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SPECTRAL REPRESENTATION OF PERIODICALLY CORRELATED SEQUENCES*

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Abstract. There are two platforms for analyzing stochastic processes: *time domain* and *spectral domain*. For periodically correlated processes both of these analyses have been discussed through invoking their close tie with multivariate stationary processes. In this note we present a direct approach to the spectral properties of periodically correlated processes.

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1. INTRODUCTION AND NOTATION

A second order stochastic process is a sequence of complex random variables with mean zero and finite second moments. Since we are interested only in correlation properties of a stochastic sequence, we adopt a slightly more general definition, however, we will keep labeling them *stochastic* to point to its origin.

DEFINITION 1.1. Let $\mathcal{H}$ be a complex Hilbert space with an inner product $(\cdot,\cdot)_{\mathcal{H}}$. A *stochastic sequence* is a sequence $\big(X(n)\big)$ of elements of $\mathcal{H}$ indexed by the set of all integers $\mathcal{Z}$. The *correlation function* of the sequence $\big(X(n)\big)$ is the function on $\boldsymbol{Z}^2$ defined by $R_X(m,n) = \big(X(m), X(n)\big)_{\mathcal{H}}$. Two stochastic sequences, $\big(X(n)\big)$ in $\mathcal{H}$ and $\big(Y(n)\big)$ in a possibly different Hilbert space $\mathcal{K}$, are said to be *equivalent* if $R_X(m,n) = R_Y(m,n)$ for every $m, n \in \mathcal{Z}$.

Historically, there are two platforms of analysis for stochastic sequences: *time domain* and *spectral domain*. Time domain analysis deals with studying properties of a sequence through its geometry, while spectral domain analysis is the study of properties of a sequence through properties of its spectrum. The milestone of the spectral analysis is to represent the process as a treatable family of functions in

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some function space related to the spectrum of the process. The notion of a spectrum stems from the fact that every bounded function $R(m,n)$ on $\mathcal{Z}^2$ is, in some sense, a mixture of fundamental harmonics $e^{-i(mu-nv)}$, $m,n\in\mathcal{Z}$. Intuitively, the spectrum of $\big(X(n)\big)$ at frequency $(u,v)\in[0,2\pi)^2$ represents the degree that the component $e^{-i(mu-nv)}$ contributes to the function $R_X(m,n)$. Mathematically, the spectrum of $\big(X(n)\big)$ is an "object" (e.g. function, measure, or Schwartz distribution in general) on $[0,2\pi)^2$ whose Fourier transform is $R_X(m,-n)$. A stochastic sequence $\big(X(n)\big)$ is *strongly harmonizable* if this "object" is a measure, that is, if there is a measure Γ such that

$$(1.1)\qquad R_X(m,n)=\int_0^{2\pi}\int_0^{2\pi} e^{-i(mu-nv)}\Gamma(du,dv),\quad m,n\in\mathcal{Z}.$$

The time domain analysis of a harmonizable sequence aims at deriving properties of the sequence from the properties of the measure Γ.

Both stationary and periodically correlated sequences are strongly harmonizable. The case of stationary $\big(X(n)\big)$ is especially pleasant because its spectral measure Γ is supported on the diagonal of the square $[0,2\pi)^2$. In this case the measure $F(\Delta)=\Gamma(\Delta\times\Delta)$ is referred to as the *spectral measure of the stationary sequence* $\big(X(n)\big)$. A basic tool for spectral analysis of stationary sequences is an observation that if $\big(X(n)\big)$ is stationary and its spectral measure has a factorization $F(du)=|h(u)|^2\mu(du)$, then $\big(X(n)\big)$ can be viewed as the trajectory $x(n)=e^{-in\cdot}h(\cdot)$ in $L^2\big([0,2\pi),\mu;\mathcal{C}\big)$ of the function h under the unitary group of multiplication by $e^{-in\cdot}$. This observation opens doors to huge variety of analytic tools available in harmonic analysis.

The main purpose of this paper is to propose an analogous spectral representation for periodically correlated (PC) sequences. This will be proceeded by constructing a simultaneous factorization of all the measures comprising the spectrum of the PC sequence.

PC sequences arise from multivariate stationary sequences by arranging the elements of the latter in one sequence. Because of this relation, one can therefore wonder why we need to study PC sequences separately. In fact, there are ample reasons to do this, the main being that prediction technique of multivariate stationary sequences does not address the question of the space-time prediction, that is, prediction of, say, element $X^k(n)$ given all the past values until moment $n-1$ and the values $X^j(n)$ for $j<k$. For that reason PC theory developed its own technique which we briefly summarize in Section 2.

In this paper $\mathcal{R}$ and $\mathcal{C}$ will denote the sets of real and complex numbers, respectively, $\mathcal{Z}$ and $\mathcal{N}$ will be the sets of integers and positive integers, and $\mathcal{H}$ and $\mathcal{K}$ will denote separable complex Hilbert spaces. By $\mathcal{C}^n$ we understand the space of row vectors $a=[a^1,a^2,\ldots,a^n]$ with complex coordinates. If $a\in\mathcal{C}^n$ then a^* will denote a column vector with components $\overline{a^k}$. If $T\in\mathcal{N}$ is fixed, then for every $m\in\mathcal{Z}$, $\mathrm{q}(m)$ and $\langle m\rangle$ will stand for the quotient and the remainder in division of m by T, $\langle m\rangle \in \{0,\dots,T-1\}$, respectively. With this notation $m = \mathrm{q}(m)T + \langle m\rangle$.

