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A REPRESENTATION OF DISTRIBUTIONS FROM CERTAIN CLASSES L_S^{id}

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Abstract. In this paper we define classes L_S^{id} of certain infinitely divisible measures on the real line. We get a representation of the characteristic functions of distributions from certain classes L_S^{id}. The method of our proof, stimulated by results of Urbanik [5] consists in finding the extreme points of a certain convex set formed by Khintchine measures of distributions from L_S^{id}. Once the extreme points are found, one can apply Choquet's theorem on representation of the points of a compact convex set as barycenters of the extreme points ([4], p. 19). From Choquet's uniqueness theorem for a metrizable space X we obtain the uniqueness of representation ([4], p. 70).

1. It is well known that the measure P on the real line R is infinitely divisible if and only if its characteristic function $\hat{P}$ has the Lévy-Khintchine representation

$$\hat{P}(t) = \exp\left\{ibt + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) \frac{1+u^2}{u^2}\, \mu(du)\right\},$$

where b is a real constant, and μ is a finite Borel measure on R ([2], p. 309), called a *Khintchine measure*.

Let μ be a finite Borel measure on $[-\infty, \infty]$. We put $\tilde{\mu}(B) = \mu(-B)$, where $-B = \{x\colon\ -x \in B\}$. We define the measure p_μ as follows:

$$p_\mu|(-\infty, 0) \cup (0, \infty) = \frac{1+u^2}{u^2}\, \mu|(-\infty, 0) \cup (0, \infty),$$

$$p_\mu|\{-\infty, 0, \infty\} = \mu|\{-\infty, 0, \infty\}.$$

For every $c \in R \setminus \{0\}$ we denote by T_c the mapping $T_c x = cx$ ($x \in [-\infty, \infty]$). Given a Borel measure λ on $[-\infty, \infty]$, $T_c \lambda$ denotes the measure defined by $T_c \lambda(B) = \lambda(T_c^{-1} B)$ for all Borel subsets B of $[-\infty, \infty]$. For Borel measures λ and ν on $[-\infty, \infty]$, $\lambda \leqslant \nu$ if and only if $\lambda(B) \leqslant \nu(B)$ for all Borel subsets B of $[-\infty, \infty]$. Further, by δ_x ($x \in [-\infty, \infty]$) we denote the probability measure concentrated at the point x.

Let P be an infinitely divisible measure on R. The *decomposability semigroup* $D^{\mathrm{id}}(P)$ corresponding to P consists of all real numbers s for which there exists an infinitely divisible measure P_s such that

$$\hat{P}(t) = \hat{P}(st)\hat{P}_s(t) \qquad (t \in R)$$

(see [3]). The *semigroup operation* is simply the multiplication of numbers. It is not difficult to prove that P is non-degenerate if and only if $D^{\mathrm{id}}(P)$ is compact (see [6]). In other words, for non-degenerate P, $D^{\mathrm{id}}(P)$ is a compact subsemigroup of the multiplicative semigroup $[-1, 1]$ containing 0 and 1 (see [7]). In [3] we proved that for every compact semigroup S containing 0 and 1 there exists an infinitely divisible measure P such that $D^{\mathrm{id}}(P) = S$. It is not difficult to prove that, for $s \neq 0$, $s \in D^{\mathrm{id}}(P)$ if and only if

$$T_s p_\mu \geqslant p_\mu, \tag{1}$$

where μ is a Khintchine measure corresponding to P (see [1] and [2]). Given a compact semigroup S containing 0 and 1, we say that the probability measure P belongs to a *class* L_S^{id} if $S \subset D^{\mathrm{id}}(P)$.

2. Throughout this paper, S is a compact semigroup containing 0 and 1. Let $M(S)$ or, shortly, M be the set of all finite Borel measures μ on $[-\infty, \infty]$ for which $T_s p_\mu \geqslant p_\mu$ for each $s \in S \setminus \{0\}$. Let M_0 be the subset of M consisting of measures concentrated on $(-\infty, \infty)$. Then, by (1), $\mu \in M_0$ if and only if μ is a Khintchine measure corresponding to a distribution from L_S^{id}. Let K be the subset of M consisting of all probability measures and put $K_0 = K \cap M_0$. Obviously, the set K is convex. The space of all probability measures on $[-\infty, \infty]$ with weak convergence is a metrizable compact space. We consider the induced topology on K. It is not difficult to show that K is closed. Thus, K is compact.

THEOREM 1. *M is a convex cone generated by K, and K is a simplex.*

Remark. Let X be a compact convex set in a real locally convex space E. Without loss of generality we may assume that X is contained in a closed hyperplane which misses the origin. Put

$$\tilde{X} = \{\alpha x \colon \alpha \geqslant 0,\ x \in X\};$$

$\tilde{X}$ is the cone generated by X. A cone $\tilde{X}$ induces a translation invariant partial ordering on E: $x \geqslant y$ if and only if $x - y \in \tilde{X}$. Then X is a simplex if and only if X is a lattice (i.e., each pair x, y in $\tilde{X}$ has the greatest lower bound, denoted by $x \wedge y$, in $\tilde{X}$ (see [4], p. 58-60)).

Theorem 1 is implied by the following

LEMMA 1. *Let Y_S be the set of all σ-finite non-negative measures p on $[-\infty, \infty]$ such that $T_s p \geqslant p$ for each $s \in S \setminus \{0\}$. Then Y_S is a lattice in its own ordering; for $p_1, p_2 \in Y_S$ we have*

$$p_1 \wedge p_2 = f \wedge g(p_1 + p_2), \tag{2}$$

where $f = dp_1/d(p_1+p_2)$, $g = dp_2/d(p_1+p_2)$, and $f \wedge g = \min(f, g)$.

Proof. Let $p_1, p_2 \in Y_S$ and $p = p_1 + p_2$. Then both p_1 and p_2 are absolutely continuous with respect to p, hence have Radon-Nikodym derivatives f and g, respectively. Let $h = f \wedge g$ (this is defined p-a.e.) and let $p_1 \wedge p_2 = hp$.

Put $C = \{x: f(x) < g(x)\}$ and $D = \{x: f(x) \geqslant g(x)\}$.

Let $s \in S \setminus \{0\}$ and let B be a Borel subset of $[-\infty, \infty]$. Then

$$T_s(p_1 \wedge p_2)(B) = \int_{T_s^{-1}B} hdp = \int_{T_s^{-1}(B\cap C)} hdp + \int_{T_s^{-1}(B\cap D)} hdp = \int_{T_s^{-1}B\cap T_s^{-1}C\cap C} fdp +$$

$$+ \int_{T_s^{-1}B\cap T_s^{-1}C\cap D} gdp + \int_{T_s^{-1}B\cap T_s^{-1}D\cap C} fdp + \int_{T_s^{-1}B\cap T_s^{-1}D\cap D} gdp$$

$$\geqslant \int_{B\cap C\cap T_sC} fdp + \int_{B\cap C\cap T_sD} gdp + \int_{B\cap D\cap T_sC} fdp + \int_{B\cap D\cap T_sD} gdp$$

$$\geqslant \int_{B\cap C\cap T_sC} fdp + \int_{B\cap C\cap T_sD} fdp + \int_{B\cap D\cap T_sC} gdp + \int_{B\cap D\cap T_sD} gdp$$

$$= \int_{B\cap C} fdp + \int_{B\cap D} gdp = \int_{B\cap C} hdp + \int_{B\cap D} hdp = \int_B hdp = p_1 \wedge p_2(B).$$

This shows that $T_s(p_1 \wedge p_2) \geqslant p_1 \wedge p_2$ for each $s \in S \setminus \{0\}$. It follows easily that $p_1 \wedge p_2$ is the greatest lower bound of p_1 and p_2, so Y_s is a lattice.

