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MULTIDIMENSIONAL CATALAN AND RELATED NUMBERS AS HAUSDORFF MOMENTS*

BY

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Abstract. We study integral representation of the so-called d-dimensional Catalan numbers $C_d(n)$, defined by $\big[\prod_{p=0}^{d-1} p!/(n+p)!\big](dn)!$, $d = 2, 3, \ldots$, $n = 0, 1, \ldots$ We prove that the $C_d(n)$'s are the nth Hausdorff power moments of positive functions $W_d(x)$ defined on $x \in [0, d^d]$. We construct exact and explicit forms of $W_d(x)$ and demonstrate that they can be expressed as combinations of $d-1$ hypergeometric functions of type ${}_{d-1}F_{d-2}$ of argument x/d^d. These solutions are unique. We analyze them analytically and graphically. A combinatorially relevant, specific extension of $C_d(n)$ for d even in the form

$$D_d(n) = \Big[\prod_{p=0}^{d-1} \tfrac{p!}{(n+p)!}\Big]\Big[\prod_{q=0}^{d/2-1} \tfrac{(2n+2q)!}{(2q)!}\Big]$$

is analyzed along the same lines.

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1. INTRODUCTION

Amongst many existing generalizations of classical Catalan numbers

$$C(n) = \frac{1}{n+1}\binom{2n}{n},$$

those that include the parameter that in a certain sense can be associated with the spatial dimension d are particularly interesting. They permit to extend to higher dimensions $d > 2$ the notions of objects enumerated by $C(n)$ in $d = 2$. In this

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note we shall be concerned with one of such generalizations, called *d-dimensional Catalan numbers* (see [1], [9], [19]), which are defined as

$$C_d(n) = \left[\prod_{p=0}^{d-1} \frac{p!}{(n+p)!}\right](dn)!, \quad n = 0, 1, \ldots, \text{ and } d = 2, 3, \ldots, \tag{1.1}$$

which for $d = 2$ clearly reduce the conventional Catalan numbers $C(n)$. The form of equation (1.1) guarantees that $C_d(0) = 1$ for all d. We shall refer to Sloane's *On-Line Encyclopedia of Integer Sequences (OEIS)* [17] and quote initial terms, for $n = 0, 1, \ldots, 7$, of several sequences $C_d(n)$, $d = 2, 3, 4$, and 5, along with the labelling of their entries in the OEIS:

- for $d = 2$: $1, 1, 2, 5, 14, 42, 132, \ldots$ (A00108), which are the Catalan numbers,
- for $d = 3$: $1, 1, 5, 42, 462, 6006, 87516, \ldots$ (A005789, A151334),
- for $d = 4$: $1, 1, 14, 462, 24024, 1662804, 140229804, \ldots$ (A005790),
- for $d = 5$: $1, 1, 42, 6006, 1662804, 701149020, 396499770810, \ldots$ (A005791),
- for general d, see A060854.

The explicit form of $C_d(n)$'s permits one to immediately write down some of their characteristics. If $\Delta(k, a) = a/k, (a + 1)/k, \ldots, (a + k - 1)/k$ denotes a special list of k elements, then the ordinary generating function of $C_d(n)$'s can be written as

$$g(d, z) = \sum_{n=0}^{\infty} C_d(n)\, z^n = {}_dF_{d-1}\left(\begin{matrix}\Delta(d, 1)\\ 2, 3, \ldots, d\end{matrix}\middle| d^d z\right). \tag{1.2}$$

Similarly, the exponential generating function of $C_d(n)$'s is of the form

$$G(d, z) = \sum_{n=0}^{\infty} C_d(n)\, \frac{z^n}{n!} = {}_dF_d\left(\begin{matrix}\Delta(d, 1)\\ 1, 2, \ldots, d\end{matrix}\middle| d^d z\right). \tag{1.3}$$