A sequence $\big(p(n)\big)$ in $\mathcal{H}$ is said to be T*-periodic* ($T \in \mathcal{N}$) if $p(n+T) = p(n)$ for every $n \in \mathcal{Z}$. The *discrete Fourier transform* of a T-periodic sequence $\big(p(n)\big)$ is the T-periodic sequence $\big(P(n)\big)$ defined by

$$P(n) = \frac{1}{T}\sum_{k=0}^{T-1} e^{-2\pi ink/T} p(k).$$

Clearly, $p(n) = \sum_{k=0}^{T-1} e^{2\pi ink/T} P(k)$. The latter operation is referred to as the *inverse discrete Fourier transform*.

In this paper we will be dealing with complex functions on $\mathcal{R}$ which are periodic with period 2π. It is convenient to identify them with complex functions on the interval $[0,2\pi)$ regarded as a group with addition modulo 2π. In order to remember this we will often call $[0,2\pi)$ to be a *circle* and, consequently, its subintervals will be called *arcs*. All sets referred in the paper will be assumed Borel, all measures in the paper will be complex Borel measures on the circle $[0,2\pi)$, unless is clearly stated otherwise, and all functions will be assumed Borel measurable. A measure ν on $[0,2\pi)$ is said to be $2\pi/T$*-invariant* if $\nu(\Delta) = \nu(\Delta + 2\pi/T)$ for every Borel subset Δ. Given a measure ν, $L^2(\nu,\mathcal{C}^n)$ will denote the Hilbert space of $\mathcal{C}^n$-valued Borel functions f on $[0,2\pi)$ (or 2π-periodic functions on R) such that

$$\|f\|^2 = \int_0^{2\pi} f(t)f(t)^*\nu(dt) < \infty. \tag{1.2}$$

If $\nu = dt$ is the Lebesgue measure, we simply write $L^2(\mathcal{C}^n)$ instead of $L^2(dt,\mathcal{C}^n)$.

A stochastic sequence was defined at the beginning of this section. Although we will be dealing with only one-variate periodically correlated sequences we need to introduce the notion of multivariate stochastic sequence. A T*-variate stochastic sequence in* $\mathcal{H}$ ($T \in \mathcal{N}$) is a sequence of (column) vectors $\boldsymbol{X}(n) = [X^k(n)]$, $n \in \mathcal{Z}$, of the length T with components in $\mathcal{H}$. The correlation function of $\big(\boldsymbol{X}(n)\big)$ is the $T\times T$-matrix valued function $R(m,n) = [R^{k,j}(m,n)]$ with entries $R^{k,j}(m,n) = \big(X^k(m), X^j(n)\big)_{\mathcal{H}}$, $k,j = 1,\dots,T$, $m,n \in \mathcal{Z}$. A T-variate stochastic sequence $\big(\boldsymbol{X}(n)\big)$ is *stationary* if $R(m,n)$ depends on $n-m$, that is, $R(m,n) = K(m-n)$. In the sequel the phrase *the correlation function of a stationary sequence* will refer to the function $K(n)$ rather than to $R(m,n)$. For every T-variate stationary sequence $\big(\boldsymbol{X}(n)\big)$ there is a unique $T\times T$-matrix valued measure $F(du) = [F^{k,j}(du)]$ on the interval $[0,2\pi)$, called the *spectral measure of the stationary sequence* $\big(\boldsymbol{X}(n)\big)$, such that

$$K^{k,j}(n) = \int_0^{2\pi} e^{-inu}F^{k,j}(du), \qquad k,j = 1,\dots,T,\ n \in \mathcal{Z}. \tag{1.3}$$

More about finite-dimensional stationary sequences can be found in [5].

2. PRIMER ON PC SEQUENCES

Let T be a fixed positive integer and $\mathcal{H}$ be a complex Hilbert space.

DEFINITION 2.1. A stochastic sequence $\big(X(n)\big)$ in $\mathcal{H}$ is called *periodically correlated with period* T (or T-PC) if $R_X(m,n) = R_X(m+T, n+T)$ for all $m, n \in \mathcal{Z}$ or, equivalently, if for every $n \in \mathcal{Z}$ the lag n correlation function $R_X(n+r, r)$ is T-periodic in r.

Write

$$a_j(n) = \frac{1}{T}\sum_{r=0}^{T-1} e^{-2\pi ijr/T} R_X(n+r,r). \tag{2.1}$$

For a fixed n, $a_j(n)$, $j \in \mathcal{Z}$, is the discrete Fourier transform of a periodic sequence $R_X(n+r,r)$, $r \in \mathcal{Z}$, and hence $a_j(n)$ is T-periodic in j and $R(n+r,r) = \sum_{j=0}^{T-1} e^{2\pi ijr/T} a_j(n)$.

There is an obvious relation between T-PC and T-variate stationary sequences.

LEMMA 2.1. *If* $\big(X(n)\big)$ *in* $\mathcal{H}$ *is* T*-PC, then the* T*-variate sequence*

$$\boldsymbol{X}(n) = [X^k(n)], \quad \textit{where } X^k(n) = X(nT+k-1),\ k = 1, \dots, T, \tag{2.2}$$

is stationary. Conversely, if $\boldsymbol{X}(m) = [X^k(m)]$ *is a* T*-variate stationary sequence in* $\mathcal{H}$*, then the sequence* $X(m) = X^{\langle m\rangle+1}\big(\mathrm{q}(m)\big)$, $m \in \mathcal{Z}$, *is periodically correlated with period* T.