Proof of Theorem 1. Let $\mu, \lambda \in K$ and $\alpha, \beta > 0$. Let $p_{\alpha\mu} \wedge p_{\mu\lambda}$ be defined as in Lemma 1. Then, by Lemma 1, $p_{\alpha\mu} \wedge p_{\beta\lambda}$ is the greatest lower bound of $p_{\alpha\mu}$ and $p_{\beta\lambda}$, and $T_s(p_{\alpha\mu} \wedge p_{\beta\lambda}) \geqslant p_{\alpha\mu} \wedge p_{\beta\lambda}$. Since $p_{\alpha\mu} \wedge p_{\beta\lambda} \leqslant p_{\alpha\mu}$, there exists a finite Borel measure ν such that $p_{\alpha\mu} \wedge p_{\beta\lambda} = p_\nu$. Then $\nu \in M$ and ν is the greatest lower bound of $\alpha\mu$ and $\beta\lambda$. Thus the theorem is proved.

Clearly, the measures concentrated on the three-point set $\{-\infty, 0, \infty\}$, which have equal masses at $-\infty$ and ∞ if $S \cap [-1, 0) \neq \emptyset$, belong to M. Moreover, $\tilde{\mu} \in M$ if and only if $\mu \in M$. Finally, it is easy to see that a measure belongs to M if and only if its restrictions to $(-\infty, 0) \cup (0, \infty)$ and $\{-\infty, \infty\}$, respectively, belong to M. if $S \cap [-1, 0) = \emptyset$, then a measure belongs to M if and only if its restrictions to $(-\infty, 0)$ and $(0, \infty)$, respectively, belong to M. Hence we get the following lemma:

LEMMA 2. *The extreme points of K are measures concentrated on one of the* *following sets:* $\{0\}$, $\{-\infty, \infty\}$, *and* $(-\infty, 0)\cup(0, \infty)$. *Moreover, if* $S\cap[-1, 0) = \emptyset$, *then the extreme points of* K *are measures concentrated on one of the following sets*: $\{0\}$, $\{-\infty\}$, $\{\infty\}$, $(0, \infty)$, *and* $(-\infty, 0)$.

By $e(K)$ we denote the set of extreme points of K.

LEMMA 3. *If* $\mu\in e(K)$, *then* μ *is absolutely continuous (i.e., absolutely continuous with respect to the Lebesgue measure) or singular continuous, or atomic.*

Proof. Let $\mu\in e(K)$. Write μ in the form

$$\mu = \alpha_1\mu_1+\alpha_2\mu_2+\alpha_3\mu_3,$$

where $\alpha_1+\alpha_2+\alpha_3 = 1$, $\alpha_i \geqslant 0$, μ_1 is an absolutely continuous probability measure, μ_2 is an atomic probability measure, and μ_3 is a singular continuous probability measure. Let E_1, E_2, and E_3 be sets such that $\mu_i(E_j) = 0$ for $i\neq j$ and $\mu_i(E_i) = 1$ $(i, j = 1, 2, 3)$. If $\alpha_i \neq 0$, then $\mu_i(B) = \mu(B\cap E_i)/\alpha_i$ and

$$T_s p_{\mu_i}(B) = \alpha_i^{-1} T_s p_\mu(B\cap E_i) \geqslant \alpha_i^{-1} p_\mu(B\cap E_i) = p_{\mu_i}(B)$$

for all Borel subsets B of $[-\infty, \infty]$, $s\in S\setminus\{0\}$. Thus, if $\alpha_i\neq 0$, then $\mu_i\in K$. Since μ is an extreme point of K, at most one α_i, say α_r, is positive. Then $\mu = \mu_r$. This proves the lemma.

Let λ be a non-negative Borel measure on $[-\infty, \infty]$ and let E be a Borel subset of $[-\infty, \infty]$ such that $\lambda(E') = 0$. Then λ is said to be S-*invariant* on E if $T_s\lambda(B) = \lambda(B)$ for all Borel subsets B of E and $s\in S\setminus\{0\}$. It is not difficult to show that if λ is S-invariant on E, then $T_s^{-1}E\subset E$ λ-a.e. for each $s\in S\setminus\{0\}$.

We shall show that if $\mu\in e(K)$, then p_μ is S-invariant on some Borel set E. First, we prove two basic lemmas on S-invariant measures.

LEMMA 4. *Let* E *be a Borel subset of* $[-\infty, \infty]$. *Suppose that* λ *and* p *are* σ-*finite measures on* $[-\infty, \infty]$, λ *is* S-*invariant on* E $(\lambda(E') = 0)$, *and* p *is absolutely continuous with respect to* λ (*with* $dp/d\lambda = f$, *say*). *Then*:

(a) *if* $T_s p\geqslant p$ *for each* $s\in S\setminus\{0\}$, *then*

$$f\circ T_s^{-1}\geqslant f \ \ \lambda\text{-a.e.} \quad \text{for each } s\in S\setminus\{0\}, \tag{3}$$

$$T_s(p-p\wedge\lambda)\geqslant p-p\wedge\lambda \quad \text{for each } s\in S\setminus\{0\}; \tag{4}$$

(b) *if* $f\circ T_s^{-1}\geqslant f$ λ-*a.e. for each* $s\in S\setminus\{0\}$, *then* $T_s p\geqslant p$ *for each* $s\in S\setminus\{0\}$.

Proof. (a) Let $T_s p\geqslant p$ for each $s\in S\setminus\{0\}$. Put

$$A = \{x\colon f(x) > f(T_s^{-1}(x))\}\cap E \quad \text{for some } s\in S\setminus\{0\}.$$

Suppose that $\lambda(A) > 0$. Then

$$p(T_s^{-1}A) = \int_{T_s^{-1}A} f d\lambda < \int_{T_s^{-1}A} f\circ T_s d\lambda = \int_A f d(\lambda\circ T_s^{-1}) = \int_A f d\lambda = p(A),$$

which contradicts $T_s p\geqslant p$. Thus $f\circ T_s^{-1}\geqslant f$ λ-a.e. for each $s\in S\setminus\{0\}$.

