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FUNCTIONAL LIMIT THEOREMS IN HÖLDER SPACE FOR RESIDUALS OF NEARLY NONSTATIONARY $\mathrm{AR}(1)$ PROCESS

BY

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Abstract. We investigate the polygonal line process built on the residuals of the first order nearly nonstationary autoregressive process. We prove functional limit theorems in Hölder space in two cases: the autoregressive coefficient ϕ_n is defined as $e^{\gamma/n}$, $\gamma < 0$ is a constant, and ϕ_n is defined as $1 - \gamma_n/n$, $\gamma_n \to \infty$, and γ_n/n tends to zero as $n \to \infty$. Also we discuss some applications of these functional limit theorems in epidemic change detection.

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1. INTRODUCTION

We analyze the polygonal line process built on the least squares residuals of a first order nearly nonstationary autoregressive process $(y_{n,k} : k = 1, \ldots, n;$ $n = 1, 2, \ldots)$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \tag{1.1}$$

where $0 < \phi_n < 1$ for fixed n, $\phi_n \to 1$ as $n \to \infty$, (ε_k) is a sequence of independent identically distributed random variables with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$ and $y_{n,0} = 0$.

The partial sums process of residuals was investigated by various authors because of its applicability in many statistical areas, like detecting structural changes or estimating probability density. A lot of studies are made for residuals partial sums of stationary and nonstationary autoregression model. For stationary processes, the weak limits of partial sums of autoregressive model's residuals were studied by Kulperger [11], and ARMA model's residuals by Bai [2]. Horváth [8] and Bai [2] proposed applications of limit theorems to change-point problems. Other authors analyzed more general processes based on residuals. For example, Yu [25] and Kulperger and Yu [12] constructed high moment partial sum processes based on residuals of ARMA and GARCH models, respectively. The residuals of stationary and nonstationary $\mathrm{AR}(1)$ process were studied by Shin [23]. He found that residuals partial sums processes converge to a standard Brownian motion when the autoregressive coefficient is strictly less than one, and it is a randomly shifted Brownian motion when the coefficient is equal to one.

Some functionals of the paths of the residuals partial sums processes are used as test statistics for the null hypothesis under certain alternatives. So one is interested in a larger choice of possible functionals, because then one has a bigger class of possible alternatives. The use of Hölder space provides functional limit theorems of a wider scope (see, e.g., Juodis et al. [10]). Račkauskas and Rastenė [19] extended Shin's [23] results establishing the convergence in Hölder spaces of polygonal line processes constructed from partial sums of residuals of the $\mathrm{AR}(1)$ model.

We establish the convergence in Hölder spaces of the polygonal line processes $\widehat{W}_n = \big(\widehat{W}_n(t), t \in [0,1], n > 0\big)$ built on the least squares residuals $(\widehat{\varepsilon}_k)$:

$$\widehat{W}_n(t) := \sum_{k=1}^{[nt]} \widehat{\varepsilon}_k + (nt - [nt])\widehat{\varepsilon}_{[nt]+1}, \quad \widehat{W}_n(0) = 0 \quad \text{and} \tag{1.2}$$

$$\widehat{W}_n(t) = 0 \quad \text{if } \sum_{k=1}^{n} y_{n,k-1}^2 = 0.$$

The residuals of the model (1.1) are defined by

$$\widehat{\varepsilon}_k = y_{n,k} - \widehat{\phi}_n y_{n,k-1} = \varepsilon_k - (\widehat{\phi}_n - \phi_n) y_{n,k-1}, \tag{1.3}$$

where $\widehat{\phi}_n$ is the least squares estimate of the coefficient ϕ_n defined by

$$\widehat{\phi}_n = \frac{\sum_{k=1}^{n} y_{n,k} y_{n,k-1}}{\sum_{k=1}^{n} y_{n,k-1}^2}. \tag{1.4}$$

We investigate Hölderian functional central limit theorems in the two situations in which ϕ_n tends to one. In the first case, we define $\phi_n = e^{\gamma/n}$ (γ is a negative constant), see Phillips [18]; in the second case, $\phi_n = 1 - \gamma_n/n$, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$, see Giraitis and Phillips [6]. Note that Markevičiūtė et al. [14] investigated first order nearly nonstationary processes and proved Hölderian limit theorems for the polygonal lines built on observations $y_{n,k}$.

The paper is organized as follows: Section 2 is devoted to some preliminaries and notation; we prove functional limit theorems for the first type model in Section 3, and for the second model in Section 4; Section 5 is devoted to applications.