Given a T-PC sequence $\big(X(n)\big)$, the T-variate stationary sequence $\boldsymbol{X}(m) = [X^k(m)]$ defined in (2.2) will be called its *associated block sequence*.

An analysis of PC sequences is based on the observation that a PC sequence is a unitary deformation of a periodic sequence. Namely, for every T-PC sequence $\big(X(m)\big)$ in $\mathcal{H}$ one can find a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a unitary operator V in $\mathcal{K}$, and a T-periodic function $\big(p(m)\big)$ in $\mathcal{K}$ such that

$$X(m) = V^m p(m), \quad m \in \mathcal{Z}. \tag{2.3}$$

If we define $Z^{j+1}(m) = V^m P(j)$, $j = 0, \dots, T-1$, $m \in \mathcal{Z}$, where $\big(P(j)\big)$ is the discrete Fourier transform of $\big(p(k)\big)$, then $\boldsymbol{Z}(m) = [Z^k(m)]$ is a T-variate stationary sequence and

$$X(m) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijm/T} Z^{j+1}(m). \tag{2.4}$$

Although $\big(\boldsymbol{Z}(m)\big)$ depends on the choice of V and $\mathcal{K}$ (which can be chosen in many different ways), the representation (2.4) turns out to be a very useful tool in analysis of PC sequences. In particular, it yields the following fundamental property of PC sequences.

THEOREM 2.1 (Gladyshev). *Let $\big(X(m)\big)$ be a PC sequence with period T. Then $\big(X(m)\big)$ is strongly harmonizable and its spectrum Γ sits on the union $L = \bigcup_{j=0}^{T-1} L_j$ of T lines*

$$L_j = \{(u,v) \in [0,2\pi)^2 : v = u + 2\pi j/T\}, \quad j = 0,\ldots,T-1,$$

parallel to the main diagonal of the torus $[0,2\pi)^2$.

Remember that the addition in $[0,2\pi)$ is modulo 2π. If now we define $\gamma_j(\Delta) = \Gamma_j(\Delta \times [\Delta + 2\pi j/T])$, where Γ_j is the restriction of Γ to the line L_j, then it is easy to see that

$$a_j(n) = \int_0^{2\pi} e^{-int}\gamma_j(dt), \quad n \in \mathcal{Z},\ j = 0,\ldots,T-1. \tag{2.5}$$

The measures γ_j, $j = 0,\ldots,T-1$, will be referred to as the spectral measures, or just the spectrum, of the T-PC sequence $\big(X(n)\big)$.

The theory of PC sequences began with Gladyshev's paper [1] and was fully developed in a series of works by Hurd. For proofs, bibliography, and more information about PC sequences we refer the reader to [2].

3. SPECTRAL REPRESENTATION OF A PC SEQUENCE

Let T be a fixed positive integer. The aim of this section is to prove the following two theorems.

THEOREM 3.1 (Factorization). *A family of measures γ_j, $j = 0,\ldots,T-1$, is a spectrum of a PC sequence if and only if there exist a $2\pi/T$-invariant nonnegative measure ν and a function $f \in L^2(\nu,\mathcal{C}^T)$ such that, for every $j \in \mathcal{Z}$ and every Borel Δ,*

$$\gamma_j(\Delta) = \frac{1}{T}\int_\Delta f(t)f(t+2\pi j/T)^*\nu(dt). \tag{3.1}$$

We want to emphasize that there is a single function f which generates all γ_j's. However, neither ν nor f are unique. In the sequel any function $f \in L^2(\nu,\mathcal{C}^T)$ satisfying (3.1) will be referred to as a *factor of measures (γ_j) with respect to ν.*

THEOREM 3.2 (Representation). *Let $\big(X(m)\big)$ be a T-PC sequence in $\mathcal{H}$ and let γ_j, $j = 0,\ldots,T-1$, be its spectrum. Further, let ν be a $2\pi/T$-invariant measure on $[0,2\pi)$ and f be a factor of measures (γ_j) with respect to ν* (see Theorem 3.1 above). *Then the* $L^2(\nu, \mathcal{C}^T)$*-valued sequence* $\big(x(m)\big)$ *defined by*

$$(3.2)\quad x(m)(u) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-im(u+2\pi j/T)} f(u + 2\pi j/T), \quad u \in [0, 2\pi),\ m \in \mathcal{Z},$$

is equivalent to the sequence $\big(X(m)\big)$.

The proofs of the two theorems are interlaced and we split them into a few lemmas. The main idea of the construction presented below stems from [4].

Given a measure μ on the interval $[0, 2\pi)$, let $\mu_{/T}$ denote the measure on $[0, 2\pi)$ defined by

$$(3.3)\qquad \mu_{/T}(\Delta) = \sum_{k=0}^{T-1} \mu_k(\Delta \cap I_k),$$

where $I_k = \big[2\pi k/T, 2\pi(k+1)/T\big)$ and μ_k is a measure supported on I_k defined by the formula

$$\mu_k(\Delta) = (1/T)\mu\big(T(\Delta \cap I_k)\big).$$

In simple words, to obtain $\mu_{/T}$ we compress μ to the arc I_0, divide it by T, replicate on consecutive arcs I_k, $k = 0, 1, \ldots, T-1$, and then add them up. Remember that the multiplication in $[0, 2\pi)$ is modulo 2π, so the precise meaning of the symbol $T\Delta$ is $T\Delta = \{t \in [0, 2\pi) : t = Tu \text{ modulo } 2\pi, u \in \Delta\}$. Below are obvious properties of the measure $\mu_{/T}$.