Now we prove inequality (4). Since both p and λ are absolutely continuous with respect to $p+\lambda$, they have Radon-Nikodym derivatives F and G, respectively. Then for all Borel subsets B of $[-\infty, \infty]$ we have

$$\int_B f\,d\lambda = p(B) = \int_B F\,d(p+\lambda) = \int_B F(f+1)\,d\lambda,$$

$$\int_B 1\,d\lambda = \lambda(B) = \int_B G\,d(p+\lambda) = \int_B G(f+1)\,d\lambda.$$

Hence $F(f+1) = f$ λ-a.e. and $G(f+1) = 1$ λ-a.e. Thus $F = f/(f+1)$, $G = 1/(f+1)$, and $F \wedge G = (f \wedge 1)/(f+1)$ λ-a.e. Then for all Borel subsets B of $[-\infty, \infty]$ we have

$$p \wedge \lambda(B) = \int_B (F \wedge G)\,d(p+\lambda) = \int_B (f \wedge 1)\,d\lambda.$$

Put $C = \{x: f(x) > 1\}$. If B is a Borel subset of E and $s \in S \setminus \{0\}$, then

$$\begin{aligned}(p-p \wedge \lambda)(B) &= \int_B (f-f \wedge 1)\,d\lambda = \int_{B\cap C} (f-1)\,d\lambda = \int_{B\cap C} (f-1)\,d(\lambda \circ T_s^{-1})\\ &= \int_{T_s^{-1}B\cap T_s^{-1}C} (f-1)\circ T_s\,d\lambda \leqslant \int_{T_s^{-1}B\cap C} (f-1)\circ T_s\,d\lambda\\ &= \int_{T_s^{-1}B\cap C} (f\circ T_s-1)\,d\lambda \leqslant \int_{T_s^{-1}B\cap C} (f-1)\,d\lambda = \int_{T_s^{-1}B} (f-f\wedge 1)\,d\lambda\\ &= (p-p\wedge\lambda)(T_s^{-1}B) = T_s(p-p\wedge\lambda)(B).\end{aligned}$$

Thus inequality (4) is proved for Borel subsets B of E. If B is a Borel subset of $[-\infty, \infty]$, then

$$\begin{aligned}(p-p \wedge \lambda)(B) &= (p-p \wedge \lambda)(B\cap E) \leqslant (p-p\wedge\lambda)\big(T_s^{-1}(B\cap E)\big)\\ &\leqslant (p-p\wedge\lambda)(T_s^{-1}B) = T_s(p-p\wedge\lambda)(B).\end{aligned}$$

(b) Suppose $f\circ T_s^{-1} \geqslant f$ λ-a.e. for each $s \in S\setminus\{0\}$. If $s \in S\setminus\{0\}$ and B is a Borel subset of $[-\infty, \infty]$, then

$$\begin{aligned}T_s p(B) &= p(T_s^{-1}B) = \int_{T_s^{-1}B} f\,d\lambda \geqslant \int_{T_s^{-1}B} f\circ T_s\,d\lambda = \int_B f\,d(\lambda\circ T_s^{-1})\\ &\geqslant \int_{B\cap E} f\,d(\lambda\circ T_s^{-1}) = \int_{B\cap E} f\,d\lambda = \int_B f\,d\lambda = p(B).\end{aligned}$$

Thus (b) is proved. This completes the proof.

LEMMA 5. *Let $\mu \in e(K)$. Suppose that λ is a σ-finite measure on $[-\infty, \infty]$, E_1 is a Borel subset of $[-\infty, \infty]$ with $\lambda(E_1') = 0$, λ is S-invariant on E_1, and μ is absolutely continuous with respect to λ. Then there exist a number $a_0 > 0$ and a set E of μ-measure 1 such that p_μ is S-invariant on E and $a_0\lambda| E = p_\mu$.*

Proof. If μ is concentrated on $\{-\infty, 0, \infty\}$, then the assertion is immediate. In the remaining cases, by Lemma 2, μ is concentrated on

$(-\infty, 0) \cup (0, \infty)$. Thus, it suffices to prove the lemma for measures μ concentrated on $(-\infty, 0) \cup (0, \infty)$.

Let E be a set such that $\lambda | E$ is absolutely continuous with respect to μ and $\lambda | E'$ is singular with respect to μ. It is easy to see that $\lambda | E$ is S-invariant on E and $\mu(E) = 1$. Clearly, $a\lambda | E$ is S-invariant on E for each $a > 0$.

Suppose that there exists $a_1 > 0$ such that, for a certain Borel set B_1, $a_1 \lambda(B_1 \cap E) < p_\mu(B_1)$ and, for a certain Borel set B_2, $a_1 \lambda(B_2 \cap E) > p_\mu(B_2)$. Setting

$$c = \int_{-\infty}^{\infty} u^2 (1+u^2)^{-1} d(p_\mu \wedge a_1 \lambda),$$

we obtain $0 < c < 1$. Put

$$\mu_1 = c^{-1}(1+u^2)^{-1} u^2 (p_\mu \wedge a_1 \lambda) \quad \text{and} \quad \mu_2 = (1-c)^{-1}(\mu - c\mu_1).$$

From Theorem 1 it follows that $\mu_1 \in K$. By Lemma 4 (a), (4), we have $\mu_2 \in K$. It is clear that $\mu_1 \neq \mu_2$ and $\mu = c\mu_1 + (1-c)\mu_2$, which contradicts the assumption that $\mu \in e(K)$. Thus, for every positive number a and for all Borel subsets B of E either $a\lambda(B) \geqslant p_\mu(B)$ or $a\lambda(B) \leqslant p_\mu(B)$. Hence there exists a positive number a_0 such that $a_0 \lambda(B) = p_\mu(B)$ for all Borel subsets B of E. Thus $a_0 \lambda | E = p_\mu$ and p_μ is S-invariant on E. This completes the proof.

The next two lemmas characterize the extreme points of the set K, which are absolutely continuous measures (i.e., absolutely continuous with respect to the Lebesgue measure) or atomic measures. Throughout the rest of the paper we denote by m the Lebesgue measure.

Let E be a Borel subset of $(-\infty, \infty)$ such that $m(E) > 0$ and $T_s^{-1} E \subset E$ m-a.e. for each $s \in S \setminus \{0\}$. We define the measure p_E by

$$p_E(B) = \int_{B \cap E} \frac{1}{|y|} dy \tag{5}$$

for Borel subsets B of $(-\infty, \infty)$. It is not difficult to prove that p_E is S-invariant on E. If, moreover, E fulfills the condition

$$\int_E |y| (1+y^2)^{-1} dy < \infty,$$

then we define the measure m_E by

$$m_E(B) = A_E \int_{B \cap E} |y| (1+y^2)^{-1} dy \tag{6}$$

for Borel subsets B of $(-\infty, \infty)$, where $(A_E)^{-1} = \int_E |y| (1+y^2)^{-1} dy$. It is easy to see that

$$p_{m_E} = A_E p_E. \tag{7}$$

LEMMA 6. *Let $\mu \in K$ and assume that μ is absolutely continuous (with $d\mu/dm = f_\mu$). Let $E = \{x\colon f_\mu(x) > 0\}$. Then:*

(a) *p_μ is absolutely continuous with respect to p_E and p_E is absolutely continuous with respect to p_μ;*

(b) *if $\mu \in e(K)$, then $\mu = m_E$;*

(c) *if $\mu \in e(K)$, then p_μ is S-invariant on E.*

Proof. (a) is obvious.