2. WEAK CONVERGENCE IN HÖLDER SPACES

By $\|f\|_\infty$ we denote the uniform norm of $f \in \mathrm{C}[0,1]$. For $\alpha \in [0,1)$ the Hölder space

$$\mathrm{H}_\alpha^o[0,1] := \{f \in \mathrm{C}[0,1] : \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t\in[0,1]\\ 0<t-s<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha},$$

is a separable Banach space. Throughout the paper, $W = \big(W(t), t \in [0,1]\big)$ is a standard Brownian motion. By the classical Lévy's result on the modulus of continuity of W, $W \in \mathrm{H}_\alpha^o[0,1]$ with probability one for every $0 \leqslant \alpha < 1/2$.

The polygonal line process $W_n = \big(W_n(t), t \in [0,1], n > 0\big)$ built on i.i.d. random variables (ε_j) is defined by

$$W_n(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nt - [nt])\varepsilon_{[nt]+1}, \quad W_n(0) = 0. \tag{2.1}$$

Račkauskas and Suquet [20] proved that for $0 < \alpha < 1/2$ the convergence

$$n^{-1/2}\sigma^{-1} W_n \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in } \mathrm{H}_\alpha^o[0,1] \tag{2.2}$$

holds if and only if

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geqslant t) = 0. \tag{2.3}$$

Condition (2.3) provides a precise relation between the strength of the convergence (2.2) and the integrability of summands. Compared with the classical Donsker invariance principle, it shows the price to be paid for functional convergence in a stronger topology. When $\alpha > 0$, condition (2.3) implies that $\mathbb{E}\,|\varepsilon_1|^p < \infty$ for $p < (1/2-\alpha)^{-1}$ and, in particular, $\mathbb{E}\varepsilon_1^2 < \infty$.

3. FIRST TYPE MODEL

3.1. Technical lemmas. To prove the main result for the first type model, we need some technical lemmas.

LEMMA 3.1. *Let N_n, D_n, N, D be real-valued random variables with D_n and D being nonnegative. Assume that $P(D=0)=0$ and that (N_n, D_n) converges in distribution on $\mathbb{R}^2$ to (N, D). Define*

$$\Phi_n := \begin{cases} N_n/D_n & \text{on } \{D_n \neq 0\},\\ 0 & \text{on } \{D_n = 0\}.\end{cases}$$

Then Φ_n converges in distribution to N/D.

We omit the proof of Lemma 3.1. The lemma below is a tool that will help to prove next two lemmas.

LEMMA 3.2. *Suppose that the process* $(y_{n,k})$ *is defined by* (1.1) *with* $\phi_n = e^{\gamma/n}$, $\gamma < 0$ *and* $y_{n,0} = 0$. *Let* (ε_k) *be i.i.d. random variables with mean zero and satisfying the condition* (2.3). *Define*

$$V_n(l) := n^{-1/2} W_n\left(\frac{l-1}{n}\right) + \gamma \int_0^{l/n} e^{(l/n-s)\gamma} n^{-1/2} W_n(s)\, ds \tag{3.1}$$

for $l \leqslant n$. *Then*

$$|n^{-1/2} y_{n,l-1} - V_n(l)| \leqslant \|n^{-1/2} W_n\|_\infty \frac{\gamma^2 e^\gamma}{2n} + \frac{e^\gamma}{n^\alpha} \omega_\alpha\left(n^{-1/2} W_n, \frac{1}{n}\right) + \frac{|2+\gamma|\, e^\gamma}{n} \|n^{-1/2} W_n\|_\infty. \tag{3.2}$$

Proof. Let us define

$$V_{l,1} := n^{-1/2} y_{n,l-1} = n^{-1/2} \sum_{j=1}^{l-1} e^{(l-1-j)\gamma/n} \varepsilon_j.$$

