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RANDOM MATRICES BY MA MODELS AND COMPOUND FREE POISSON LAWS

BY

AYAKO HASEGAWA (TOKYO), NORIYOSHI SAKUMA (KARIYA-SHI), AND HIROAKI YOSHIDA (TOKYO)

Abstract. Recently, Pfaffel and Schlemm have investigated the Marchenko–Pastur type limit ($n \to \infty$ and $\lim_{n\to\infty} n/p = \lambda > 0$) of the sample covariance matrix $p^{-1}\boldsymbol{X}_n{}^t\boldsymbol{X}_n$, where $\boldsymbol{X}_n$ is the $p \times n$ random matrix with dependence such that each row of $\boldsymbol{X}_n$ is given by a certain linear process. They have also determined the limit spectral measure by giving the functional equation for its Stieltjes transform.

In this paper, we will see that such a limit spectral measure is a compound free Poisson law and, in the case where dependence is given by MA modeled Gaussian process, the sample covariance matrix can be regarded as compound Wishart matrix and, hence, gives the random matrix model for a compound free Poisson law. We will also give an application of compound Wishart matrix to the statistical data analysis of times series.

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1. INTRODUCTION

Assume that $\{X_{i,j}\}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, n)$ is the family of independent random variables with common mean and unit variance. Let $\boldsymbol{X}_n = (X_{i,j})_{p\times n}$ be a random matrix and we put the $p \times p$ symmetric matrix as $\boldsymbol{W}_n = p^{-1}\, \boldsymbol{X}_n{}^t\boldsymbol{X}_n$, which is called the *sample covariance matrix* of the sample size n of p-dimensional data $\{X_{i,j}\}$. The spectral analysis of the sample covariance matrix $\boldsymbol{W}_n$ has been studied since the work of Marchenko and Pastur [6] was appeared. The main problems in the spectral analysis of large dimensional random matrices (the major monograph in this topic is [1] by Bai and Silverstein) are to investigate the empirical spectral measure $\mu_{\boldsymbol{W}_n}$ of $\boldsymbol{W}_n$ defined by

$$\mu_{\boldsymbol{W}_n}(dx) = \frac{1}{p}\sum_{i=1}^{p} \delta_{\lambda_i}(x),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the sample covariance matrix$\boldsymbol{W}_n$, and to determine the limit spectral measure as $n \to \infty$. Here in the limit $n \to \infty$, we assume that the dimension p is of the same order as the sample size n, that is, $p = p_n \to \infty$ such that $\lim_{n\to\infty} n/p = \lambda \in (0, \infty)$.

The empirical spectral measure of Gaussian sample covariance matrix was first calculated by Wishart in [18]. Hence the above sample covariance matrix of independent Gaussian random variables $\{X_{i,j}\}$ is called the *Wishart matrix*.

A few decades later, Marchenko and Pastur [6] considered the case where the random variables $\{X_{i,j}\}$ are (not restricted to Gaussians but more general) independently identically distributed with finite second moment. Under weak conditions on $\{X_{i,j}\}$, it was shown by Silverstein [11] that the empirical spectral measure of the sample covariance matrix $\boldsymbol{W}_n$ converges almost surely, as $n \to \infty$ and $n/p \to \lambda > 0$, to the compactly supported probability measure μ_{MP} given by

$$\mu_{\text{MP}}(dx) = \frac{1}{2\pi x}\sqrt{-(x-\lambda_-)(x-\lambda_+)}\,\chi_{[\lambda_-,\lambda_+]}(x)\,dx + \max\{0, 1-\lambda\}\,\delta_0(x),$$

where χ_I is the indicator function for the interval I and δ_0 denotes the Dirac unit mass at zero. In this formula, $\lambda_\pm = \big(1 \pm \sqrt{\lambda}\big)^2$ and there is point mass at zero if $\lambda < 1$. The limit spectral measure μ_{MP} is called the *Marchenko–Pastur law*.