LEMMA 3.1. *Let* μ *be a nonnegative measure on* $[0, 2\pi)$ *and let* $\mu_{/T}$ *be as above. Then* $\mu_{/T}$ *is* $2\pi/T$*-invariant and for any* $f \in L^2(\mu, \mathcal{C}^m)$, $m \in \mathcal{N}$,

$$(3.4)\qquad \int_0^{2\pi} f(u)\mu(du) = \int_0^{2\pi} f(Tu)\mu_{/T}(du).$$

Consequently, the mapping $S : (Sf)(u) = f(Tu)$ *is an isometry from* $L^2(\mu, \mathcal{C}^m)$ *into* $L^2(\mu_{/T}, \mathcal{C}^m)$. *The range of* S *consists of all* $2\pi/T$*-periodic functions* $g \in L^2(\mu_{/T}, \mathcal{C}^m)$. *Moreover, for every* $2\pi/T$*-invariant measure* ν *on* $[0, 2\pi)$ *there is a measure* μ *such that* $\nu = \mu_{/T}$.

Proof. If $f = \mathbf{1}_\Delta$, then, using the notation above,

$$\int_0^{2\pi} \mathbf{1}_\Delta(Tu)\mu_{/T}(du) = \sum_{k=0}^{T-1} \mu_k(\Delta/T + 2\pi k/T) = \mu(\Delta) = \int_0^{2\pi} \mathbf{1}_\Delta(u)\mu(du),$$

which implies (3.4). Consequently, $\|Sf\|^2_{L^2(\mu_{/T}, \mathcal{C}^m)} = \|f\|^2_{L^2(\mu, \mathcal{C}^m)}$. Obviously, every $2\pi/T$-periodic function $g \in L^2(\mu_{/T}, \mathcal{C}^m)$ is of the form $g(u) = f(Tu)$, where $f(u) = g(u/T)$, $u \in [0, 2\pi)$, and $f \in L^2(\mu, \mathcal{C}^m)$. If ν is $2\pi/T$-invariant and we define $\mu(\Delta) = \nu(\Delta/T)$, then $\mu_{/T} = \nu$. ∎

Let $\big(X(n)\big)$ be a T-PC sequence and let $F = [F^{j,k}]$ be the spectral measure of the multivariate sequence $\boldsymbol{X}(n) = [X^k(n)]$, $n \in \boldsymbol{Z}$, associated with $\big(X(n)\big)$ as in Lemma 2.1. Suppose that all $(F^{j,k})$ are absolutely continuous with respect to a nonnegative measure μ (for example, one can take $\mu = \sum_{k=0}^{T-1} F^{k,k}$). Denote $f(t) = [f^{j,k}(t)]$ to be the Radon–Nikodym derivative of F with respect to μ. Since $f(t)$ is nonnegative definite μ-a.e., there exists a $T \times T$-matrix valued Borel measurable function $h(t)$ such that $h(t)h(t)^* = f(t)$ μ-a.e. If now $h^k(t)$ denotes the k-th row of $h(t)$, $k = 1, \ldots, T$, then $h^k \in L^2(\mu, \mathcal{C}^T)$ and

$$f^{j,k}(t) = h^j(t)\big(h^k(t)\big)^* \ \mu\text{-a.e.} \tag{3.5}$$

In the sequel, functions $h^k \in L^2(\mu, \mathcal{C}^T)$ that satisfy (3.5) will be referred to as *factors of F with respect to μ.* The equation (3.5) shows that the sequence $\boldsymbol{X}(n) = [X^k(n)]$ is equivalent to the $L^2(\mu, \mathcal{C}^T)$-valued sequence $\boldsymbol{h}(n) = [h^k(n)]$ defined as $h^k(n)(\cdot) = e^{-in\cdot}h^k(\cdot)$, $k = 1, \ldots, T$, $n \in \mathcal{Z}$. Remembering that $X(m) = X^{\langle m\rangle+1}\big(\mathrm{q}(m)\big)$ (see Lemma 2.1) we infer that the PC sequence $\big(X(m)\big)$ is equivalent to the sequence $\big(h(m)\big)$ in $L^2(\mu, \mathcal{C}^T)$ given by

$$h(m)(\cdot) = e^{-i\mathrm{q}(m)\cdot}h^{\langle m\rangle+1}(\cdot), \qquad m \in \mathcal{Z}. \tag{3.6}$$

This representation turns out to be difficult to work with and we propose its certain modification.

LEMMA 3.2. *Let $\big(X(m)\big)$ be a T-PC sequence and μ and h^k be as above. Set $\nu = \mu_{/T}$, and for every $u \in [0, 2\pi)$, $m \in \mathcal{Z}$, and $k = 1, \ldots, T$, define*

$$g^k(u) = h^k(Tu), \tag{3.7}$$

$$f(u) = \sum_{k=0}^{T-1} e^{iku}g^{k+1}(u), \tag{3.8}$$

$$x(m)(u) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-im(u+2\pi j/T)}f(u + 2\pi j/T). \tag{3.9}$$

Then $\big(x(m)\big)$ is a sequence in $L^2(\nu, \mathcal{C}^T)$ equivalent to the sequence $\big(X(m)\big)$.