Noting that $\varepsilon_l = W_n(l/n) - W_n\big((l-1)/n\big)$, we can write

$$y_{n,l-1} = W_n\left(\frac{l-1}{n}\right) + \frac{\gamma}{n} \sum_{j=1}^{l-2} e^{(l-1-j)\gamma/n} W_n\left(\frac{j}{n}\right) + \frac{\gamma^2 u_n}{2n^2} \sum_{j=1}^{l-2} e^{(l-1-j)\gamma/n} W_n\left(\frac{j}{n}\right),$$

where u_n is defined by

$$u_n = -1 + \frac{2n^2}{\gamma^2} o\left(\frac{1}{n^2}\right), \tag{3.3}$$

and $u_n \to -1$ as $n \to \infty$. Then we define

$$V_{l,2} := n^{-1/2} W_n\left(\frac{l-1}{n}\right) + \frac{\gamma}{n} \sum_{j=1}^{l-2} e^{(l-1-j)\gamma/n} n^{-1/2} W_n\left(\frac{j}{n}\right),$$

and for the approximation error we obtain the bound

$$|V_{l,2} - V_{l,1}| \leqslant \|n^{-1/2} W_n\|_\infty \frac{\gamma^2 e^\gamma}{2n}.$$

Further, we approximate the Riemann sum by the integral (see (3.1))

$$V_n(l) := n^{-1/2} W_n\left(\frac{l-1}{n}\right) + \gamma \int_0^{l/n} e^{(l/n-s)\gamma} n^{-1/2} W_n(s)\,ds.$$

Now we estimate the error. For any $f \in \mathrm{C}[0,1]$, we have (for details see Markevičiūtė et al. [14])

$$\left|\frac{1}{n}\sum_{j=1}^{l-2} f\left(\frac{j+j_0}{n}\right) - \int_0^{l/n} f(s)\,ds\right| \leqslant \omega_0\left(f, \frac{1+j_0}{n}\right) + \|f\|_\infty \frac{2}{n}. \tag{3.4}$$

Moreover,

$$\text{if } f \in \mathrm{H}^o_\alpha[0,1], \quad \omega_0(f,\delta) \leqslant \omega_\alpha(f,\delta)\delta^\alpha. \tag{3.5}$$

If $f(t) = g(t)h(t)$ with g of class C^1 and $h \in \mathrm{C}[0,1]$, we get

$$\omega_0(gh,\delta) \leqslant \|g\|_\infty \omega_0(h,\delta) + \|g'\|_\infty \|h\|_\infty \delta. \tag{3.6}$$

Thus, from (3.4)–(3.6) we obtain the uniform bound

$$|V_n(l) - V_{l,2}| \leqslant \frac{e^\gamma}{n^\alpha}\omega_\alpha\left(n^{-1/2}W_n, \frac{1}{n}\right) + \frac{|2+\gamma|\,e^\gamma}{n}\|n^{-1/2}W_n\|_\infty. \quad \blacksquare$$

We will use the following functionals in the proofs of the next two lemmas and the main result of this section:

$$N(x) := \frac{1}{2}\left(x(1) + \gamma\int_0^1 e^{(1-s)\gamma}x(s)\,ds\right)^2 - \gamma\int_0^1 \left(x(r) + \gamma\int_0^r e^{(r-s)\gamma}x(s)\,ds\right)^2 dr - \frac{1}{2}, \tag{3.7}$$

$$D(x) := \int_0^1 \left(x(r) + \gamma\int_0^r e^{(r-s)\gamma}x(s)\,ds\right)^2 dr, \tag{3.8}$$

$$F(x)(t) := \int_0^t \left(x(r) + \gamma\int_0^r e^{(r-s)\gamma}x(s)\,ds\right) dr, \quad t \in [0,1], \tag{3.9}$$

for $x \in \mathrm{H}^o_\alpha[0,1]$.

LEMMA 3.3. *Suppose that the process* $(y_{n,k})$ *is defined by* (1.1) *with* $\phi_n = e^{\gamma/n}$, $\gamma < 0$ *and* $y_{n,0} = 0$. *Let* (ε_k) *be i.i.d. random variables with mean zero*, $\mathbb{E}\varepsilon_1^2 = 1$, *and satisfying the condition* (2.3) *for some* $\alpha \in (0,1/2)$. *Define*

$$A_{n,0} := n^{-2}\sum_{k=1}^n y_{n,k-1}^2.$$

Then

$$|D(n^{-1/2}W_n) - A_{n,0}| = o_{\mathrm{P}}(n^{-\alpha}). \tag{3.10}$$

P r o o f. Using Lemma 3.2, we approximate $A_{n,0} := n^{-2}\sum_{k=1}^{n} y_{n,k-1}^2$ by

$$A_{n,1} := \frac{1}{n}\sum_{k=1}^{n}\left(n^{-1/2}W_n\left(\frac{k-1}{n}\right) + \gamma\int_0^{k/n} e^{(k/n-s)\gamma} n^{-1/2}W_n(s)\,ds\right)^2.$$