The use of Stirling's formula gives the leading term of $n \to \infty$ asymptotics for $C_d(n)$:

$$C_d(n) \xrightarrow{n\to\infty} n^{-(d^2-1)/2} d^{dn} + \ldots, \quad d = 2, 3, \ldots \tag{1.4}$$

In equations (1.2) and (1.3) we have used the standard notation for the generalized hypergeometric function ${}_pF_q\left(\begin{smallmatrix}(\alpha_p)\\ (\beta_q)\end{smallmatrix}\middle| x\right)$, with (α_p) and (β_q) the lists of p "upper" and q "lower" parameters, respectively, see [16]. Observe that since in ${}_dF_d$ of equation (1.3) there is a pair of lower and upper parameters differing by one, the appropriate function ${}_dF_d$ can be reduced to a combination of ${}_{d-1}F_{d-1}$'s, see [16], the formula 7.2.13.17 on page 439.

Inspired by the very fruitful interpretation of Catalan numbers $C(n)$ as moments of a positive function on the interval $[0, 4]$, which is intimately related to the famous Wigner's semicircle law [2], we set out to consider the sequences $C_d(n)$,

$d > 2$, as Hausdorff power moments and have defined an objective of obtaining for $d > 2$ the equivalents of the solution for $d = 2$, quoted in equation (2.8) below.

The paper is organized as follows: in Section 2 we describe the method of obtaining exact and explicit solutions for $d \geqslant 2$. Subsequently we write down the general solution for d arbitrary and quote the specific cases of $d = 2, 3, 4$, and 5. In Section 3 we discuss some possible generalizations of $C_d(n)$'s. In Section 4 we close the note with short conclusions and comments about possible applications of the probability distributions found here.

2. SOLUTIONS OF THE HAUSDORFF MOMENT PROBLEM

We want to find the solutions of the Hausdorff moment problem

$$\int_0^{R(d)} x^n W_d(x)\, dx = C_d(n), \quad n = 0, 1, \ldots, \text{ and } d = 2, 3, \ldots, \tag{2.1}$$

where $R(d)$, the upper edge of the support of $W_d(x)$, will be determined below. The conventional estimate $R(d) = \lim_{n\to\infty}[C_d(n)]^{1/n} = d^d$ will be confirmed later by the Mellin transform analysis. As a preliminary step we shall demonstrate that the desired $W_d(x)$ defined in (2.1) is positive. Applying the Gauss–Legendre multiplication formula for gamma function to equation (1.1) and introducing complex s such that $n = s - 1$ we obtain

$$C_d(s-1) = (2\pi)^{(1-d)/2} d^{1/2-d} \Big(\prod_{k=0}^{d-1} k!\Big) (d^d)^s \prod_{j=0}^{d-1} \frac{\Gamma\big(s - 1 + (j+1)/d\big)}{\Gamma(s+j)}, \tag{2.2}$$

which should be interpreted as the Mellin transform of $W_d(x)$, i.e. the integral $\int_0^\infty x^{s-1} W_d(x) dx$, denoted by $\mathcal{M}\,[W_d(x); s]$, see [18]. Since for all $0 \leqslant j \leqslant d-1$ the inequality $j > (j + 1 - d)/d$ is true, the individual term labelled by j in the second product of equation (2.2) has the inverse Mellin transform [18] (see [16], the formula 8.4.2.3 on page 631),

$$\mathcal{M}^{-1}\left[\frac{\Gamma\big(s - 1 + (j+1)/d\big)}{\Gamma(s+j)}; x\right] = \frac{x^{(j+1)/d-1}\,(1-x)^{j-(j+1)/d}}{\Gamma\big(1 + j - (j+1)/d\big)}, \tag{2.3}$$

$j = 0, 1, \ldots, d-1$, e.g., it is proportional to the standard probabilistic beta distribution [10] in the variable x, which is a positive and absolutely continuous function for $0 \leqslant x \leqslant 1$. We perceive now $C_d(s-1)$ as a product of d such individual terms. Then the weight $W_d(x)$ is a positive and absolutely continuous function on $[0, R(d)]$, since it is a d-fold Mellin convolution of positive and absolutely continuous functions on $[0, 1]$. In the final result we accommodate the prefactor $(d^d)^s$ which indicates, via elementary property of the Mellin transform [18], that the solution of (2.1) will depend on x/d^d.