Recently, Pfaffel and Schlemm [9] have investigated the Marchenko–Pastur type limit of the sample covariance matrix $\boldsymbol{W}_n$ in the case where there is dependence in the rows of $\boldsymbol{X}_n$. Especially, they have treated the case where the ith row of $\boldsymbol{X}_n$ is given by a linear process of the form

$$(X_{i,j})_{j=1}^n = \Big(\sum_{\ell=0}^{\infty} c_\ell Z_{i,j-\ell}\Big)_{j=1}^n, \quad c_\ell \in \mathbb{R}.$$

Here, in the paper [9], the set $\{Z_{i,j}\}$ is given as the family of independent standardized (mean zero and variance one) random variables with uniformly bounded fourth moment and the Lindeberg-type condition. Later in this paper, we will assume $\{Z_{i,j}\}$ to be the set of independent standard Gaussians, which corresponds to the case where each row of the data matrix $\boldsymbol{X}_n$ is given as an MA (moving average) modeled Gaussian process.

It has been shown by Pfaffel and Schlemm [9] that when $n \to \infty$ and $\lim_{n\to\infty} n/p = \lambda$, the empirical spectral measure of the sample covariance matrix $\boldsymbol{W}_n$ converges almost surely to the compactly supported probability measure μ, and they have derived the exact formula of non-linear equation for the Stieltjes transform $m_\mu(z)$ of the measure μ. Here the Stieltjes transform of μ is defined by

$$m_\mu(z) = \int \frac{1}{x-z}\,\mu(dx) \quad \text{for all } z \in \mathbb{C}^+.$$

Moreover it might be worth to note that Xie [19] extends the results of Pfaffel and Schlemm in [9] to the case where $\lambda = 0$ is allowed.

The theory of free probability initiated by Voiculescu (see, for instance, [17], [16], [4]) gives the powerful tools for random matrices. For instance, the asymptotic freeness of independent random matrices well helps us to calculate the limit spectral measures for the sum and the product of random matrices by using the free additive [14] and multiplicative [15] convolutions, respectively.

The Marchenko–Pastur law can be regarded as the free analogue of Poisson law because it is characterized by the property that it has the constant ($= \lambda$) free cumulant of all orders. Furthermore, the free analogue of Lévy–Khintchine formula will allow us to consider the compound free Poisson laws (see [7], [4]), which is the free counterpart of the classical notion of compound Poisson. We should note, however, that in the classical probability the compound Poisson laws can rely on more general measures (see, for instance, [10], Definition 4.1).

In this paper, we shall see that the limit spectral measure of the sample covariance matrix obtained in [9] is given as the compound free Poisson law. We shall also make a suggestion that it can be applied to the statistical data analysis of time series.

Each section is constituted as follows. In Section 2, we shall first give the random matrix models for compound free Poisson laws related to symmetric Toeplitz matrices, and as the special case of this result we will see that the limit spectral measures of the sample covariance matrices derived in [9] are given as compound free Poisson laws in the case of Gaussians. We also see the random matrix model for the compound free Poisson of k-point discrete probability measures on $\mathbb{R}$ by our model of compound Wishart matrix. In Section 3, we shall show an application of the result in the previous section to the time series analysis, especially to MA models. Namely, by using the relation between the free cumulants of a compound free Poisson law and the moments of the compounding law, we will introduce criteria for the goodness of the estimated parameters in an MA model.

2. RANDOM MATRIX MODEL FOR COMPOUND FREE POISSON LAWS

We shall begin this section with recalling the definition and some properties of the free cumulants of compound free Poisson laws.

DEFINITION 2.1. The notion of free Poisson law can be generalized in the following way (see [4], [7]). For any compactly supported probability measure ρ on $\mathbb{R}$ and $\lambda > 0$, we put

$$\mathcal{R}(z) = \lambda \int \frac{x}{1 - xz} \, d\rho(x).$$

Then, by the free analogue of Lévy–Khintchine formula for the $\boxplus$-infinite divisibility [17], it can be found that there exists a compactly supported probability measure μ on $\mathbb{R}$ such that its R-transform satisfies $\mathcal{R}(z) = R_\mu(z)$. Such a measure μ is called a *compound free Poisson law* and we denote it by $\pi(\rho, \lambda)$. When $d\rho(x) = \delta_1(x)$ (Dirac point mass at one), the corresponding compound free Poisson law is, of course, reduced to the free Poisson (Marchenko–Pastur).