Proof. Since $\big(X(m)\big)$ is equivalent to the $L^2(\mu, \mathcal{C}^T)$-valued sequence $\big(h(m)\big)$ defined in (3.6) and the mapping S from Lemma 3.1 is an isometry from $L^2(\mu, \mathcal{C}^T)$ into $L^2(\nu, \mathcal{C}^T)$, we conclude that $\big(X(m)\big)$ is equivalent to an $L^2(\nu, \mathcal{C}^T)$-valued sequence $x(m) = S\big(h(m)\big)$. Because $\mathrm{q}(m)T = m - \langle m\rangle$,

$$x(m)(u) = S\big(h(m)\big) = e^{-imu}e^{i\langle m\rangle u}g^{\langle m\rangle+1}(u), \tag{3.10}$$

where $g^k = S(h^k)$. Write $p(k)(u) = e^{iku}g^{k+1}(u)$, $k = 0, \ldots, T-1$. Because each g^{k+1} is $2\pi/T$-periodic, the inverse discrete Fourier transform $\big(P(j)\big)$ of $\big(p(k)\big)$

takes the form

$$\begin{aligned}P(j)(u) &= \sum_{k=0}^{T-1} e^{2\pi ikj/T} p(k)(u) = \sum_{k=0}^{T-1} e^{ik(u+2\pi j/T)} g^{k+1}(u) \\ &= \sum_{k=0}^{T-1} e^{ik(u+2\pi j/T)} g^{k+1}(u+2\pi j/T) = f(u+2\pi j/T), \qquad j \in \boldsymbol{Z}.\end{aligned}$$

Therefore, for any $k = 0, \dots, T-1$,

$$\begin{aligned}e^{iku} g^{k+1}(u) = p(k)(u) &= \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi ikj/T} P(j)(u) \\ &= \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi ikj/T} f(u+2\pi j/T).\end{aligned} \tag{3.11}$$

In particular,

$$\begin{aligned}e^{i\langle m\rangle u} g^{\langle m\rangle+1}(u) &= \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi i\langle m\rangle j/T} f(u+2\pi j/T) \\ &= \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi imj/T} f(u+2\pi j/T).\end{aligned}$$

Substituting the above to (3.10) gives (3.9). ∎

In what follows we will study properties of sequences of the form (3.9).

LEMMA 3.3. *Let ν be a $2\pi/T$-invariant measure on $[0,2\pi)$ and $f \in L^2(\nu, \mathcal{C}^T)$. Define*

$$x(m)(u) = \frac{1}{T} \sum_{j=0}^{T-1} e^{-im(u+2\pi j/T)} f(u+2\pi j/T), \quad u \in [0,2\pi), m \in \mathcal{Z}. \tag{3.12}$$

Then:

1. *$\big(x(m)\big)$ be a T-PC sequence in $L^2(\nu, \mathcal{C}^T)$.*
2. *Define $\xi(\Delta) = (1/T) \sum_{j=0}^{T-1} \mathbf{1}_\Delta(u+2\pi j/T) f(u+2\pi j/T)$. Then ξ is a vector measure in $L^2(\nu, \mathcal{C}^T)$ and for every $m \in \boldsymbol{Z}$*

$$x(m) = \int_0^{2\pi} e^{-imt} \xi(dt). \tag{3.13}$$

3. *The spectrum (γ_p) of $\big(x(m)\big)$ is given by*

$$\gamma_p(\Delta) = \frac{1}{T} \int_\Delta f(t) f(t+2\pi p/T)^* \nu(dt). \tag{3.14}$$

4. *If* $\boldsymbol{x}(n) = [x^k(n)]$ *is the block sequence generated by* $\big(x(n)\big)$, $x^{k+1}(n) = x(nT+k)$, *then its spectral measure* F *is absolutely continuous with respect to the measure* μ *defined as* $\mu(\Delta) = \nu(\Delta/T)$, *and its Radon–Nikodym derivatives admit factorizations*

$$\frac{dF^{j,k}}{d\mu}(u) = h^j(u)h^k(u)^*, \tag{3.15}$$

where

$$h^{k+1}(u) = \frac{1}{T}\sum_{p=0}^{T-1} e^{-ik(u/T+2\pi p/T)} f(u/T + 2\pi p/T) \ \mu\text{-a.e.}$$

P r o o f. 1. The lag m correlation function of $\big(x(m)\big)$,

$$R_x(m+k,k) = \frac{1}{T^2}\int_0^{2\pi} e^{-imu}\left[\sum_{j=0}^{T-1}\sum_{l=0}^{T-1} e^{-2\pi i(m+k)j/T} e^{2\pi ikl/T} f\left(u+\frac{2\pi j}{T}\right) f\left(u+\frac{2\pi l}{T}\right)^*\right]\nu(du),$$

is clearly T-periodic in k, and hence the sequence $\big(x(m)\big)$ is T-PC.