It turns out that such a d-fold Mellin convolution can be carried out explicitly. The key step is first to identify the weight $W_d(x)$ as a special case of the Meijer G-function $G^{m,n}_{p,q}$ (see [16]). This is a direct consequence of (2.2), i.e.,

$$W_d(x) = (2\pi)^{(1-d)/2} d^{1/2-d} \Big(\prod_{k=0}^{d-1} k!\Big)\, G^{d,0}_{d,d}\left(\frac{x}{d^d}\Big|\begin{matrix} 0,1,\ldots,d-1 \\ -\Delta(d,0)\end{matrix}\right), \tag{2.4}$$

where $\Delta(n,a) = a/n, (a+1)/n, \ldots, (a+n-1)/n$. Next, the Meijer G-function is converted to the hypergeometric form by using the formulas 16.17.2 and 17.17.3 of [13], which is the Slater theorem. We quote only the final result which is of the form

$$W_d(x) = \sum_{j=1}^{d-1} \frac{c_j(d)}{x^{j/d}} \times \tag{2.5}$$
$$\times {}_{d-1}F_{d-2}\left(\begin{matrix} -\frac{j}{d}, -1-\frac{j}{d}, \ldots, -d+2-\frac{j}{d} \\ 1-\frac{1}{d}, 1-\frac{2}{d}, \ldots, 1-\frac{j-1}{d};\ 1+\frac{1}{d}, 1+\frac{2}{d}, \ldots, 1+\frac{d-j-1}{d}\end{matrix}\Big|\frac{x}{d^d}\right)$$

defined for $0 \leqslant x \leqslant d^d$, which implies $R(d) = d^d$ in equation (2.1). (For the reader's convenience we point out that in equation (2.5), in the lower list of parameters of ${}_{d-1}F_{d-2}$, there are *two* sequences of numbers, which contain $j-1$ and $d-1-j$ terms, respectively). The numerical coefficient $c_j(d)$ is equal to

$$c_j(d) = (2\pi)^{(1-d)/2} d^{j-d+1/2} \times \tag{2.6}$$
$$\times \left[\prod_{p=1}^{d-1} \frac{p!}{\Gamma\big(p+j/d\big)}\right]\left[\prod_{k=1}^{j-1}\Gamma\Big(\frac{k}{d}\Big)\right]\left[\prod_{k=j+1}^{d-1}\Gamma\Big(\frac{j-k}{d}\Big)\right],$$

where $j = 1, \ldots, d-1$ and $d = 2, 3, \ldots$

The structure of parameter list of the Meijer G-function in equation (2.4) warrants that the assumptions of the formula 2.24.2.1 in [16] are satisfied:

$$-\frac{1}{d}\sum_{k=0}^{d-1} k - \sum_{k=0}^{d-1} k = -\frac{d^2-1}{2} < 0, \quad d = 2, 3, \ldots \tag{2.7}$$

Therefore, the Mellin transform of $W_d(x)$ is well defined for $\Re(s) > (d-1)/d$.

We shall explicitly write down the solutions for $d = 2, 3, 4$, and 5, starting with $W_2(x)$,

$$W_2(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \quad 0 < x \leqslant 4, \tag{2.8}$$

which is obtained in many references (see [11], [14], and Figure 1a below). It is the only density that can be expressed by an elementary function. Furthermore,

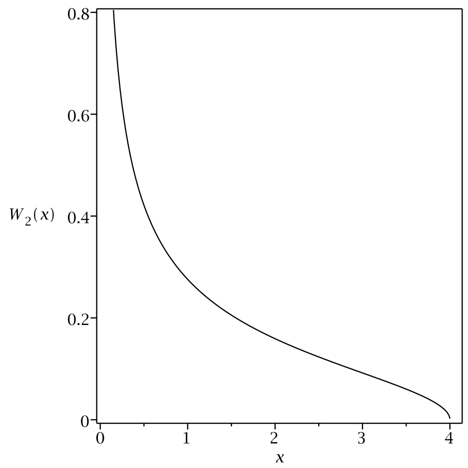

FIGURE 1a. The density $W_2(x)$, see equation (2.8)