REMARK 2.1. *Let ρ be a compactly supported probability measure on $\mathbb{R}$ and denote its moment generating function by*

$$M_\rho(z) = \sum_{k=0}^{\infty} m_k(\rho)\, z^k,$$

where $m_k(\rho)$ stands for the kth moment of ρ. Then it can be seen (see, for instance, [4]) *that the compound free Poisson law $\mu = \pi(\rho, \lambda)$ is $\boxplus$-infinitely divisible and its R-transform $R_\mu(z)$ satisfies the relation*

$$zR_\mu(z) = \lambda\,\big(M_\rho(z) - 1\big),$$

that is, the kth free cumulant $r_k(\mu)$ of μ is given by $\lambda\, m_k(\rho)$.

Let $\boldsymbol{X}(n)$ be the $p \times n$ random matrix such that the ith row of $\boldsymbol{X}(n)$ is given by an MA-modeled Gaussian process of the form

$$(X_{i,j})_{j=1}^{n} = \Big(\sum_{\ell=0}^{\infty} c_\ell Z_{i,j-\ell}\Big)_{j=1}^{n}, \quad c_\ell \in \mathbb{R}, \tag{2.1}$$

where $\{Z_{i,j}\}_{i,j}$ is a family of independent standard Gaussian random variables.

It has been shown in [9] that when $n \to \infty$ and $\lim_{n\to\infty} n/p = \lambda$, the empirical spectral measure μ_n of the sample covariance matrix $p^{-1}\boldsymbol{X}(n)^t\boldsymbol{X}(n)$ converges almost surely to the compactly supported probability measure μ which is determined by the requirement that its Stieltjes transform $m_\mu(z)$ satisfies the functional equation

$$\frac{1}{m_\mu(z)} = -z + \lambda \int \frac{x}{1 + x m_\mu(z)}\, d\rho(x). \tag{2.2}$$

Here ρ is the compactly supported probability measure given as the weak limit as $n \to \infty$ of the spectral measure of the $n \times n$ Toeplitz matrix $\big(\gamma(i-j)\big)_{i,j}^{n}$, where $\gamma(h)$ is the autocovariance function of the MA model (2.1) given by

$$\gamma(h) = \sum_{j=0}^{\infty} c_j c_{j+|h|}. \tag{2.3}$$

As we have mentioned, we will give an interpretation of the equation (2.2) in the frame of free probability in this section. For this purpose, we shall see the following proposition which gives the random matrix model for a compound free Poisson law.

Let $\boldsymbol{Z}(n)$ be a $p \times n$ random matrix of independent standard Gaussian random variables. We consider an $n \times n$ symmetric Toeplitz matrix $\boldsymbol{\Gamma}(n) = (\gamma_{ij})_{i,j}^{n}$, where $\gamma_{ij} = r_{|i-j|}$ for some real sequence $\{r_k\}_{k\geqslant 0}$, and assume that it has the weak limit spectral measure ρ as $n \to \infty$. The conditions for the convergence of spectra of symmetric Toeplitz matrices related to orthogonal polynomials can be found in [3], Chapter 6, and see [5] for the theorem of Kac, Murdock, and Szegö on spectral measures of Hermitian Toeplitz matrices (see also, for instance, [13]).

PROPOSITION 2.1. *Assume that $\boldsymbol{Z}(n)$ and $\boldsymbol{\Gamma}(n)$ are as above. Then in the limit $n \to \infty$ and $\lim_{n\to\infty} n/p = \lambda > 0$, the almost sure limit of the empirical spectral measure of*

$$\frac{1}{p}\, \boldsymbol{Z}(n)\, \boldsymbol{\Gamma}(n)\, {}^t\boldsymbol{Z}(n) \tag{2.4}$$

is given by the compound free Poisson law $\pi(\rho, \lambda)$.