2. The assertion follows from the standard measure-theoretical argument. First of all, the σ-additivity of ξ is an obvious consequence of the vector version of Lebesgue's convergence theorem. If $\phi = \sum_p a_p \mathbf{1}_{\Delta_p}$ is a simple scalar function, then by definition of the integral we have

$$\Big[\int_0^{2\pi} \phi(t)\xi(dt)\Big](u) = \sum_p a_p\, \xi(\Delta_p)(u) = \frac{1}{T}\sum_{j=0}^{T-1} \phi(u+2\pi j/T) f(u+2\pi j/T).$$

The Lebesgue convergence theorem implies that the above formula holds true for any bounded Borel function ϕ. Taking $\phi(t) = e^{-imt}$ we obtain

$$\Big[\int_0^{2\pi} e^{-imt}\xi(dt)\Big](u) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-im(u+2\pi j/T)} f(u+2\pi j/T) = x(m)(u).$$

3. Since $\big(x(m)\big)$ is PC, it follows from Gladyshev's theorem that it is harmonizable. Therefore, the measure Γ in the representation (1.1) of the correlation function $R_x(m,n)$ of $\big(x(n)\big)$ satisfies $\Gamma(\Delta \times D) = \big(\xi(\Delta), \xi(D)\big)_{L^2(\nu,\mathcal{C}^T)}$. From (3.13) we therefore infer that

$$\Gamma(\Delta\times D) = \frac{1}{T^2}\sum_{j=0}^{T-1}\sum_{k=0}^{T-1}\left[\int_0^{2\pi} \mathbf{1}_\Delta\left(u+\frac{2\pi j}{T}\right)\mathbf{1}_D\left(u+\frac{2\pi k}{T}\right) f\left(u+\frac{2\pi j}{T}\right) f\left(u+\frac{2\pi k}{T}\right)^* \nu(du)\right].$$

Note that if Δ is *small*, meaning Δ is included in an interval of the length smaller than $2\pi/T$, and if $D = \Delta + 2\pi p/T$ for some p, then

$$\mathbf{1}_\Delta(u+2\pi j/T)\mathbf{1}_{\Delta+2\pi p/T}(u+2\pi k/T) = \mathbf{1}_{[\Delta-2\pi j/T]}(u)\mathbf{1}_{[\Delta-2\pi(k-p)/T]}(u) = 0,$$

except when $k = j + p$ modulo T. Therefore, for such small Δ we have

$$\begin{aligned}\gamma_p(\Delta) &= \Gamma(\Delta\times[\Delta+2\pi p/T])\\ &= (1/T^2)\sum_{j=0}^{T-1}\int_0^{2\pi}\mathbf{1}_\Delta(u+2\pi j/T)f(u+2\pi j/T)f\big(u+2\pi(p+j)/T\big)^*\nu(du).\end{aligned}$$

Since ν is $2\pi/T$ invariant, each integral above equals

$$\int_0^{2\pi}\mathbf{1}_\Delta(t)f(t)f(t+2\pi p/T)^*\nu(dt).$$

Consequently, for small Δ,

$$\gamma_p(\Delta) = (1/T)\int_\Delta f(t)f(t+2\pi p/T)^*\nu(dt).$$

Since γ_p is a measure, the formula above holds for any Borel set Δ.

4. Define

$$g^{k+1}(u) = (1/T)\sum_{j=0}^{T-1}e^{-ik(u+2\pi j/T)}f(u+2\pi j/T),\quad k=0,\dots,T-1. \tag{3.16}$$

Then, clearly, each g^{k+1} is $2\pi/T$-periodic and, in terms of g^k, the sequence $x(m)$ defined in (3.12) equals

$$x(m)(u) = e^{-i\mathrm{q}(m)Tu}g^{\langle m\rangle+1}(u).$$

Consequently, the coordinates of the block sequence $\boldsymbol{x}(m) = [x^k(m)]$ associated with $\big(x(m)\big)$ are

$$x^{k+1}(m)(u) = x(mT+k)(u) = e^{-imTu}g^{k+1}(u),\quad k=0,\dots,T-1,\ m\in\mathcal{Z}.$$

Since each function above is $2\pi/T$-periodic and $\mu_{/T} = \nu$, in view of Lemma 3.1 the correlation function $K(m) = [K^{j,k}(m)]$ of $\big(\boldsymbol{x}(m)\big)$ equals

$$\begin{aligned}K^{j+1,k+1}(m) &= \int_0^{2\pi}e^{-imTu}g^{k+1}(u)g^{j+1}(u)^*\nu(du)\\ &= \int_0^{2\pi}e^{-imu}h^{k+1}(u)h^{j+1}(u)^*\mu(du),\end{aligned}$$

where

$$h^{k+1}(u) = g^k(u/T) = (1/T) \sum_{p=0}^{T-1} e^{-ik(u/T+2\pi p/T)} f(u/T + 2\pi p/T).$$

Therefore

$$\frac{dF^{j,k}}{d\mu}(u) = h^j(u)h^k(u)^* \ \nu\text{-a.e.} \ \blacksquare$$

P r o o f o f T h e o r e m 3.1. Suppose that (γ_p) is the spectrum of a T-PC sequence $\big(X(n)\big)$. Let $F = [F^{j,k}]$ be the spectral measure of the block sequence $\boldsymbol{X}(n) = [X^k(n)]$, $n \in \boldsymbol{Z}$, associated with $\big(X(n)\big)$ as in Lemma 2.1. Suppose that all $(F^{j,k})$ are absolutely continuous with respect to a nonnegative measure μ. Take $\nu = \mu_{/T}$. Then from Lemma 3.2 it follows that $\big(X(n)\big)$ is equivalent to the sequence (3.9), and from Lemma 3.3, part 3, we conclude (3.1). Conversely, suppose that γ_p, $p = 0, \dots, T-1$, are given by (3.1), where ν is $2\pi/T$-invariant. Define the sequence $\big(x(m)\big)$ as in (3.12). Lemma 3.3, parts 1 and 3, tells us that $\big(x(m)\big)$ is T-PC and (γ_p) is its spectrum. $\blacksquare$

