Z_n^*(t)}{|t|} \right| \leqslant \frac{1}{2} \left\| \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (X_j X_j^T - I_d) \right\| \cdot \left| A_n(t) + t \exp\left(-\frac{|t|^2}{2}\right) \right|$$

$$+ |A_n(t)|\, o_P(1) + \sqrt{n}\, \|S_n^{-1/2} - I_d\| \cdot |\bar{X}| \cdot |B_n(t)| + \sqrt{n}\, |\bar{X}| \cdot \left| B_n(t) - i \exp\left(-\frac{|t|^2}{2}\right) \right|,$$

where

$$A_n(t) = \frac{1}{n} \sum_{j=1}^{n} X_j (i \cos\langle t, X_j\rangle - \sin\langle t, X_j\rangle),$$

$$B_n(t) = \frac{1}{n} \sum_{j=1}^{n} (i \cos\langle t, X_j\rangle - \sin\langle t, X_j\rangle).$$

Now, using

$$\sup_{|t|<r} \left| A_n(t) + t \exp\left(-\frac{|t|^2}{2}\right) \right| \to 0 \text{ a.s.}, \qquad \sup_{|t|<r} \left| B_n(t) + i \exp\left(-\frac{|t|^2}{2}\right) \right| \to 0 \text{ a.s.},$$

we have (3).

In order to prove (1), one should obtain weak convergence of finite-dimensional distributions (which is a consequence of Theorem 3.1) and tightness of the process $Z_n^*(\cdot)/|\cdot|$.

The tightness of this process can be proved by applying Corollary 7.17 from [1] to the real and imaginary part of this process. The entropy condition used in that corollary is shown in [8].

We define the following auxiliary functions:

$$g(t, x) = \frac{\cos\langle t, x\rangle - \exp(-|t|^2/2)}{|t|} + \left(\frac{1}{2}\frac{\langle t, x\rangle^2}{|t|} - \frac{|t|}{2}\right)\exp\left(-\frac{|t|^2}{2}\right),$$

$$h(t, x) = \frac{\sin\langle t, x\rangle - \langle t, x\rangle \exp(-|t|^2/2)}{|t|}.$$

Now, we ought to show that there exist real-valued random variables M_1 and M_2 such that $E(M_1^2) < \infty$, $E(M_2^2) < \infty$, and

$$|g(t, X) - g(s, X)| \leqslant M_1 |t-s|, \quad |h(t, X) - h(s, X)| \leqslant M_2 |t-s|.$$

Straightforward (but somewhat lengthy) calculations show that these inequalities hold for

$$M_1 = 2 + 3|X|^2 + r^2 + \tfrac{1}{2}r^2|X|, \quad M_2 = \tfrac{1}{2}|X|^2 + 2r|X| + r^3|X| + \tfrac{1}{2}r|X|^2. \ \blacksquare$$

Theorem 3.2 and continuity of the norm yield

COROLLARY 3.3. *Under the assumptions of Theorem* 3.1,

$$\sup_{|t|<r} \frac{\sqrt{n}\,|\tilde{C}_n(t) - \exp(-|t|^2/2)|}{|t|} \xrightarrow{d} \left\| \frac{Z(\cdot)}{|\cdot|} \right\|_{C(B_r)}.$$

Unfortunately, there are no results about the distribution of the limit random variable, since it is the supremum of modulus of non-stationary complex-valued d-dimensional Gaussian random process.

4. CONSISTENCY

Consistency of the test based on the statistic T_n for every non-normal alternative is implied by Corollary 3.3 and the following theorem:

THEOREM 4.1. *If P is not a d-dimensional non-degenerate normal distribution, then there exists a constant $D > 0$ such that*

$$\liminf_{n\to\infty} \frac{|\tilde{C}_n(t) - \exp(-|t|^2/2)|}{|t|} \geqslant D \ \text{a.s.}$$