P r o o f. The symmetric Toeplitz matrix $\boldsymbol{\Gamma}(n)$ can be diagonalized by an orthogonal matrix $\boldsymbol{P}(n)$ as $\boldsymbol{D}(n) = {}^t\boldsymbol{P}(n)\, \boldsymbol{\Gamma}(n)\, \boldsymbol{P}(n)$, where $\boldsymbol{D}(n)$ is a diagonal matrix of dimension n.

Since an orthogonal transformation on a system of independent standard Gaussian random variables preserves independence and standard Gaussianity, the product $\widetilde{\boldsymbol{Z}}(n) = \boldsymbol{Z}(n)\, \boldsymbol{P}(n)$ yields a $p \times n$ random matrix of independent standard Gaussians again. Hence the random matrix in (2.4) becomes the *compound Wishart matrix* in [4], Section 4.4. Then the proof is an application of Proposition 4.4.11 in [4] to the case where $Z(n) = p^{-1/2} \widetilde{\boldsymbol{Z}}(n)$ and $B(n) = \boldsymbol{D}(n)$. ■

Here we briefly recall the proof of the main results in Pfaffel and Schlemm [9]. In their proof, they first dealt with the truncated process

$$\widetilde{X}_{i,j} = \sum_{\ell=0}^{n} c_\ell Z_{i,j-\ell},$$

and considered the $p \times n$ matrix $\widetilde{\boldsymbol{X}}(n) = (\widetilde{X}_{i,j})$, $i = 1, \ldots, p$, $j = 1, \ldots, n$. An important observation is the decomposition $\widetilde{\boldsymbol{X}}(n) = \boldsymbol{Z}(n)\boldsymbol{H}(n)$, where $\boldsymbol{Z}(n) = (Z_{i,j})$ is the $p \times 2n$ matrix of independent standard Gaussians and $\boldsymbol{H}(n)$ is the $2n \times n$ deterministic (non-random) matrix given by

$$\boldsymbol{H}(n) = \begin{pmatrix} c_n & 0 & \cdots & 0 \\ c_{n-1} & c_n & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ c_1 & & & c_n \\ c_0 & c_1 & \cdots & c_{n-1} \\ 0 & c_0 & & c_{n-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_0 \end{pmatrix} \in \mathbb{R}^{2n \times n}.$$

Hence $\widetilde{\boldsymbol{X}}(n)\, {}^t\widetilde{\boldsymbol{X}}(n)$ can be reformulated as

$$\widetilde{\boldsymbol{X}}(n)\, {}^t\widetilde{\boldsymbol{X}}(n) = \boldsymbol{Z}(n)\, \boldsymbol{H}(n)\, {}^t\boldsymbol{H}(n)\, {}^t\boldsymbol{Z}(n).$$

Pfaffel and Schlemm [9] have shown that the spectral measure of the $2n \times 2n$ matrix $\boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$ weakly converges, as $n \to \infty$, to

$$\frac{1}{2}\delta_0 + \frac{1}{2}\rho,$$

where ρ is the weak limit of the spectral measure of the symmetric Toeplitz matrix $\boldsymbol{\Gamma}(n) = \big(\gamma(i-j)\big)_{i,j}^{n}$ with the autocovariance function $\gamma(h)$ defined by (2.3).

The rigorous proof of this fact takes many pages, but we can expect it by the following observation: Since the rank of the $2n \times 2n$ matrix $\boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$ is at most n, a half of eigenvalues of the matrix $\boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$ must be zero for any n. Because the $n \times n$ matrices ${}^t\boldsymbol{H}(n)\,\boldsymbol{H}(n)$ and $\boldsymbol{\Gamma}(n)$ are asymptotically the same in elementwise operations and the set of the non-trivial eigenvalues of $\boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$ coincides with that of ${}^t\boldsymbol{H}(n)\,\boldsymbol{H}(n)$, hence the weak limit of the spectral measure of the $2n \times 2n$ matrix $\boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$ has the point mass at zero with weight $1/2$ and the half-weighted measure of ρ.