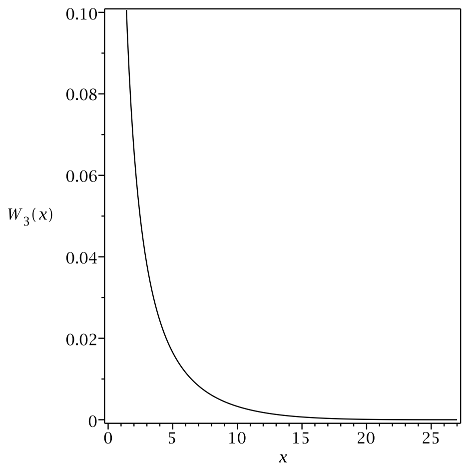

FIGURE 1b. The density $W_3(x)$, see equation (2.9)

for $d > 2$ no density can be expressed by *standard* special functions, and the hypergeometric form is the final one. For $d = 3, \ldots, 5$ the solutions take the form:

(2.9)

$$W_3(x) = \frac{c_1(3)}{x^{1/3}}\, {}_2F_1\left(\begin{matrix} -\frac{4}{3}, -\frac{1}{3} \\ \frac{4}{3} \end{matrix} \Big| \frac{x}{3^3}\right) + \frac{c_2(3)}{x^{2/3}}\, {}_2F_1\left(\begin{matrix} -\frac{5}{3}, -\frac{2}{3} \\ \frac{2}{3} \end{matrix} \Big| \frac{x}{3^3}\right), \quad 0 < x \leqslant 3^3,$$

(2.10)

$$W_4(x) = \frac{c_1(4)}{x^{1/4}}\, {}_3F_2\left(\begin{matrix} -\frac{9}{4}, -\frac{5}{4}, -\frac{1}{4} \\ \frac{5}{4}, \frac{3}{2} \end{matrix} \Big| \frac{x}{4^4}\right) + \frac{c_2(4)}{x^{1/2}}\, {}_3F_2\left(\begin{matrix} -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2} \\ \frac{3}{4}, \frac{5}{4} \end{matrix} \Big| \frac{x}{4^4}\right)$$

$$+ \frac{c_3(4)}{x^{3/4}}\, {}_3F_2\left(\begin{matrix} -\frac{11}{4}, -\frac{7}{4}, -\frac{3}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix} \Big| \frac{x}{4^4}\right), \quad 0 < x \leqslant 4^4,$$

(2.11) $W_5(x) =$

$$= \frac{c_1(5)}{x^{1/5}}\, {}_4F_3\left(\begin{matrix} -\frac{16}{5}, -\frac{11}{5}, -\frac{6}{5}, -\frac{1}{5} \\ \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \end{matrix} \Big| \frac{x}{5^5}\right) + \frac{c_2(5)}{x^{2/5}}\, {}_4F_3\left(\begin{matrix} -\frac{17}{5}, -\frac{12}{5}, -\frac{7}{5}, -\frac{2}{5} \\ \frac{4}{5}, \frac{6}{5}, \frac{7}{5} \end{matrix} \Big| \frac{x}{5^5}\right)$$

$$+ \frac{c_3(5)}{x^{3/5}}\, {}_4F_3\left(\begin{matrix} -\frac{18}{5}, -\frac{13}{5}, -\frac{8}{5}, -\frac{3}{5} \\ \frac{3}{5}, \frac{4}{5}, \frac{6}{5} \end{matrix} \Big| \frac{x}{5^5}\right) + \frac{c_4(5)}{x^{4/5}}\, {}_4F_3\left(\begin{matrix} -\frac{19}{5}, -\frac{14}{5}, -\frac{9}{5}, -\frac{4}{5} \\ \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \end{matrix} \Big| \frac{x}{5^5}\right),$$