Proof. Csörgő [4] has showed the existence of a constant $D_0 > 0$ such that for every $t \in \boldsymbol{R}^d$

$$\liminf_{n\to\infty} \big||\tilde{C}_n(t)|^2 - \exp(-|t|^2)\big| \geqslant D_0 \ \text{a.s.}$$

Since, almost surely

$$\frac{|\tilde{C}_n(t)-\exp(-|t|^2/2)|}{|t|} \geqslant \frac{||\tilde{C}_n(t)|-\exp(-|t|^2/2)|}{|t|}$$
$$= \frac{||\tilde{C}_n(t)|^2-\exp(-|t|^2)|}{|t|\cdot|\tilde{C}_n(t)+\exp(-|t|^2/2)|} \geqslant \frac{||\tilde{C}_n(t)|^2-\exp(-|t|^2)|}{2r},$$

the conclusion follows immediately. ∎

5. ASYMPTOTIC BEHAVIOUR OF THE TEST STATISTIC UNDER CONTIGUOUS ALTERNATIVES

Consider a triangular array $X_{n1}, \ldots, X_{nn}$, $n \geqslant d+1$, of rowwise independent and identically distributed random vectors with probability density function $f_n(x) = \Phi_d(x)\left(1+n^{-1/2}p(x)\right)$, where Φ_d is the density of a d-dimensional standard normal vector and p is a bounded measurable function such that $\int_{\mathbf{R}^d} p(x)\,\Phi_d(x)\,dx = 0$. We assume that n is large enough to guarantee that $f_n(x)$ is non-negative.

From the following theorem we deduce that the considered test is able to detect alternatives that converge to the normal distribution at the rate $n^{-1/2}$.

THEOREM 5.1. *If all the above assumptions hold, we have*

$$\sup_{|t|<r} \frac{|Z_n(\cdot)|}{|\cdot|} \xrightarrow{d} \left\|\frac{Z(\cdot)+C(\cdot)}{|\cdot|}\right\|_{C(B_r)},$$

where the function C is defined by

$$C(t) = \int g(x, t)\,p(x)\,\Phi_d(x)\,dx + i\int h(x, t)\,p(x)\,\Phi_d(x)\,dx,$$

and g and h are defined in Section 3.

Proof. Analogously to [8], one can show that

$$\frac{Z_n(\cdot)}{|\cdot|} \xrightarrow{d} \frac{Z(\cdot)+C(\cdot)}{|\cdot|} \quad \text{in } C(B_r).$$

Thus, the continuous mapping theorem completes the proof.

6. SIMULATIONS*

In this section, we present the results of a simulation study in which we computed the empirical power of the test for some specified alternatives. Since in some recent studies (see [7] and [6]) the BHEP test was recommended as

* All the computations were performed in ACK Cyfronet AGH in Kraków under Grant KBN/SGI/2800/PK/029/2003 using Mathematica 4.2.

a reasonable choice for the omnibus test for multivariate normality, we compare the estimated powers with the estimated powers of this test.

We simulated the critical values and powers of the tests for the sample sizes $n = 20$, 50 and 100 upon 10000 runs in each case. The significance level was $\alpha = 0.05$.

In Table 1, we present the empirical critical values for the test statistic. Table 2 and Table 3 contain the empirical powers for the test based on statistic T_n with radius $r = 1, 2, 5$ and the BHEP test with parameter $\beta = 0.5, 1, 2$, respectively. Table 2 presents the powers for the case $d = 1$, whereas Table 3 contains the results for the case $d = 2$. In these tables we have used the following symbols: $N(0, 1)$ and $Ex(1)$ denote the standard normal and exponential distribution, $U(0, 1)$ is the uniform distribution on the unit interval, $LN(0, 1)$ is the lognormal distribution corresponding to the standard normal distribution, and $Log(a, b)$, $B(a, b)$ and $G(a, b)$ stand for the logistic, beta and gamma distributions, respectively.