Then Pfaffel and Schlemm [9] have applied the formula (1.2) in [8] to the case where the $p \times 2n$ matrix $\mathbf{X}_n = \boldsymbol{Z}(n)$, the $2n \times 2n$ matrix $\mathbf{T}_n = \boldsymbol{H}(n)\,{}^t\boldsymbol{H}(n)$, and $\mathbf{A}_n = \mathbf{0}$ (zero matrix), and derived the functional equation (2.2) for the Stieltjes transform of the limit spectral measure μ of the sample covariance matrix.

Although they derived the equation (2.2) in the above manner, in the case of Gaussians, we can infer simply by Proposition 2.1 that the measure μ can be given as the compound free Poisson. Namely, with the above observation on the relation between the Toeplitz matrix $\boldsymbol{\Gamma}(n)$ and ${}^t\boldsymbol{H}(n)\,\boldsymbol{H}(n)$ in our mind, we obtain the following theorem:

THEOREM 2.1. *Let $\boldsymbol{X}(n)$ be the $p \times n$ random matrix defined by* (2.1), *and let $\gamma(h)$ be the autocovariance function as in* (2.3). *We assume that the spectral measure of the symmetric Toeplitz matrix $\boldsymbol{\Gamma}(n) = \big(\gamma(i-j)\big)_{i,j}^{n}$ converges weakly, as $n \to \infty$, to ρ.*

Then in the limit $n \to \infty$ and $\lim_{n\to\infty} p/n = \lambda > 0$, the empirical spectral measure of $p^{-1}\boldsymbol{X}(n)^t\boldsymbol{X}(n)$ converges almost surely to the compound free Poisson law $\pi(\rho, \lambda)$.

REMARK 2.2. *We should also note that the functional equation* (2.2) *for the Stieltjes transform of the limit spectral measure μ of the sample covariance matrix yields the equation for the free cumulant series* (*Voiculescu's R-transform*) *of μ for the compound free Poisson as follows*:

For a compactly supported probability measure μ on $\mathbb{R}$, the Cauchy transform $G_\mu(z)$ of μ is given by

$$G_\mu(z) = \int \frac{1}{z-x}\,\mu(dx).$$

Then Voiculescu's R-transform (*the free cumulant series*) *$R_\mu(z)$ of μ is related to* *the Cauchy transform $G_\mu(z)$ by the functional equation*

$$G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z.$$

Since the Cauchy transform $G_\mu(z)$ is the opposite signature of the Stieltjes transform $m_\mu(z)$, that is, $m_\mu(z) = -G_\mu(z)$, and the inverse function $G_\mu^{-1}(z)$ is given by $R_\mu(z) + 1/z$, one can easily find that the inverse of the Stieltjes transform is given by

$$m_\mu^{-1}(z) = R_\mu(-z) - \frac{1}{z}.$$

Using this fact, we invert the function $m_\mu(z)$ in the functional equation (2.2). *Then it follows that*

$$\frac{1}{z} = -\left(R_\mu(-z) - \frac{1}{z}\right) + \lambda \int \frac{x}{1 + x\,z}\,\rho(dx).$$

Replacing the variable z with $-z$ we obtain

$$R_\mu(z) = \lambda \int \frac{x}{1 - x\,z}\,d\rho(x),$$

which means that μ is the compound free Poisson law $\pi(\rho, \lambda)$ by Definition 2.1.

The explicit formula of the limit spectral measure of $\mathbf{\Gamma}(n) = \big(\gamma(i-j)\big)_{i,j}^n$ has been also investigated in [9], which is formulated in terms of the Fourier transform of the autocovariance function $\gamma(h)$ (the spectral density of the process),

$$f(\omega) = \sum_{h \in \mathbb{Z}} \gamma(h) e^{-\sqrt{-1} h\omega}. \tag{2.5}$$

Namely, they have written the functional equation for the Stieltjes transform of the limit spectral measure μ by using the function $f(\omega)$.