$$0 < x \leqslant 5^5.$$

TABLE 1. The coefficients $c_j(d)$ (see equations (2.5) and (2.6)) for $d = 3, \ldots, 6$ and $j = 1, \ldots, d-1$. To simplify the notation in $c_j(5)$ and $c_j(6)$ we set $A = \sin\left(\frac{\pi}{5}\right)$, $B = \sin\left(\frac{2\pi}{5}\right)$ and $E = \Gamma\left(\frac{5}{6}\right)^6$, $D = \Gamma\left(\frac{2}{3}\right)^6$

j	$c_j(3)$	$c_j(4)$	$c_j(5)$	$c_j(6)$
1	$-\frac{3^3\sqrt{3}}{16\pi^3}\Gamma\left(\frac{2}{3}\right)^3$	$\frac{4^4 2}{75\pi^4}\Gamma\left(\frac{3}{4}\right)^4$	$-\frac{5^9\sqrt{5}\Gamma\left(\frac{4}{5}\right)^5 A^4}{2^5 3^2 11^2 \pi^5 B}$	$\frac{2^{13}3^{16}E}{7^4 13^3 1805\pi^6}$
2	$\frac{3^2}{10}\Gamma\left(\frac{2}{3}\right)^{-3}$	$-\frac{4^3}{15\pi^2}$	$\frac{5^9\sqrt{5}\Gamma\left(\frac{3}{5}\right)^5 B^4}{2^6 7^3 17\pi^5 A}$	$-\frac{3^{19}D}{2^{10}7^3 65\pi^6}$
3	–	$\frac{4^6}{4851}\Gamma\left(\frac{3}{4}\right)^{-4}$	$-\frac{5^9\sqrt{5}(AB)^{-1}}{2^7 3^4 13^2\Gamma\left(\frac{3}{5}\right)^5}$	$\frac{2^{21}}{7^2 5^2 3^4\pi^3}$
4	–	–	$\frac{5^9\sqrt{5}(AB)^{-1}}{2^7 3^4 7^2 19\Gamma\left(\frac{4}{5}\right)^4}$	$-\frac{3^{17}}{11^2 5^3 2^8 7D}$
5	–	–	–	$\frac{2^{22}3^{17}}{11^4 5^4 17^3 23^2 29E}$

The coefficients $c_j(d)$, $j = 1, \ldots, d-1$, for $d = 3, \ldots, 6$ are collected in Table 1. With $c_j(6)$'s given there and using equations (2.5) and (2.6), the reader can easily reconstruct $W_6(x)$, which will not be reproduced here. The solution $W_3(x)$ is represented in Figure 1b.

3. GENERALIZATION OF MULTIDIMENSIONAL CATALAN NUMBERS

In this section we analyze the extension of $C_4(n)$ obtained by replacing $(4n)!$ in equation (1.1) by $(2n)!(2n+2)!$. The corresponding sequence

$$D_4(n) \equiv 6(2n)!(2n+2)!\big[\prod_{r=0}^{3}(n+r)!\big]^{-1}$$

has attracted attention in several contexts, as it appears in [1], [3], [4], [6].

The initial terms of $D_4(n)$ are 1, 1, 4, 30, 330, 4719, 81796, 1643356 for $n = 0, 1, \ldots, 7$. It is listed as A006149 in OEIS where also additional information can be found. It turns out that the ordinary generating function of $D_4(n)$ can be expressed by the elliptic functions $\mathbb{E}(y)$ and $\mathbb{K}(y)$ (see [13]):

$$\sum_{n=0}^{\infty} D_4(n)\, z^n = \frac{1+6z}{4z^2} + \frac{(1-16z)(1+112z)}{240\,\pi\, z^3}\,\mathbb{K}(4\sqrt{z}) - \frac{1+224z+256z^2}{240\,\pi\, z^3}\,\mathbb{E}(4\sqrt{z}). \tag{3.1}$$