(b) Since p_E is S-invariant on E and p_μ is absolutely continuous with respect to p_E, there exists, by Lemma 5, a positive number a_0 such that $p_\mu = a_0 p_E$. Since μ is probability, $a_0 = A_E$. Clearly, a measure p determines uniquely a measure ν such that $p = p_\nu$ (if such a measure ν exists). Thus, by (7), $\mu = m_E$.

(c) follows immediately from (6) and (7). This completes the proof.

Let S satisfy the condition

$$\sum_{s \in S} s^2 < \infty.$$

Let $E \subset (-\infty, \infty) \setminus \{0\}$ be a non-empty countable set such that $T_s^{-1} E \subset E$ for each $s \in S \setminus \{0\}$. We define the measure p_E as

$$p_E = \sum_{x \in E} \delta_x. \tag{8}$$

Clearly, p_E is S-invariant on E. If, moreover,

$$\sum_{x \in E} \frac{x^2}{1+x^2} < \infty,$$

then we define the measure m_E as

$$m_E = A_E \sum_{x \in E} \frac{x^2}{1+x^2} \delta_x, \tag{9}$$

where

$$A_E = \left(\sum_{x \in E} \frac{x^2}{1+x^2} \right)^{-1}.$$

Then it is not difficult to prove that (7) holds.

LEMMA 7. *Let $\mu \in K$ and assume that μ is atomic. Let $E = \{x\colon \mu(\{x\}) > 0\}$. Then conditions* (a), (b), *and* (c) *of Lemma* 6 *are fulfilled.*

The proof is analogous to that of Lemma 6.

Given S, we say that S fulfills *condition* $(*)$ if for every singular continuous measure μ from $e(K)$ there exist a set E of μ-measure 1 and a measure λ such that λ is S-invariant on E and μ is absolutely continuous with respect to λ.

LEMMA 8. *Assume that S fulfills condition $(*)$ and $\mu \in e(K)$. Then there exists a set E of μ-measure 1 such that p_μ is S-invariant on E.*

Proof. If μ is absolutely continuous or atomic, then the lemma follows from Lemmas 6 and 7, respectively. In the remaining cases, by Lemma 3, μ is singular continuous. Since S fulfills condition $(*)$, an application of Lemma 5 completes the proof.

Given S, we say that a probability measure μ on $[-\infty, \infty]$ *belongs to the set* $\mathcal{Q}$ if there exists a set E of μ-measure 1 such that

(a) p_μ is S-invariant on E;

(b) for any sets E_1, E_2 such that $T_s^{-1}E_1 \subset E_1$ μ-a.e., $T_s^{-1}E_2 \subset E_2$ μ-a.e. for each $s \in S \setminus \{0\}$, and $E_1 \cup E_2 = E$ μ-a.e., if $\mu(E_1) > 0$ and $\mu(E_2) > 0$, then $\mu(E_1 \cap E_2) > 0$.

Now we shall prove that if S fulfills condition $(*)$, then the sets $e(K)$ and $\mathcal{Q}$ coincide.

LEMMA 9. $\mathcal{Q} \subset e(K)$.

Proof. Let $\mu \in \mathcal{Q}$ and let E be the set of μ-measure 1 which fulfills conditions (a) and (b) of the definition of $\mathcal{Q}$.

Suppose that there exist μ_1 and μ_2 from K such that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2,$$

where $\mu_1, \mu_2 \in K$ and $0 < \alpha < 1$. Since both p_{μ_1} and p_{μ_2} are absolutely continuous with respect to p_μ, they have Radon-Nikodym derivatives f and g, respectively. Then

$$\int_B 1 dp_\mu = p_\mu(B) = \alpha p_{\mu_1}(B) + (1-\alpha) p_{\mu_2}(B)$$
$$= \alpha \int_B f dp_\mu + (1-\alpha) \int_B g dp_\mu = \int_B (\alpha f + (1-\alpha) g) dp_\mu$$

for all Borel subsets B of $[-\infty, \infty]$. Hence

$$\alpha f + (1-\alpha) g = 1 \quad \mu\text{-a.e.} \tag{10}$$

By Lemma 4 we have

$$f \circ T_s^{-1} \geqslant f \quad \text{and} \quad g \circ T_s^{-1} \geqslant g \ \ \mu\text{-a.e.} \quad \text{for each } s \in S \setminus \{0\}. \tag{11}$$

Setting $A = \{x\colon g(x) > 1\}$ and $C = \{x\colon f(x) > 1\}$, by (11) we get

$$T_s^{-1}A \subset A \text{ and } T_s^{-1}C \subset C \ \ \mu\text{-a.e.} \quad \text{for each } s \in S \setminus \{0\}. \tag{12}$$

Put $D = \{x\colon f(x) = 1\}$. By (10), $D = \{x\colon g(x) = 1\}$. Clearly, the sets A, C, and D are mutually disjoint. By (11), we have

$$T_s^{-1}D \subset (C \cup D) \cap (A \cup D) \ \ \mu\text{-a.e.} \quad \text{for each } s \in S \setminus \{0\}.$$

Taking into account the equality $(C \cup D) \cap (A \cup D) = D$ we obtain

(13) $$T_s^{-1} D \subset D \ \ \mu\text{-a.e.} \quad \text{for each } s \in S \setminus \{0\}.$$

Since μ_1 and μ_2 are probability measures, we get

(14) $$\mu(A) > 0 \quad \text{if and only if} \quad \mu(C) > 0.$$

Setting $E_1 = E \cap A$ and $E_2 = E \cap (C \cup D)$, by (12) and (13) we have $T_s^{-1} E_1 \subset E_1$ and $T_s^{-1} E_2 \subset E_2$ μ-a.e. for each $s \in S \setminus \{0\}$.

Suppose that $\mu(A) > 0$. Then, by (14), $\mu(C) > 0$. Consequently, $\mu(E_1) > 0$ and $\mu(E_2) > 0$. Since $E_1 \cap E_2 \subset A \cap (C \cup D)$ and $A \cap (C \cup D) = \emptyset$, we have $\mu(E_1 \cap E_2) = 0$. This contradicts the assumption that $\mu \in \mathcal{Q}$.

Thus $\mu(A) = 0$, and taking into account (14) we get $\mu(C) = 0$. Hence $f = g = 1$ μ-a.e. Consequently, $\mu_1 = \mu_2 = \mu$, which completes the proof.

LEMMA 10. *Let S fulfill condition* $(*)$. *Then* $e(K) \subset \mathcal{Q}$.

Proof. Let $\mu \in e(K)$. By Lemma 8 there exists a set E of μ-measure 1 such that p_μ is S-invariant on E. Suppose that there exist two sets E_1 and E_2 such that $T_s^{-1} E_1 \subset E_1$ and $T_s^{-1} E_2 \subset E_2$ μ-a.e. for each $s \in S \setminus \{0\}$, $E_1 \cup E_2 = E$ μ-a.e., $\mu(E_1) > 0$, $\mu(E_2) > 0$, and $\mu(E_1 \cap E_2) = 0$. Then

$$\mu = \alpha \mu_1 + (1-\alpha) \mu_2,$$

where $\alpha = \mu(E_1)$, $\mu_1(B) = \alpha^{-1} \mu(B \cap E_1)$, $\mu_2(B) = (1-\alpha)^{-1} \mu(B \cap E_2)$. Since $\mu_1, \mu_2 \in K$, $\mu_1 \neq \mu_2$, and $0 < \alpha < 1$, this contradicts the assumption that $\mu \in e(K)$. Thus, if $\mu \in e(K)$, then $\mu \in \mathcal{Q}$. This completes the proof.