REMARK 2.3. *The assumptions in* [7] *cover only two special cases of the limit spectral measure of ρ with smooth spectral density and ρ with piecewise constant spectral density mentioned in Lemma* 2.1 *below.*

In the case of Gaussians, however, our model of a compound Wishart matrix in Proposition 2.1 *can relax the assumptions on the spectral density.*

LEMMA 2.1. *If the Fourier transform $f(\omega)$ of the autocovariance function $\gamma(h)$ is given by the piecewise constant function on $[0, 2\pi]$ of the form*

$$f(\omega) = \sum_{j=1}^{k} \alpha_j\, I_{A_j}(\omega)$$

for some positive real numbers α_j *and a measurable partition*

$$[0, 2\pi] = A_1 \cup A_2 \cup \ldots \cup A_k, \qquad k \in \mathbb{N},$$

then the empirical spectral measure μ_n *of* $p^{-1}\boldsymbol{X}(n)^t\boldsymbol{X}(n)$ *converges almost surely to the compactly supported probability measure* μ, *as* $n \to \infty$ *and* $\lim_{n\to\infty} n/p = \lambda > 0$, *the Stieltjes transform* $m_\mu(z)$ *of which satisfies the functional equation*

$$\frac{1}{m_\mu(z)} = -z + \frac{\lambda}{2\pi}\sum_{j=1}^{k}\frac{|A_j|\,\alpha_j}{1+\alpha_j m_\mu(z)}, \tag{2.6}$$

where $|A_j|$ *denotes the Lebesgue measure of the set* A_j.

In the same manner as we have shown in Remark 2.2, the functional equation (2.6) can be reformulated as

$$R_\mu(z) = \lambda \sum_{j=1}^{k}\frac{p_j\,\alpha_j}{1-\alpha_j z},$$

where $p_j = |A_j|/(2\pi)$ $(j = 1, 2, \ldots, k)$, and hence $\sum_{j=1}^{k} p_j = 1$. Namely, μ is given as the compound free Poisson law $\pi(\nu, \lambda)$ of the *positive real* k-point measure ν of the form

$$\nu(dx) = \sum_{j=1}^{k} p_j\,\delta_{\alpha_j}(x), \tag{2.7}$$

where δ_{α_j} stands for the Dirac unit mass at α_j.

Now we shall see a slight extension as an application of the compound Wishart matrix model of Proposition 2.1. In Lemma 2.1 the parameters α_j $(j = 1, 2, \ldots, k)$ are restricted to *positive reals*, our compound Wishart matrix model, however, can extend to *any reals*.

Actually, given a k-point measure (2.7) with α_j being possibly negative, we make the piecewise constant function on $[0, 2\pi]$ of the form

$$f(\omega) = \sum_{j=1}^{k}\alpha_j\,I_{A_j}(\omega),$$

where a measurable partition $\{A_j\}_{j=1}^{k}$ should be taken so that the function $f(\omega)$ becomes an even function in periodic continuation to $\mathbb{R}$, that is, $f(\omega + 2n\pi) = f(\omega)$ for $\omega \in [0, 2\pi]$ and $n \in \mathbb{Z}$. Then we calculate the inverse Fourier transform of (2.5) to obtain the sequence $\{\gamma(h)\}_{h\in\mathbb{Z}}$, that is,

$$\gamma(h) = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)e^{\sqrt{-1}h\omega}\,d\omega, \qquad h \in \mathbb{Z}.$$

The above choice of a measurable partition ensures that $\gamma(-h) = \gamma(h)$, which implies that $\mathbf{\Gamma}(n)$ is symmetric. Using this function γ, we construct the Toeplitz matrix $\mathbf{\Gamma}(n) = \big(\gamma(i-j)\big)_{i,j}^{n}$. Then the compound Wishart matrix

$$\frac{1}{p}\,\boldsymbol{Z}(n)\,\mathbf{\Gamma}(n)\,{}^{t}\boldsymbol{Z}(n)$$

gives a random matrix model for the compound free Poisson law of *any real* k-point measure.

EXAMPLE 2.1. We consider the two-point measure ν,

$$\nu(dx) = \frac{1}{2}\,\delta_{-1}(x) + \frac{1}{2}\,\delta_{1}(x),$$

and make the corresponding function $f(\omega)$, for instance, as

$$f(\omega) = \begin{cases} \;\;1, & \omega \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right], \\ -1, & \omega \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \end{cases}$$

that is, the measurable partition of $[0, 2\pi]$ is $A_1 = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and $A_2 = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and $|A_1| = |A_2| = \pi$.