In fact, the sequence $D_4(n)$ allows for the same kind of analysis as does the ensemble of $C_d(n)$'s. The Hausdorff moment problem for $D_4(n)$, namely

$$\int_0^h x^n\, V_4(x)\, dx = D_4(n) = \frac{6(2n)!(2n+2)!}{\prod_{r=0}^{3}(n+r)!}, \quad n = 0, 1, \ldots, \tag{3.2}$$

can be solved by using the method of Mellin convolution and the Meijer G-function elucidated above. The weight can be proven to be positive on $x \in [0, h]$ with $h = 16$ and takes the form

$$V_4(x) = \frac{1}{15\pi^2}\Bigg[\left(\frac{64}{\sqrt{x}} + 56\sqrt{x} + \frac{x^{3/2}}{4}\right)\times \\ \times\, \mathbb{E}\left(\sqrt{1-\frac{x}{16}}\right) - 2\sqrt{x}(16+x)\,\mathbb{K}\left(\sqrt{1-\frac{x}{16}}\right)\Bigg]. \tag{3.3}$$

The function $V_4(x)$ is plotted in Figure 2a.

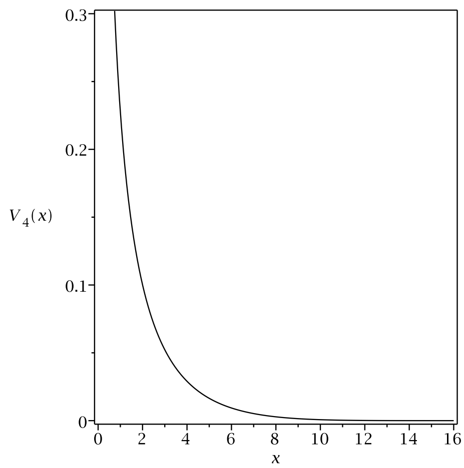

FIGURE 2a. The density $V_4(x)$, see equation (3.3)

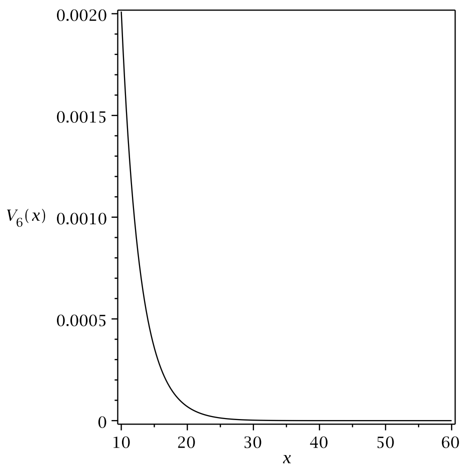

FIGURE 2b. The density $V_6(x)$, see equation (3.8) for $d = 6$

The sequence $D_4(n)$ analyzed above is a special case $d = 4$ of the following generalization of $C_d(n)$ defined for *even d*:

$$D_d(n) = \left[\prod_{r=0}^{d-1}\frac{r!}{(n+r)!}\right]\left[\prod_{s=0}^{d/2-1}\frac{(2n+2s)!}{(2s)!}\right], \quad d = 2, 4, 6, \ldots \tag{3.4}$$

Here the parameter d should not be associated anymore with the spatial dimension. Several exact characteristics of the sequences $D_d(n)$ are available. The ordinary generating function takes the form

$$\tilde{g}(d,z) = {}_{d/2}F_{d/2-1}\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-1}{2}, 1 \\ \frac{d}{2}+1, \frac{d}{2}+2, \ldots, d \end{matrix} \Big| 2^d z\right), \tag{3.5}$$

whereas the corresponding exponential generating function is equal to

$$\tilde{G}(d,z) = {}_{d}F_{d}\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-1}{2}, 1, 2, \ldots, \frac{d}{2} \\ 1, 2, 3, \ldots, d \end{matrix} \Big| 2^d z\right). \tag{3.6}$$