Now, we are ready to prove the representation of the characteristic functions of distributions from the classes L_S^{id} for which S fulfills condition $(*)$.

THEOREM 2. *Let S fulfill condition* $(*)$. *An infinitely divisible measure P belongs to the class L_S^{id} if and only if its characteristic function $\hat{P}$ has the representation*

(15) $$\hat{P}(t) = \exp\left\{ibt + \int_{\mathcal{Q}_0} \left[\int_{-\infty}^{\infty} \left(e^{ity} - 1 - \frac{ity}{1+y^2}\right) \frac{1+y^2}{y^2}\, \mu(dy)\right] \nu(du)\right\},$$

where b is a real constant, $\mathcal{Q}_0$ is the set of all probability measures from $\mathcal{Q}$ concentrated on $(-\infty, \infty)$, and ν is a finite Borel measure on $\mathcal{Q}_0$. Moreover, the function $\hat{P}$ determines b and ν uniquely.

Proof. By Lemmas 9 and 10, $e(K) = \mathcal{Q}$. Now, we can apply Choquet's theorem on representation of the points of a compact convex set as barycenters of the extreme points ([4], p. 19). Consequently, for every measure $\tau \in K$ there exists a probability measure λ on $\mathcal{Q}$ such that for all continuous functions on $[-\infty, \infty]$ we have

(16) $$\int_{[-\infty,\infty]} f(y)\,\tau(dy) = \int_{\mathcal{Q}} \Big(\int_{[-\infty,\infty]} f(y)\,\mu(dy)\Big)\,\lambda(d\mu).$$

Moreover, the measure τ assigns zero mass to the set $\{-\infty, \infty\}$ if and only if λ has zero mass at every μ from $\mathcal{Q}$ which is not concentrated on $(-\infty, \infty)$. Further, formula (16) holds for all bounded continuous functions on $(-\infty, \infty)$ whenever $\tau \in K_0$. Hence we get the following statement: $\tau \in M_0$ if and only if there exists a finite Borel measure ν on $\mathcal{Q}_0$ such that

$$\int_{-\infty}^{\infty} f(y)\tau(dy) = \int_{\mathcal{Q}_0} \Big(\int_{-\infty}^{\infty} f(y)\mu(dy)\Big)\nu(d\mu) \tag{17}$$

for all continuous bounded functions f on $(-\infty, \infty)$. Setting

$$f_t(y) = \left(e^{ity} - 1 - \frac{ity}{1+y^2}\right)\frac{1+y^2}{y^2}$$

into (17), we obtain the formula

$$\int_{-\infty}^{\infty} f_t(y)\tau(dy) = \int_{\mathcal{Q}_0}\left[\int_{-\infty}^{\infty}\left(e^{ity} - 1 - \frac{ity}{1+y^2}\right)\frac{1+y^2}{y^2}\,\mu(dy)\right]\nu(d\mu),$$

which implies representation (15).

Since K is a simplex (see Theorem 1), from Choquet's uniqueness theorem for a metrizable space X we infer that ν is determined uniquely ([4], p. 70). Hence b is also determined uniquely. This completes the proof.

3. Using Theorem 2 we give the representation of the characteristic functions of distributions from the classes L_S^{id} in two cases: for $m(S) > 0$ and in the special case of a discrete semigroup S.

The following lemma implies that if $m(S) > 0$, then S fulfills condition $(*)$.

LEMMA 11. *Let P be an infinitely divisible measure. If the decomposability semigroup $D^{\mathrm{id}}(P)$ has the positive Lebesgue measure, then the Khintchine measure corresponding to P is except on $\{0\}$ absolutely continuous with respect to the Lebesgue measure.*

Proof. Given a finite Borel measure λ on R, we put

$$\bar{p}_\lambda(B) = p_\lambda(\exp\{B\})$$

for all Borel subsets B of R. Then, by (1), we have the inequality

$$\bar{p}_\lambda(B+\log s) \geqslant \bar{p}_\lambda(B) \tag{18}$$

for all Borel subsets B of R and for each positive number s from a decomposability $D^{\mathrm{id}}(Q)$ of an infinitely divisible measure Q for which λ is a Khintchine measure.

Let μ be a Khintchine measure corresponding to P. Clearly, it suffices to prove the lemma in the case of μ concentrated on $(-\infty, 0)\cup(0, \infty)$. Then, by Theorem 1.2 in [3], p_μ is non-atomic. Suppose that

$$p_\mu = \alpha p_\tau + (1-\alpha) p_\nu, \tag{19}$$

where $0<\alpha<1$, p_τ is absolutely continuous, and p_ν is singular continuous. Since $\alpha<1$, there exists $x_0 \neq 0$ such that

$$\lim_{n\to\infty} \sup_{0<h<1/n} \frac{p_\nu((x_0, x_0+h))}{h} = \infty. \tag{20}$$

Without loss of generality we may assume that $x_0>0$. Then

$$\lim_{n\to\infty} \sup_{0<h<1/n} \frac{\bar{p}_\nu((\log x_0, \log x_0+h))}{h} = \infty. \tag{21}$$

Since $\bar{p}_\nu$ is singular continuous and finite on bounded sets, we have

$$\lim_{n\to\infty} \sup_{0<h<1/n} \frac{\bar{p}_\nu((\log x, \log x+h))}{h} = 0 \quad m\text{-a.e.} \tag{22}$$

If $s\in D^{\mathrm{id}}(P)\cap(0, 1)$, then by (18) we obtain

$$\bar{p}_\nu((\log x_0+\log s, \log x_0+h+\log s)) \geqslant \bar{p}_\nu((\log x_0, \log x_0+h)). \tag{23}$$

Then by (21) and (23) we get

$$\lim_{n\to\infty} \sup_{0<h<1/n} \frac{\bar{p}_\nu((\log x, \log x+h))}{h} = \infty \tag{24}$$

for $x\in x_0(D^{\mathrm{id}}(P)\cap(0, 1))$. Since $m(x_0(D^{\mathrm{id}}(P)\cap(0, 1)))>0$, equality (24) contradicts (22). Thus $\alpha=1$. This completes the proof.

From Lemma 11 it follows that in the case of $m(S)>0$ the set of extreme points of K which are singular continuous measures is empty. Thus condition $(*)$ is fulfilled. In this case $\mathcal{Q}_0$ consists of δ_0 and of all probability measures m_E (defined by (6)), where $E\in X$ and X is the set of all Borel subsets E of $(-\infty, \infty)$ satisfying the following conditions:

(25a) $0<\int_E |y|(1+y^2)^{-1}dy<\infty$;

(25b) $T_s^{-1}E\subset E$ m-a.e. for each $s\in S\setminus\{0\}$;

(25c) for all sets E_1, E_2 such that $T_s^{-1}E_1\subset E_1$, $T_s^{-1}E_2\subset E_2$ m-a.e. for each $s\in S\setminus\{0\}$, and $E_1\cup E_2=E$ m-a.e., if $m(E_1)>0$ and $m(E_2)>0$, then $m(E_1\cap E_2)>0$.