The approximation error is bounded by

$$(3.11)\qquad \begin{aligned}|A_{n,1} - A_{n,0}| &\leqslant \max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1} - V_n(k)|\\ &\leqslant \max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1}| + \max_{1\leqslant k\leqslant n}|V_n(k)|.\end{aligned}$$

By Lemma 3.2, $\max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1} - V_n(k)| = o_{\mathrm{P}}(n^{-\alpha})$. As $V_n(l)$ is the image of $n^{-1/2}W_n$ by a continuous functional on H_α^o, we infer from the continuous mapping theorem and Hölderian invariance principle that $\max_{1\leqslant k\leqslant n}|V_n(k)|$ is stochastically bounded. Also, by [18], $\max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1}|$ is stochastically bounded.

Further, $A_{n,1}$ can be approximated by A_n, and the bound of approximation error is

$$|A_n - A_{n,1}| \leqslant \omega\left(f, \frac{1}{n}\right),$$

where $f(r) := \left(W_n(r) + \gamma\int_0^r e^{(r-s)\gamma}W_n(s)\,ds\right)^2$. Define $g(r) := f^{1/2}(r)$. Then

$$(3.12)\quad \omega\left(f,\frac{1}{n}\right) \leqslant \frac{1}{n^\alpha}\omega_\alpha\left(f,\frac{1}{n}\right) \leqslant \frac{2}{n^\alpha}\|g\|_\infty\,\omega_\alpha\left(g,\frac{1}{n}\right)$$
$$\leqslant \frac{2}{n^\alpha}\cdot\|n^{-1/2}W_n\|_\infty e^\gamma\left(\omega_\alpha\left(n^{-1/2}W_n,\frac{1}{n}\right) + \frac{1}{n^{1-\alpha}}e^\gamma\|n^{-1/2}W_n\|_\infty\right).$$

So we obtain $|A_n - A_{n,0}| = o_{\mathrm{P}}(n^{-\alpha})$. ∎

LEMMA 3.4. *Suppose that the process* (y_k) *is defined by* (1.1) *with* $\phi_n = e^{\gamma/n}$, $\gamma < 0$ *and* $y_{n,0} = 0$. *Let* (ε_k) *be i.i.d. random variables with mean zero,* $\mathbb{E}\varepsilon_1^2 = 1$, *and satisfying the condition* (2.3) *for some* $\alpha \in (0, 1/2)$. *Define*

$$B_{n,0} := n^{-1}\sum_{k=1}^{n}\varepsilon_k y_{n,k-1}.$$

Then

$$(3.13)\qquad |N(n^{-1/2}W_n) - B_{n,0}| = o_{\mathrm{P}}(n^{-\alpha}).$$

P r o o f. By squaring the equation (1.1), subtracting $y_{n,k-1}^2$ from both sides and summing both sides over k, we obtain

$$y_{n,n}^2 = (e^{2\gamma/n} - 1) \sum_{k=1}^{n} y_{n,k-1}^2 + 2e^{\gamma/n} \sum_{k=1}^{n} y_{n,k-1}\varepsilon_k + \sum_{k=1}^{n} \varepsilon_k^2.$$

Then multiplying everything by n^{-1}, we get

$$\begin{aligned} B_{n,1} &:= 2n^{-1} \sum_{k=1}^{n} y_{n,k-1}\varepsilon_k \\ &= \frac{1}{e^{\gamma/n}} \left(n^{-1} y_{n,n}^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 - \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 - \frac{\gamma^2 u_n}{n^3} \sum_{k=1}^{n} y_{n,k-1}^2 \right), \end{aligned}$$

where $u_n \to -1$ as $n \to \infty$. Further, we can approximate $B_{n,1}$ by

$$B_{n,2} := \frac{1}{e^{\gamma/n}} \left(n^{-1} y_{n,n}^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 - \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 \right),$$

and the bound of the approximation error is

$$|B_{n,2} - B_{n,1}| \leqslant \frac{\gamma^2}{n} \left| \frac{1}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

because, by [18], $\left|n^{-2} \sum_{k=1}^{n} y_{n,k-1}^2\right|$ is stochastically bounded on $\mathbb{R}$ and $\gamma^2/n \to 0$ as $n \to \infty$. Further, we can approximate $B_{n,2}$ by

$$B_{n,3} := \frac{1}{e^{\gamma/n}} \left(n^{-1} y_{n,n}^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 - 1 \right).$$