The leading term of the $n \to \infty$ asymptotics for $D_d(n)$ can be obtained by using the Stirling formula and it has the following form:

$$D_d(n) \xrightarrow{n\to\infty} n^{-d\,(d-1)/4}\, 2^{d\,n}, \quad d = 4, 6, \ldots \tag{3.7}$$

It is remarkable that the Hausdorff moment problem for $D_d(n)$, i.e.

$$\int_0^{\kappa(d)} x^n\, V_d(x)\, dx \;=\; D_d(n), \qquad n = 0, 1, \ldots, \text{ and } d = 4, 6, \ldots,$$

can be exactly solved as well in terms of positive functions $V_d(x)$ defined for $x \in [0, 2^d]$, i.e., $\kappa(d) = 2^d$, which equals

$$V_d(x) = \frac{2^{-d} \prod_{r=0}^{d-1} r!}{\prod_{k=0}^{d/2-1} \Gamma\left(k+\frac{1}{2}\right) k!}\, G^{d/2,0}_{d/2,d/2}\left(\frac{x}{2^d} \Big| \begin{matrix} \frac{d}{2}, \frac{d}{2}+1, \ldots, d-1 \\ -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, \frac{d-3}{2} \end{matrix}\right). \tag{3.8}$$

Here the condition 2.24.2.1 in [16] implies $-\frac{d}{4}(d+1) < 0$, $d = 4, 6, \ldots$, which is always satisfied. In addition, the Mellin transform of $V_d(x)$ is well defined for $\Re(s) > \frac{1}{2}$. The proof of positivity of $V_d(x)$ can be carried out along the lines exposed in the previous section.

Since in (3.8) in both parameter lists in the Meijer G-function there are index pairs that differ by an integer, this Meijer G-function cannot be represented by a sum of generalized hypergeometric functions. However, the expression (3.8) can be easily manipulated algebraically and represented graphically (in this work we have made an extensive use of Maple®). In Figure 2b we display $V_6(x)$ in the range $x \in [10,\ 60]$. Observe the rapid decrease of this function for $x \gtrsim 25$, followed by a large region where it is practically flat and equals zero. A similar behavior is observed for higher values of d.

4. DISCUSSION AND CONCLUSIONS

In this work we have treated essentially two generalizations of conventional Catalan numbers, which are related to such notions as Young tableaux, hook lengths, generalized Dyck paths, etc. [7]. They all turn out to be moments of positive functions on supports included in the positive half line. The relevant weight functions have been obtained explicitly and analyzed graphically. All these positive functions are unique solutions of Hausdorff moment problems. The key tool in this approach has been the inverse Mellin transform and the encoding with Meijer G-functions. The positivity of solutions has been rigorously proven using the method of Mellin convolution, applied to related problems previously (see [8], [12], [15]).

It should be specified that the function $W_2(x)$ of equation (2.8) is the known Marchenko–Pastur distribution [11], [15], which describes the level statistic of random Wishart matrices $W = GG^\dagger$, where G is a square, $N \times N$ random Ginibre matrix. As far as applications for random matrices are concerned two problems appear to be relevant for the distributions found in the present work.

First, it would be intriguing to know if the distributions $W_d(x)$ for $d \geqslant 3$, and $V_d(x)$ for $d = 4, 6, \ldots$ would correspond to limit spectral densities of certain (if any) ensembles of random matrices. A second possibility is to extend the analysis of products of square random matrices to products of rectangular $N \times M$ random matrices with $r = N/M$. A case in point is a detailed analysis of products of rectangular Gaussian random matrices carried out in [5]. Therefore, once the relevant matrix ensemble has been properly identified, it is quite feasible to undertake the analysis of appropriate products of rectangular matrices. This would lead, in the spirit of [5], to, for instance, $W_3^{(r)}(x)$ parametrized by r, with $W_3^{(1)}(x) \equiv W_3(x)$. Both of these problems are under active consideration.

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