In this case, we obtain the sequence $\{\gamma(h)\}_{h\in\mathbb{Z}}$ as

$$\gamma(h) = \begin{cases} \dfrac{(-1)^m\,2}{(2m+1)\,\pi}, & |h| = 2m+1, \\ 0, & |h| = 2m, m = 0, 1, 2, \ldots \end{cases}$$

Namely,

$$\gamma(0) = 0,\ \gamma(\pm 1) = \frac{2}{\pi},\ \gamma(\pm 2) = 0,\ \gamma(\pm 3) = -\frac{2}{3\pi}, \\ \gamma(\pm 4) = 0,\ \gamma(\pm 5) = \frac{2}{5\pi},\ \ldots,$$

and the $n \times n$ matrix $\mathbf{\Gamma}(n)$ becomes the following Toeplitz matrix:

$$\Gamma(n) = \frac{2}{\pi} \underbrace{\begin{pmatrix} 0 & 1 & 0 & -\frac{1}{3} & 0 & \cdots & \\ 1 & 0 & 1 & 0 & -\frac{1}{3} & \ddots & \vdots \\ 0 & 1 & 0 & 1 & 0 & & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 1 & \ddots & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 1 & 0 & & 0 \\ \vdots & \ddots & & \ddots & & \ddots & 1 \\ & \cdots & 0 & -\frac{1}{3} & 0 & 1 & 0 \end{pmatrix}}_{n} \left.\vphantom{\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0 \end{pmatrix}}\right\} n.$$

REMARK 2.4. *The compound free Poisson law $\pi(\nu, \lambda)$ of the two-point measure ν in Example* 2.1 *is known as the free Bessel law of the parameter $s = 2$ and $t = \lambda$. The free Bessel laws have been deeply investigated by Banica et al.* [2], *where they have also given another random matrix model for the free Bessel laws in case of $t = 1$.*

3. AN APPLICATION TO TIME SERIES ANALYSIS

In this section, we shall show an application of Theorem 2.1 related with the compound free Poisson law, which is devoted to the statistical data analysis of time series. Namely, we shall give criteria of the goodness of the estimated parameters in an MA model.

For a given stationary time series $\{x_j\}_{j=1}^{N}$, we shall approximate it by an MA model,

$$X_j = c_0\, Z_j + c_1\, Z_{j-1} + c_2\, Z_{j-2} + \ldots + c_q\, Z_{j-q}, \tag{3.1}$$

where $\{Z_j\}$ is the family of independent standard Gaussian random variables and $q \in \mathbb{N}$ is the order of the MA model. That is, we shall regard the time series data $\{x_j\}$ as a sample path of the MA process $\{X_j\}$ in (3.1).

At first, we construct the $p \times n$ matrix $\boldsymbol{x}$ from given data $\{x_j\}$ such that

$$\boldsymbol{x} = (\xi_{ij}), \quad \text{with } \xi_{i,j} = x_{(i-1)n+j} \text{ and } pn \leqslant N.$$

In this construction, the bigger sizes of p and n are the better, and the closer to the square ratio ($p = n$) is also the more acceptable.

We calculate the first few (for instance, five) moments $\{m_k\}$ of the covariance matrix $p^{-1}\boldsymbol{x}^t\boldsymbol{x}$, that is,

$$m_k = \frac{1}{p}\mathrm{Tr}\big((p^{-1}\boldsymbol{x}^t\boldsymbol{x})^k\big).$$

Then we calculate the first few free cumulants $\{r_k\}$ from the moments calculated above by using the free cumulant-moment formula (see, for instance, [12]):

$$\begin{aligned}
r_1 &= m_1,\\
r_2 &= m_2 - m_1^2,\\
r_3 &= m_3 - 3m_1m_2 + 2m_1^3,\\
r_4 &= m_4 - 4m_1m_3 - 2m_2^2 + 10m_1^2m_2 - 5m_1^4,\\
r_5 &= m_5 - 5m_1m_4 - 5m_2m_3 + 15m_1^2m_3 + 15m_1m_2^2 - 35m_1^3m_2 + 14m_1^5,
\end{aligned}$$

where, of course, the first k moments can produce the first k free cumulants.