In this case, for the approximation error we have

$$|B_{n,3} - B_{n,2}| \leqslant \left| \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 - 1 \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

by the weak law of large numbers since $\mathbb{E}\varepsilon_0^2 = 1$. Next, we approximate $B_{n,3}$ by

$$B_{n,4} := n^{-1} y_{n,n}^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 - 1.$$

As $\left|n^{-1} y_{n,n}^2 - (2\gamma/n^2) \sum_{k=1}^{n} y_{n,k-1}^2 - 1\right|$ is stochastically bounded by Lemma 1 in [18], we obtain

$$|B_{n,4} - B_{n,3}| = \left| n^{-1} y_{n,n}^2 - \frac{2\gamma}{n^2} \sum_{k=1}^{n} y_{n,k-1}^2 - 1 \right| \cdot \left| 1 - \frac{1}{e^{\gamma/n}} \right| \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Finally, put $g(r) = W_n(r) + \gamma \int_0^r e^{(r-s)\gamma} W_n(s)\,ds$ and define

$$B_n = \frac{1}{2}\bigl(g(1)\bigr)^2 - \gamma \int_0^1 \bigl(g(r)\bigr)^2\,dr - \frac{1}{2}.$$

Then, using Lemma 3.2, we obtain

$$|B_n - B_{n,4}| \leqslant \frac{1}{2}\Bigl|(n^{-1/2}y_{n,n})^2 - \Bigl(W_n(1) + \gamma \int_0^1 e^{(1-s)\gamma} W_n(s)\,ds\Bigr)^2\Bigr|$$
$$+ \gamma\left|\frac{1}{n^2}\sum_{k=1}^{n} y_{n,k-1}^2 - \int_0^1 \Bigl(W_n(r) + \gamma \int_0^r e^{(r-s)\gamma} W_n(s)\,ds\Bigr)^2\,dr\right|.$$

Further, the first summand is bounded by (3.11) and the second one is bounded by (3.11) and (3.12). ■

3.2. Invariance principle. For the process $\widehat{W}_n$ defined by (1.2) we prove an invariance principle under the necessary and sufficient condition (2.3).

THEOREM 3.1. *Let* $\alpha \in (0, 1/2)$. *Suppose that* $(y_{n,k})$ *is generated by* (1.1), $\phi_n = e^{\gamma/n}$, *and* $\gamma < 0$ *is a constant. Assume that* (ε_k) *are i.i.d. random variables with* $\mathbb{E}\varepsilon_0 = 0$, $\mathbb{E}\varepsilon_0^2 = \sigma^2$ *and* $y_{n,0} = 0$. *Then*

$$n^{-1/2}\sigma^{-1}\widehat{W}_n \xrightarrow[n\to\infty]{\mathcal{D}} W - A^{-1}BJ \quad \textit{in } \mathrm{H}_\alpha^o[0,1] \tag{3.14}$$

if and only if condition (2.3) *holds. Here* $A = \int_0^1 U_\gamma^2(t)\,dt$, $B = \int_0^1 U_\gamma(t)\,dW(t)$, *and* $J(t)$ *is an integrated Ornstein–Uhlenbeck process defined by*

$$J(t) := \int_0^t U_\gamma(s)\,ds, \tag{3.15}$$

where $U_\gamma(r)$ *is an Ornstein–Uhlenbeck process*,

$$U_\gamma(r) := \int_0^r e^{(r-s)\gamma}\,dW(s), \quad U_\gamma(0) = 0. \tag{3.16}$$

REMARK 3.1. *If variance is unknown*, *then by Slutsky's theorem it can be replaced in* (3.14), *via Theorem* 1 *in Phillips* [18], *by its estimator defined by*

$$\widehat{\sigma}^2 := \frac{1}{n}\sum_{k=1}^{n} \widehat{\varepsilon}_k^2. \tag{3.17}$$

Proof of Theorem 3.1. Without loss of generality we assume that $\sigma^2 = 1$.