Since $X\cup\{0\}$ is homeomorphic to $\mathcal{Q}_0$ up to m-null sets ($E\to m_E$, $0\to\delta_0$), we obtain the following theorem as a corollary to Theorem 2:

THEOREM 3. *Let $m(S) > 0$. An infinitely divisible measure P belongs to the class L_S^{id} if and only if its characteristic function $\hat{P}$ has the representation*

$$\hat{P}(t) = \exp\left\{ibt - \frac{Gt^2}{2} + \int\limits_X \left(\int\limits_Z \frac{|y|}{1+y^2}\, dy\right)^{-1} \int\limits_Z \left(e^{ity} - 1 - \frac{ity}{1+y^2}\, dy\right) \nu(dZ)\right\},$$

where b is a real constant, G is a non-negative real constant, X consists (up to m-null sets) of all Borel subsets B of R which satisfy conditions (25), and ν is a finite Borel measure on X. Moreover, the function $\hat{P}$ determines b, G, and ν uniquely.

Setting $S = [0, 1]$ in Theorem 3, we obtain as a corollary the Urbanik theorem ([5], p. 209).

THEOREM 4. *P is a self-decomposable distribution (i.e., $[0, 1] \subset D^{\mathrm{id}}(P)$) if and only if its characteristic function $\hat{P}$ has the representation*

$$\hat{P}(t) = \exp\left\{ibt + \int\limits_{-\infty}^{\infty} \left(\int\limits_0^{tu} \frac{e^{iv}-1}{v}\, dv - it \arctan u\right) \frac{1}{\log(1+u^2)}\, \nu(du)\right\},$$

where b is a real constant, ν is a finite Borel measure on R, and the integrand is defined as $-\frac{1}{4}t^2$ when $u = 0$.

In fact,

$$X = \{[x, 0]:\ x \in (-\infty, 0)\} \cup \{[0, x]:\ x \in (0, \infty)\}$$

up to m-null sets and the mapping $[x, 0] \to x$, $[0, x] \to x$ is a homeomorphism between X and $(-\infty, 0) \cup (0, \infty)$.

Setting $S = [q, 1]$, where $-1 \leqslant q < 0$, in Theorem 3 we obtain

COROLLARY 1. *Let $S = [q, 1]$, where $-1 \leqslant q < 0$. An infinitely divisible measure P belongs to the class L_S^{id} if and only if its characteristic function $\hat{P}$ has the representation*

$$\hat{P}(t) = \exp\left\{ibt + \int\limits_0^{\infty} \int\limits_{u/q}^{qu} \left(\int\limits_v^u \frac{e^{ity}-1}{|y|}\, dy - it \arctan u - it \arctan v\right) \times \right.$$
$$\left. \times \frac{1}{\log(1+u^2)(1+v^2)}\, \nu(du, dv)\right\},$$

where b is a real constant and ν is a finite Borel measure on the set $\{(u, v):\ 0 \leqslant u < \infty,\ u/q \leqslant v \leqslant qu\}$. Moreover, the function $\hat{P}$ determines b and ν uniquely.

Now we give the representation of the characteristic functions of distributions from the classes L_S^{id} if S is a semigroup of the form

(26) $$\{(-1)^{n_j} s_0^{k_j}\}_{j=0}^{\infty} \cup \{0\},$$

where $0 < s_0 < 1$, k_j is an increasing sequence of positive integers, $k_0 = 0$, and $n_j \in \{1, 2\}$. Without loss of generality we may assume that for sufficiently large j either $k_{j+1} = k_j$ or $k_{j+1} = k_j + 1$.

THEOREM 5. *Let S be a semigroup of the form* (26). *An infinitely divisible measure P belongs to the class L_S^{id} if and only if its characteristic function $\hat{P}$ has the representation*

$$\hat{P}(t) = \exp\left\{ibt - \frac{Gt^2}{2} + \right.$$
$$\left. + \int\limits_{Y\times\{1,\ldots,n\}} \left\{\left(\sum_{s\in S_k} \frac{s^2u^2}{1+s^2u^2}\right)^{-1} \sum_{s\in S_k} \left[\left(e^{itsu} - 1 - \frac{itsu}{1+s^2u^2}\right)\frac{1+s^2u^2}{s^2u^2}\right]\right\} \nu(d(u, k))\right\},$$

where b is a real constant, G is a non-negative real constant, $Y = (0, \infty)$ if $-1 \in S$ and $Y = R\setminus\{0\}$ if $-1 \notin S$, n is an integer greater than or equal to 1, ν is a finite Borel measure on $Y \times \{1, \ldots, n\}$, and the sequence $\{S_k\}_{k=1}^{\infty}$ such that $S \subset S_k$ and $T_s^{-1} S_k \subset S_k$ for each $s \in S\setminus\{0\}$ is defined in the following way:

(i) *if there exists a sequence of positive integers $\{m_j\}$ such that $S = \{s_1^{m_j}\}_{j=0}^{\infty} \cup \{0\}$, then $\{S_k\}_{k=1}^{\infty}$ consists of all the sets of the form $\{s_1^{l_j}\}_{j=0}^{\infty} \cup \{0\}$, where $s_1 = s_0$ or $s_1 = -s_0$, and $\{l_j\}$ is a sequence of positive integers;*

(ii) *in the remaining cases, $\{S_k\}_{k=1}^{\infty}$ consists of all the sets of the form $\{(-1)^{r_j} s_0^{l_j}\}_{j=0}^{\infty} \cup \{0\}$, where $\{l_j\}$ is a sequence of positive integers.*

Moreover, the function $\hat{P}$ determines b, G, and ν uniquely.

Proof. Let $\mu \in e(K)$. If μ is concentrated on $\{0\}$, then $\mu = \delta_0$. If μ is concentrated on $\{-\infty, \infty\}$ and $S \cap [-1, 0) = \emptyset$, then $\mu = \delta_{-\infty}$ or $\mu = \delta_{\infty}$. If μ is concentrated on $\{-\infty, \infty\}$ and $S \cap [-1, 0) \neq \emptyset$, then $\mu = \frac{1}{2}\delta_{-\infty} + \frac{1}{2}\delta_{\infty}$. In the remaining cases, by Lemma 2, μ is concentrated on $(-\infty, 0) \cup (0, \infty)$.

Let us consider the case $S = \{(-s_0)^{m_j}\}_{j=0}^{\infty} \cup \{0\}$. Put

$$A(u, v) = \bigcup_{k=-\infty}^{\infty} \left(v(-s_0)^{2k+2}, u(-s_0)^{2k}\right] \cup \bigcup_{k=-\infty}^{\infty} \left[u(-s_0)^{2k+1}, v(-s_0)^{2k+3}\right),$$

where $v > 0$ and $u \in (vs_0^2, v)$.