P r o o f o f T h e o r e m 3.2. Suppose that the spectrum (γ_p) of $\big(X(m)\big)$ satisfies (3.1), where ν is $2\pi/T$-invariant. Let $\big(x(m)\big)$ be given by (3.2). Then, by Lemma 3.3, the sequences $\big(x(m)\big)$ and $\big(X(m)\big)$ have the same spectrum, and hence they are equivalent. $\blacksquare$

4. COROLLARIES AND REMARKS

In Lemmas 3.2 and 3.3 we established an explicit correspondence between factors of the spectrum (γ_p) of a PC sequence and factors of the spectral measure of its associated block sequence.

COROLLARY 4.1. *Let* $\big(X(n)\big)$ *be* T*-PC*, (γ_p) *be its spectrum*, *and* F *be the spectral measure of its associated block sequence.*

(A) *Suppose that each* γ_p *is absolutely continuous with respect to a* $2\pi/T$*-invariant nonnegative measure* ν *and let* $f \in L^2(\nu, \mathcal{C}^T)$ *be a factor of measures* (γ_p) *with respect to* ν. *Then* F *is absolutely continuous with respect to the measure* μ *defined by* $\mu(\Delta) = \nu(\Delta/T)$, *and for every* $j, k = 0, \dots, T-1$

$$\frac{dF^{j,k}}{d\mu}(u) = h^j(u)h^k(u)^* \ \mu\text{-a.e.},$$

where $h^{k+1}(u) = (1/T) \sum_{p=0}^{T-1} e^{-ik(u/T+2\pi p/T)} f(u/T + 2\pi p/T)$.

(B) *Conversely*, *if* F *is absolutely continuous with respect to a nonnegative measure* μ *and* $h^k \in L^2(\mu, \mathcal{C}^T)$ *are factors of* F *with respect to* μ, *then each* γ_p *is* *absolutely continuous with respect to* $\nu = \mu_{/T}$, *and for every* p

$$\frac{d\gamma_p}{d\nu}(u) = (1/T)f(u)f(u + 2\pi p/T)^* \ \nu\text{-a.e.}, \quad \textit{where } f(u) = \sum_{k=0}^{T-1} e^{iku}h^{k+1}(Tu).$$

As a byproduct we have obtained an explicit, though rather complex, relation between the spectra of the two sequences. This completes an unfinished attempt to describe this relationship undertaken in [3].

$$\frac{d\gamma_p}{d\nu}(u) = \frac{1}{T}\sum_{k=0}^{T-1}\sum_{j=0}^{T-1} e^{-2\pi ijp/T}e^{i(k-j)u}\frac{dF^{j,k}}{d\mu}(Tu)\ \nu\text{-a.e.}, \tag{4.1}$$

$$\frac{dF^{j,k}}{d\mu}(u) = \frac{e^{iu(k-j)/T}}{T^2}\sum_{p=0}^{T-1}\sum_{q=0}^{T-1} e^{2\pi i(qk-jp)/T}\frac{d\gamma_{q-p}}{d\nu}\left(\frac{u}{T} + \frac{2\pi p}{T}\right)\ \mu\text{-a.e.} \tag{4.2}$$

Corollary 4.1 allows us to address the question of uniqueness of a factor f appearing in Theorem 3.1. A $T \times T$ complex matrix U is called a *partial isometry* if $\|xU\| = \|x\|$ for all $x \in \mathcal{N}(U)^\perp$, where $\mathcal{N}(U)$ stands for the null space of U defined as the set of all $x \in \mathcal{C}^T$ such that $xU = 0$. If U is a partial isometry, then $xUU^* = x$ for all $x \in \mathcal{N}(U)^\perp$.

COROLLARY 4.2. *Let* $\big(X(n)\big)$ *be* T*-PC*, (γ_p) *be its spectrum, and let* ν *be a* $2\pi/T$*-invariant nonnegative measure on* $[0, 2\pi)$ *such that each* γ_p *is absolutely continuous with respect to* ν. *If* f_1 *and* f_2 *are two factors of* (γ_p) *with respect to* ν, *then there exists a* $2\pi/T$*-periodic* $T \times T$*-matrix function* $V(u)$ *such that* $V(u)$ *is a partial isometry and* $f_2(u) = f_1(u)V(u)$ ν*-a.e.*