TABLE 1. Empirical critical values

	$\alpha = 0.01$					
	$d = 1$			$d = 2$		
	$n = 20$	$n = 50$	$n = 100$	$n = 20$	$n = 50$	$n = 100$
$r = 1.0$	0.6762	0.6521	0.6594	0.8329	0.8417	0.8395
$r = 2.0$	0.9811	0.9750	0.9666	1.1888	1.2139	1.2022
$r = 5.0$	0.9847	0.9525	0.9761	1.1940	1.2151	1.2074
	$\alpha = 0.05$					
	$d = 1$			$d = 2$		
	$n = 20$	$n = 50$	$n = 100$	$n = 20$	$n = 50$	$n = 100$
$r = 1.0$	0.5032	0.4933	0.4997	0.6710	0.6939	0.6917
$r = 2.0$	0.8021	0.7959	0.7968	1.0087	1.0279	1.0384
$r = 5.0$	0.8137	0.8083	0.8085	1.0172	1.0362	1.0526
	$\alpha = 0.1$					
	$d = 1$			$d = 2$		
	$n = 20$	$n = 50$	$n = 100$	$n = 20$	$n = 50$	$n = 100$
$r = 1.0$	0.4180	0.4155	0.4158	0.5928	0.6236	0.6211
$r = 2.0$	0.7058	0.6964	0.7027	0.9219	0.9426	0.9517
$r = 5.0$	0.7263	0.7152	0.7219	0.9393	0.9545	0.9642

TABLE 2. Estimated powers for $d = 1$, $\alpha = 0.05$

Alternatives	$n = 20$					
	T_n			BHEP		
	$r = 1$	$r = 2$	$r = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$
$LN(0, 1)$	90	90	90	91	91	87
$Ex(1)$	73	75	74	74	75	69
$U(0, 1)$	1	15	18	2	11	21
$Log(0, 0.6)$	12	10	10	12	10	7
χ^2_{10}	24	22	21	24	22	16
t_4	26	23	23	26	23	17
t_6	17	13	13	17	14	10
$G(2, 1)$	49	48	47	49	48	38
$B(2.5, 1.5)$	6	12	12	7	11	13
	$n = 50$					
$LN(0, 1)$	100	100	100	100	100	100
$Ex(1)$	100	99	99	100	100	99
$U(0, 1)$	3	63	62	5	53	60
$Log(0, 0.6)$	21	17	16	21	16	12
χ^2_{10}	64	53	52	63	54	37
t_4	47	44	43	47	44	35
t_6	29	25	24	29	24	17
$G(2, 1)$	94	90	90	94	91	82
$B(2.5, 1.5)$	22	37	36	24	36	32
	$n = 100$					
$LN(0, 1)$	100	100	100	100	100	100
$Ex(1)$	100	100	100	100	100	100
$U(0, 1)$	10	97	97	30	94	94
$Log(0, 0.6)$	28	25	25	28	24	18
χ^2_{10}	91	84	83	91	84	68
t_4	67	67	67	68	67	57
t_6	43	39	38	43	39	28
$G(2, 1)$	100	100	100	100	100	99
$B(2.5, 1.5)$	53	74	73	59	73	65