Suppose that there exist real numbers $v_0 > 0$ and $u_0 \in (v_0 s_0^2, v_0)$ such that $0 < \mu(A(v_0, v_0)) < 1$. Put

$$c = \mu(A(u_0, v_0)), \quad \mu_1 = c^{-1}\mu| A(u_0, v_0), \quad \mu_2 = (1-c)^{-1}(\mu - c\mu_1).$$

It is clear that $\mu_1 \neq \mu_2$, $\mu_1, \mu_2 \in K$, and $\mu = c\mu_1 + (1-c)\mu_2$, which contradicts the assumption that $\mu \in e(K)$.

Hence for every positive number v and for every number $u \in (vs_0^2, v)$ we have either $A(u, v) = 0$ or $A(u, v) = 1$. Consequently, μ is atomic and there exists a real number $x \neq 0$ such that

$$\mu(\{x(-s_0)^k\}_{k=-\infty}^{\infty}) = 1.$$

Since for sufficiently large j we have $k_{j+1} = k_j + 1$, Lemma 7 and Lemma 6 (b) imply that there exists an integer j_1 such that

$$\sup_{-\infty < k < \infty} \{|x(-s_0)^k|: \mu(\{x(-s_0)^k\}) > 0\} = |x(-s_0)^{j_1}|$$

and $\mu = m_{w_1 s_{i_1}}$, where $w_1 = x(-s_0)^{j_1}$, $1 \leqslant i_1 \leqslant n$.

In the case $S = \{s_0^{m_j}\}_{j=0}^{\infty} \cup \{0\}$, we put

$$B(u, v) = \bigcup_{k=-\infty}^{\infty} (vs_0^{k+1}, us_0^k] \quad \text{for } v > 0,\ u \in (vs_0, v)$$

and

$$C(u, v) = \bigcup_{k=-\infty}^{\infty} [us_0^k, vs_0^{k+1}) \quad \text{for } v < 0,\ u \in (vs_0, v).$$

Then, either for every $v > 0$ and for every $u \in (vs_0, v)$

$$\mu(B(u, v)) = 0 \quad \text{or} \quad \mu(B(u, v)) = 1$$

or for every $v < 0$ and for every $u \in (v, vs_0)$

$$\mu(C(u, v)) = 0 \quad \text{or} \quad \mu(C(u, v)) = 1$$

(because either $\mu((0, \infty)) = 1$ or $\mu((-\infty, 0)) > 1$).

Hence there exists a real number $y \neq 0$ such that $\mu(\{ys_0^k\}_{k=-\infty}^{\infty}) = 1$. By Lemma 7 and Lemma 6 (b) there exist $w_2 \neq 0$ and $i_2 \in \{1, \ldots, n\}$ such that $\mu = m_{w_2 s_{i_2}}$.

Assume now that S is not of the form (i). Put

$$D(u, v) = \bigcup_{k=-\infty}^{\infty} (vs_0^{k+1}, us_0^k] \cup \bigcup_{k=-\infty}^{\infty} [-us_0^k, -vs_0^{k+1}) \quad \text{for } v > 0,\ u \in (vs_0, v).$$

Then for every $v > 0$ and for every $u \in (us_0, v)$ either $D(u, v) = 0$ or $D(u, v) = 1$. Hence there exists $z \neq 0$ such that $\mu(\pm\{zs_0^k\}_{k=-\infty}^{\infty}) = 1$. By Lemma 7 and Lemma 6 (b) there exist $w_3 \neq 0$ and $i_3 \in \{1, \ldots, n\}$ such that $\mu = m_{w_3 s_{i_3}}$.

Further, we note that if $-1 \in S$, then $m_{-ws_i} = m_{ws_i}$, and if $-1 \notin S$, then $m_{-ws_i} \neq m_{ws_i}$. Thus the mapping $m_{(w,s_i)} \to (w, i)$ is a homeomorphism between the extreme points of K which are measures concentrated on $(-\infty, 0) \cup (0, \infty)$ and $Y \times \{1, \ldots, n\}$. Therefore, Theorem 5 follows from Theorem 3.

Setting $S = \{s^k\}_{k=0}^{\infty} \cup \{0\}$, where $0 < |s| < 1$, in Theorem 5 we obtain

COROLLARY 2. *Let P be an infinitely divisible measure and $0 < |s| < 1$. Then $s \in D^{id}(P)$ if and only if the characteristic function $\hat{P}$ of P has the representation*

$$\hat{P}(t) = \exp\left\{ibt + \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{s^{2k}u^2}{1+s^{2k}u^2}\right)^{-1} \sum_{k=0}^{\infty} \left[\left(e^{its^k u} - 1 - \frac{its^k u}{1+s^{2k}u^2}\right) \frac{1+s^{2k}u^2}{s^{2k}u^2}\right] \nu(du)\right\}$$

where b is a real constant and ν is a finite Borel measure on R. Moreover, the function $\hat{P}$ determines b and ν uniquely.

4. In Lemmas 6 and 7 we considered S-invariant measures which were either absolutely continuous or atomic. Now we prove the lemma on S-invariant singular continuous measures.

Let $m(S) = 0$. Given $0 < \varepsilon < 1$ and a Borel subset B of R, we put

$$B_\varepsilon = \bigcup_{x \in B \cap [0,\infty)} [x\varepsilon, x\varepsilon^{-1}] \cup \bigcup_{x \in B \cap (-\infty, 0)} [x\varepsilon^{-1}, x\varepsilon].$$

Let E be a perfect and nowhere dense subset of R such that $T_s^{-1}E \subset E$ for each $s \in S \setminus \{0\}$ and $E \subset (-a, a)$ for some $a > 0$. Note that $T_s^{-1}E_\varepsilon \subset E_\varepsilon$ for each $s \in S \setminus \{0\}$. Choose $\varepsilon_n \uparrow 1$. Since the set of probability measures on $[-\infty, \infty]$ is compact, without loss of generality we may assume that $\{m_{E_{\varepsilon_n}}\}$ is convergent as $\varepsilon_n \uparrow 1$ (by passing to a subsequence if necessary). Let m_E be the limit of this sequence.

LEMMA 12. *Let $m(S) = 0$, $0 < \varepsilon_n < \varepsilon_{n+1} < 1$, and $\varepsilon_n \to 1$. Let E be a perfect and nowhere dense m-null subset of R such that $T_s^{-1}E \subset E$ for each $s \in S \setminus \{0\}$ and $E \subset [-1, 1]$. Suppose that m_E is the limit of $\{m_{E_{\varepsilon_n}}\}$ as $n \to \infty$. Then $m_E(E) = 1$ and p_{m_E} is S-invariant on E.*

Proof. Clearly, p_{m_E} is non-atomic. Let $\mathcal{N}$ be the set of all Borel subsets of E. Put

$$\mathcal{G} = \{[a, b] \cap E\colon a < b,\ 0 \notin [a, b]\}.$$

Then $\mathcal{C}_\sigma(\mathcal{G}) = \mathcal{N}$. Put $E_n = E_{\varepsilon_n}$, $p = p_{m_E}$, and $p_n = p_{m_{E_n}}$. Let

$$\mathcal{K} = \{B\colon B \subset E \text{ and } T_s p(B) = p(B) \text{ for each } s \in S \setminus \{0\}\}.$$