Proof. Let F be the spectral measure of the block sequence associated with $\big(X(m)\big)$, and let (h_i^{k+1}), $i = 1, 2$, be two factors of F with respect to μ. Then the $T \times T$-matrix functions $h_i(u) = [h_i^{k+1}(u)]$, $i = 1, 2$, satisfy

$$\frac{dF}{d\mu}(u) = h_1(u)h_1(u)^* = h_2(u)h_2(u)^*\ \mu\text{-a.e.}$$

It is known that a matrix factor of a nonnegative definite matrix is unique up to multiplication by a partial isometry. It follows that there exists a $T \times T$-matrix function $U(u)$ such that $U(u)$ is a partial isometry and $h_2(u) = h_1(u)U(u)$ μ-a.e. The later equation says that $h_2^{j+1}(u) = h_1^{j+1}(u)U(u)$, $j = 0, \ldots, T-1$. In view of the correspondence described in Corollary 4.1 we have

$$f_2(u) = \sum_{k=0}^{T-1} e^{iku}h_2^{k+1}(Tu) = \sum_{k=0}^{T-1} e^{iku}h_1^{k+1}(Tu)U(Tu) = f_1(u)V(u)$$

with $V(u) = U(Tu)$. ∎

If a measure ϱ on $[0, 2\pi)$ is absolutely continuous with respect to the Lebesgue measure dt, then we shortly say that ϱ is *absolutely continuous*. If this is the case then the Radon–Nikodym derivative $(d\varrho/dt)(u) = \varrho'(u)$ will be called the *density of* ϱ and will be denoted by ϱ'. The spectrum (γ_p) of a PC is said to be *absolutely continuous* if all γ_p's are absolutely continuous. Since $|\gamma_p(\Delta)|^2 \leqslant \gamma_0(\Delta)\gamma_0(\Delta + 2\pi p/T)$, for absolute continuity of (γ_p) it is enough that the diagonal measure γ_0 is absolutely continuous. Note that the Lebesgue measure is $2\pi/T$-invariant for every $T \in \mathcal{N}$. Moreover, if μ is the Lebesgue measure then $\mu_{/T}$ is also, and *vice versa*, if ν is the Lebesgue measure then $\mu(\Delta) = T\nu(\Delta/T)$ is also. As an immediate consequence of relations (4.1) and (4.2) we obtain

COROLLARY 4.3. *A T-PC sequence $\big(X(n)\big)$ has an absolutely continuous spectrum iff the spectrum of its associated block sequence is absolutely continuous.*

Now we can state the absolute continuous version of Theorem 3.1.

THEOREM 4.1. *Suppose that $\big(X(m)\big)$ is T-PC with absolutely continuous spectrum (γ_p) and let $\gamma_p'(t)$ denote the density of γ_p. Then there exists a function $f \in L^2(\mathcal{C}^T)$ such that for every $p \in \mathcal{Z}$*

$$\gamma_p'(u) = (1/T)f(u)f(u + 2\pi p/T)^* \ du\text{-a.e.} \tag{4.3}$$

The last theorem in this paper links properties of f with regularity properties of the sequence $\big(X(m)\big)$ in exactly the same way as for stationary sequences. Recall that a stochastic sequence $\big(X(m)\big)$ in $\mathcal{H}$ is called *regular* if $\bigcap_{n\in\mathcal{Z}} M_X(n) = \{0\}$, where $M_X(n)$ is the closed subspace of $\mathcal{H}$ spanned by the set $\{X(m) : m \leqslant n\}$. A function $f \in L^2(\mathcal{C}^n)$ is called *analytic* if its Fourier transform is supported on the set of nonnegative integers, that is, if its *Fourier coefficients*

$$\widehat{f}(n) = (1/2\pi)\int_0^{2\pi} e^{-int} f(t)dt$$

are zero for all $n < 0$. It what follows, by $L^2_r(\mathcal{C}^T)$, $r \in \mathcal{Z}$, we denote the subspace of $L^2(\mathcal{C}^T)$ consisting of all functions for which $\widehat{f}(n) = 0$ for all $n < r$. With this notation $L^2_0(\mathcal{C}^T)$ is the space of analytic functions and $L^2_r(\mathcal{C}^T) = e^{ir\cdot}L^2_0(\mathcal{C}^T)$.

THEOREM 4.2. *A T-PC sequence $\big(X(m)\big)$ is regular if and only if its spectrum is absolutely continuous and a function f in* (4.3) *can be chosen analytic.*

P r o o f. It is well known that a T-variate stationary sequence is regular iff its spectral measure is absolutely continuous and its spectral density $f(t)$ admits an analytic factor, i.e. there is a $T \times T$-matrix valued function $h(t)$ whose rows $h^k(t)$ are analytic and such that $h(t)h(t)^* = f(t)$ (see, for example, [5]). Suppose first that $\big(X(m)\big)$ is regular. Then its block sequence is also regular, and hence in Lemma 3.2 we can choose μ to be the Lebesgue measures and functions h^k and

g^k to be analytic. Consequently, $\nu = \mu_{/T}$ is the Lebesgue measure and the function f in (3.8), which by Lemma 3.3 is a factor of the spectrum of $\big(X(m)\big)$, is analytic. Conversely, suppose that there exists an analytic $f \in L_0^2(\mathcal{C}^T)$ that satisfies (4.3). Let $\big(x(m)\big)$ be defined by (3.12). Then by Lemma 3.3, part 3, $\big(x(m)\big)$ is equivalent to the sequence $\big(X(m)\big)$. Since the past $M_x(-n)$ of $\big(x(m)\big)$ is a subspace of $L_n^2(\mathcal{C}^T)$, the intersection $\bigcap_{n\in\mathcal{Z}} M_x(n) \subseteq \bigcap_{n\in\mathcal{Z}} L_n^2(\mathcal{C}^T) = \{0\}$, and hence $\big(x(n)\big)$, and so $\big(X(n)\big)$, is regular. ■

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