TABLE 3. Estimated powers for $d = 2$, $\alpha = 0.05$

Alternatives	$n = 20$					
	T_n			BHEP		
	$r = 1$	$r = 2$	$r = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$
$LN(0, 1)^2$	93	96	95	97	97	94
$Ex(1)^2$	76	81	80	85	88	79
$U(0, 1)^2$	1	4	8	1	8	19
$Log(0, 0.6)^2$	15	13	13	16	13	9
$(\chi^2_{10})^2$	24	22	22	28	26	16
$(t_4)^2$	32	29	29	34	30	20
$(t_6)^2$	20	17	17	21	17	11
$G(2, 1)^2$	50	52	51	58	60	43
$B(2.5, 1.5)^2$	3	6	7	4	9	11
$LN(0, 1)\otimes N(0, 1)$	76	79	79	75	76	65
$Ex(1)\otimes N(0, 1)$	52	57	56	53	55	42
$U(0, 1)\otimes N(0, 1)$	2	4	6	3	6	11
$Log(0, 0.6)\otimes N(0, 1)$	9	8	8	10	8	6
$\chi^2_{10}\otimes N(0, 1)$	15	14	14	15	15	10
$t_4\otimes N(0, 1)$	20	18	17	20	17	11
$t_6\otimes N(0, 1)$	13	11	11	13	11	8
$G(2, 1)\otimes N(0, 1)$	31	31	31	32	31	21
$B(2.5, 1.5)\otimes N(0, 1)$	4	5	6	4	6	8
	$n = 50$					
$LN(0, 1)^2$	100	100	100	100	100	100
$Ex(1)^2$	100	100	100	100	100	100
$U(0, 1)^2$	0	35	44	1	48	57
$Log(0, 0.6)^2$	23	19	18	25	19	12
$(\chi^2_{10})^2$	64	56	55	71	62	38
$(t_4)^2$	57	53	53	61	56	41
$(t_6)^2$	35	29	29	37	30	18
$G(2, 1)^2$	95	94	94	98	97	88
$B(2.5, 1.5)^2$	9	26	27	16	34	29
$LN(0, 1)\otimes N(0, 1)$	100	100	100	100	100	99
$Ex(1)\otimes N(0, 1)$	97	97	97	96	95	87
$U(0, 1)\otimes N(0, 1)$	2	22	28	3	18	26
$Log(0, 0.6)\otimes N(0, 1)$	15	12	12	14	11	7
$\chi^2_{10}\otimes N(0, 1)$	41	36	35	41	33	19
$t_4\otimes N(0, 1)$	36	34	33	36	31	20
$t_6\otimes N(0, 1)$	22	18	18	21	17	10
$G(2, 1)\otimes N(0, 1)$	78	76	76	77	71	51
$B(2.5, 1.5)\otimes N(0, 1)$	7	15	16	9	15	13

Alternatives	$n = 100$					
	T_n			BHEP		
	$r = 1$	$r = 2$	$r = 5$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$
$LN(0, 1)^2$	100	100	100	100	100	100
$Ex(1)^2$	100	100	100	100	100	100
$U(0, 1)^2$	0	94	94	13	96	96
$Log(0, 0.6)^2$	30	27	25	35	30	19
$(\chi_{10}^2)^2$	94	89	88	97	93	74
$(t_4)^2$	79	78	77	85	82	69
$(t_6)^2$	50	45	43	56	48	33
$G(2, 1)^2$	100	100	100	100	100	100
$B(2.5, 1.5)^2$	28	67	65	55	77	66
$LN(0, 1)\otimes N(0, 1)$	100	100	100	100	100	100
$Ex(1)\otimes N(0, 1)$	100	100	100	100	100	100
$U(0, 1)\otimes N(0, 1)$	2	78	78	6	57	62
$Log(0, 0.6)\otimes N(0, 1)$	19	17	16	19	16	11
$\chi_{10}^2\otimes N(0, 1)$	78	68	66	74	61	38
$t_4\otimes N(0, 1)$	55	54	52	55	50	37
$t_6\otimes N(0, 1)$	31	27	26	31	25	16
$G(2, 1)\otimes N(0, 1)$	99	98	98	98	97	87
$B(2.5, 1.5)\otimes N(0, 1)$	18	44	42	23	38	31

χ_k^2 and t_k are the chi-square and t-Student distributions with k degrees of freedom.

$P_1\otimes P_2$ is the distribution having independent marginals P_1 and P_2, and by P^2 we denote $P\otimes P$.

It might be observed from the simulations that in one dimension both tests behave much the same. In case of $d = 2$, the results depend on the type of alternative distribution. When both marginals are the same, the BHEP test is slightly superior, while in the case when one of the marginals is normal, the test based on T_n behaves slightly better.

It must be noticed that the power performance of both procedures heavily depend on the choice of test parameters.

It is worth noticing that the tests based on the empirical characteristic function behave poorly when the alternative is the uniform distribution. For instance, the most powerful invariant test (specialized against the uniform distribution) has power 60 in the case $d = 2$, $n = 20$ (see [10]).

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