We prove that $\mathcal{G} \subset \mathcal{K}$. Choose and fix positive numbers $a < b$ and ε_k. Put $C = [a, b] \cap E$ and $C_n = C_{\varepsilon_n}$. Then

$$\limsup_{n\to\infty} p_n\big([a\varepsilon_k, a) \cup (b, b\varepsilon_k^{-1}] \cap E_k\big) = \limsup_{n\to\infty} p_n([a\varepsilon_k, a] \cup [b, b\varepsilon_k^{-1}])$$

$$\leqslant p([a\varepsilon_k, a] \cup [b, b\varepsilon_k^{-1}]) = p\big([a\varepsilon_k, a) \cup (b, b\varepsilon_k^{-1}]\big).$$

Since $[a\varepsilon_k, a) \cup (b, b\varepsilon_k^{-1}] \downarrow \emptyset$ as $k \to \infty$ and $p\big([a\varepsilon_k, a) \cup (b, b\varepsilon_k^{-1}]\big) < \infty$ (because of $0 \notin [a\varepsilon_k, b\varepsilon_k^{-1}]$), we have

$$p\big([a\varepsilon_k, a) \cup (b, b\varepsilon_k^{-1}]\big) \to 0 \quad \text{as } k \to \infty.$$

Hence

(27) $$\lim_{k\to\infty}\limsup_{n\to\infty} p_n\big(([a\varepsilon_k, a)\cup(b, b\varepsilon_k^{-1}])\cap E_k\big) = 0.$$

A similar argument shows that

(28) $$\lim_{k\to\infty}\limsup_{n\to\infty} T_s p_n\big(([a\varepsilon_k, a)\cup(b, b\varepsilon_k^{-1}])\cap E_k\big) = 0.$$

Taking into account (27) and (28) we obtain

$$\begin{aligned}
p(C) &= p\Big(\bigcap_{k\to\infty} C_k\Big) = \lim_{k\to\infty} p(C_k) \geqslant \lim_{k\to\infty}\limsup_{n\to\infty} p_n(C_k) = \lim_{k\to\infty}\lim_{n_j\to\infty} p_{n_j}(C_k)\\
&= \lim_{k\to\infty}\lim_{n_j\to\infty} p_{n_j}(E_k\cap[a, b])\\
&\quad - \lim_{k\to\infty}\lim_{n_j\to\infty} p_{n_j}(T_s^{-1}E_k\cap T_s^{-1}[a, b])\\
&\quad - \lim_{k\to\infty}\lim_{n_j\to\infty} p_{n_j}(T_s^{-1}E_k\cap T_s^{-1}[a, b])\\
&= \lim_{k\to\infty}\Big(\lim_{n_j\to\infty} p_{n_j}(T_s^{-1}E_k\cap T_s^{-1}[a, b]) +\\
&\qquad + \limsup_{n_j\to\infty} p_{n_j}\big(T_s^{-1}(E_k\cap[a\varepsilon_k, a)\cup E_k\cap(b, b\varepsilon_k^{-1}])\big)\Big)\\
&\geqslant \lim_{k\to\infty}\limsup_{n_j\to\infty} p_{n_j}(T_s^{-1}C_k)\\
&\geqslant \lim_{k\to\infty}\liminf_{n_j\to\infty} p_{n_j}(T_s^{-1}C_k) \geqslant \lim_{k\to\infty}\liminf_{n_j\to\infty} p_{n_j}(\operatorname{int} T_s^{-1}C_k)\\
&\geqslant \lim_{k\to\infty} p(\operatorname{int} T_s^{-1}C_k) = p\Big(\bigcap_{k\to\infty} \operatorname{int} T_s^{-1}C_k\Big) = p(T_s^{-1}C).
\end{aligned}$$

Thus $p(C)\geqslant p(T_s^{-1}C)$. Repeating this procedure we prove that $p(T_s^{-1}C)\geqslant p(C)$. Thus $p(C) = p(T_s^{-1}C)$ for $C\in\mathscr{G}$.

It is not difficult to verify that the class $\mathscr{K}$ is closed under finite disjoint unions, and if $A_1, A_2, \ldots\in\mathscr{K}$, $A_1\subset A_2\subset\ldots$, then $\bigcup_n A_n\in\mathscr{K}$. Clearly, the class $\mathscr{G}$ is closed under intersections. Taking into account that, for Borel subsets B of R,

$$p(B) = \lim_{n\to\infty} p\big(B\setminus(-\varepsilon_n, \varepsilon_n)\big) \quad\text{and}\quad p\big(B\setminus(-\varepsilon_n, \varepsilon_n)\big) < \infty,$$

it not difficult to prove that if $A_1, A_2\in\mathscr{K}$ and $A_1\subset A_2$, then $A_2\setminus A_1\in\mathscr{K}$. Since $\mathscr{G}$ is closed under intersections, we have $\mathscr{C}_\sigma(\mathscr{G}) = \mathscr{K}$.

For the rest of the proof we see that

$$m_E(E) = m_E\Big(\bigcap_{n\to\infty} E_n\Big) = \lim_{n\to\infty} m_E(E_n) \geqslant \lim_{n\to\infty}\lim_{k\to\infty} m_{E_k}(E_n) = 1$$

(because if $\varepsilon_k > \varepsilon_n$, then $m_{E_k}(E_n) = 1$), which completes the proof.

In [3] (Theorem 1.2) we proved that for every infinitely divisible measure

P there exists Q with atomic Khintchine measure for which $D^{id}(P) = D^{id}(Q)$ if and only if

$$\sum_{s \in D^{id}(P)} s^2 < \infty .$$

Further, for every infinitely divisible measure P there exists Q with absolutely continuous Khintchine measure such that $D^{id}(P) = D^{id}(Q)$ ([3], Theorem 1.3). If $m(D^{id}(P)) > 0$, then the Khintchine measure of P is except on $\{0\}$ absolutely continuous (see Lemma 11). From the next theorem it follows (setting $S = D^{id}(P)$) that if $m(D^{id}(P)) = 0$, then there exists Q with singular continuous Khintchine measure such that $D^{id}(Q) \supset D^{id}(P)$.

THEOREM 6. *Let $m(S) = 0$. Then there exists an infinitely divisible measure Q such that the Khintchine measure corresponding to Q is singular continuous and $D^{id}(Q) \supset S$.*

Proof. Clearly, there exists a perfect and nowhere dense m-null set E such that $T_s^{-1}E \subset E$ for each $s \in S \setminus \{0\}$ and $E \subset [-1, 1]$. By Lemma 12 there exists a probability measure m_E on R such that $m_E(E) = 1$ and p_{m_E} is S-invariant on E. From the proof of Lemma 12 it follows that m_E is singular continuous.

Let B be a Borel subset of R and $s \in S \setminus \{0\}$. Then

$$p_{m_E}(T_s^{-1}B) \geqslant p_{m_E}(T_s^{-1}(B \cap E)) = p_{m_E}(B \cap E) = p_{m_E}(B).$$

Hence it follows that if Q is an infinitely divisible measure for which the corresponding Khintchine measure is equal to m_E, then $D^{id}(Q) \supset S$. This completes the proof.

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