Next we shall make the Toeplitz matrix $\boldsymbol{\gamma} = \big(\gamma(i-j)\big)_{i,j}$ based on the autocovariance function of the MA model in (3.1), that is,

$$\gamma(h) = \sum_{j=0}^{q} c_j c_{j+|h|}.$$

Here we should note that the size of the matrix $\boldsymbol{\gamma}$ can be taken as large as we want because the matrix $\boldsymbol{\gamma}$ depends only on the autocovariance function. Then we shall also calculate the first few moments $\widetilde{m_k}$ of the matrix $\boldsymbol{\gamma}$.

As we have mentioned in the previous section about the random matrix model related with the compound free Poisson law, if a given time series $\{x_i\}$ is well-approximated by the MA model (3.1), in other words, the MA parameters $\{c_\ell\}_{\ell=0}^{q}$ are well-estimated from the data $\{x_i\}$, then the kth free cumulant r_k of the covariance matrix $p^{-1}\boldsymbol{x}^t\boldsymbol{x}$ and the kth moment $\widetilde{m_k}$ of the Toeplitz matrix $\boldsymbol{\gamma}$ constituted from the MA parameters should satisfy

$$\frac{r_k}{\lambda\,\widetilde{m_k}} \approx 1, \quad k = 1, 2, \ldots, \tag{3.2}$$

for large p and n, where $\lambda = n/p$. Now we will use the approximations (3.2) as criteria for the goodness of the estimated parameters for an MA model.

EXAMPLE 3.1. We shall here illustrate an example numerically by simulated data. We consider the following MA model of order five:

$$X_i = 1.0\,Z_i - 0.3\,Z_{i-1} - 0.5\,Z_{i-2} + 0.7\,Z_{i-3} + 0.3\,Z_{i-4} - 0.5\,Z_{i-5}. \tag{3.3}$$

According to this model, the simulated time series data of length 40,000 is generated numerically. Then we make the $p \times n = 200 \times 200$ data matrix $\boldsymbol{x}$ and calculate the first five free cumulants of the covariance matrix $p^{-1}\boldsymbol{x}^t\boldsymbol{x}$.

k	1	2	3	4	5
r_k	2.15	10.25	61.10	400.07	2771.64

On the other hand, based on the MA parameters $c_0 = 1.0$, $c_1 = -0.3$, $c_2 = -0.5$, $c_3 = 0.7$, $c_4 = 0.3$, $c_5 = -0.5$, we make the parameter matrix $\boldsymbol{\gamma}$ and also numerically calculate the first five moments of $\boldsymbol{\gamma}$. In this example, we take the size of the matrix $\boldsymbol{\gamma}$ as 500×500.

k	1	2	3	4	5
$\widetilde{m}_k$	2.17	10.38	60.44	374.48	2392.61

Since $\lambda = n/p = 1$, we obtain the following table of the approximations:

k	1	2	3	4	5
$r_k/(\lambda\widetilde{m}_k)$	0.9914	0.9875	1.0107	1.0683	1.1584

REMARK 3.1. Since the parameter matrix $\boldsymbol{\gamma}$ depends only on the autocovariance function, the above criteria can be extended to AR and ARMA models without any change.

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Department of Information Sciences
Ochanomizu University
2-1-1, Otsuka, Bunkyo, Tokyo 112-8610 Japan
E-mail: hasegawa.ayako@is.ocha.ac.jp

Department of Mathematics
Aichi University of Education
1 Hirosawa, Igaya-cho
Kariya-shi 448-8542 Japan
E-mail: sakuma@auecc.aichi-edu.ac.jp

Department of Information Sciences
Ochanomizu University
2-1-1, Otsuka, Bunkyo, Tokyo 112-8610 Japan
E-mail: yoshida@is.ocha.ac.jp

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