To prove the sufficiency, at first, we express $\widehat{W}_n$ in terms of W_n and the polygonal line process S_n built on observations y_k:

$$n^{-1/2}\widehat{W}_n = n^{-1/2}W_n - \frac{n^{-1}\sum\limits_{k=1}^{n} \varepsilon_k y_{n,k-1}}{n^{-2}\sum\limits_{k=1}^{n} y_{n,k-1}^2} \cdot n^{-3/2}S_n, \tag{3.18}$$

where $S_n = \big(S_n(t), t \in [0,1], n > 0\big)$, and

$$S_n(t) = \sum_{j=1}^{[nt]} y_{n,j-1} + (nr - [nt])y_{n,[nt]}, \quad S_n(0) = 0. \tag{3.19}$$

Next, using the definition of U_γ, we obtain, by Itô's formula,

$$\int_0^1 U_\gamma(r)\,dW(r) = \frac{1}{2}\big(U_\gamma^2(1) - 1 - 2\gamma\int_0^1 U_\gamma^2(r)\,dr\big).$$

Further, define the operator T,

$$T(W) := W - A^{-1}BJ,$$

so that

$$T : \mathrm{H}_\alpha^o[0,1] \to \mathrm{H}_\alpha^o[0,1], \quad T(x) := x - \frac{N(x)}{D(x)}F(x),$$

where $N(x), D(x)$ and $F(x)$ are defined by (3.7), (3.8) and (3.9), respectively. It is obvious that the domain of the operator T is

$$H_T := \{x \in \mathrm{H}_\alpha^o[0,1] : D(x) \neq 0\}.$$

Further note that H_T is the Hölder space without the zero functions. Indeed, from the equations $D(x) = 0$, recalling that x is a continuous function on $[0,1]$, we obtain for every $r \in [0,1]$

$$x(r) + \gamma\int_0^r e^{(r-s)\gamma}x(s)\,ds = 0. \tag{3.20}$$

Thus, any continuous solution x of $D(x) = 0$ satisfies

$$x(r) = -\gamma e^{r\gamma}\int_0^r e^{-s\gamma}x(s)\,ds. \tag{3.21}$$

Further, from the continuity of x it follows that the right-hand side of (3.21) is obviously differentiable; consequently, x is itself differentiable, and for all $r \in (0,1)$ we obtain $x'(r) = 0$. This implies that x is a constant on $[0,1]$ (it is continuous at zero and at one). Let r tend to zero in (3.21). Then by the continuity of x we obtain $x(0) = 0$, and since x is constant, $x(r) = 0$ for every $r \in [0,1]$. Thus we obtain

$$\mathbb{P}(W \in \mathrm{H}_\alpha^o[0,1] \setminus H_T) = \mathbb{P}(W = 0) = 0.$$

Next, we obtain the convergence (3.14) by proving that

(a) T is a continuous operator on H_T, and $\mathbb{P}(W \in \mathrm{H}_\alpha^o[0,1] \setminus H_T) = 0$;

(b) $\|n^{-1/2}\widehat{W}_n - T(n^{-1/2}W_n)\|_\alpha \xrightarrow[n\to\infty]{\mathrm{P}} 0$.

We start with the continuity of T. The operator T is the difference of two operators. The first one is the identity on $\mathrm{H}_\alpha^o[0,1]$, obviously continuous. The second one is

$$\widetilde{T}(x) := \frac{N(x)}{D(x)} \cdot F(x), \quad x \in H_T.$$

First we show that $N : \mathrm{H}_\alpha^o[0,1] \to \mathbb{R}$ and $D : \mathrm{H}_\alpha^o[0,1] \to \mathbb{R}$ are continuous. Let us check first the continuity of D. By the triangular inequality of L_2-norm applied to the function $f(x)(r) = x(r) + \gamma \int_0^r e^{(r-s)\gamma} x(s)\, ds$, we have

$$\begin{aligned}
|D^{1/2}(x) - D^{1/2}(y)| &= \left| \Bigl(\int_0^1 \bigl(f(x)(r)\bigr)^2 dr \Bigr)^{1/2} - \Bigl(\int_0^1 \bigl(f(y)(r)\bigr)^2 dr \Bigr)^{1/2} \right| \\
&\leqslant \left| \Bigl(\int_0^1 \bigl(|x(r) - y(r)| + \gamma \int_0^r e^{(r-s)\gamma} |x(s) - y(s)|\, ds \bigr)^2 dr \Bigr)^{1/2} \right| \\
&\leqslant \|x - y\|_\infty \left(\frac{1}{2\gamma}(e^{2\gamma} - 1) \right)^{1/2}.
\end{aligned}$$

Since $\|\cdot\|_\infty \leqslant \|\cdot\|_\alpha$, we obtain

$$|D^{1/2}(x) - D^{1/2}(y)| \leqslant \left(\frac{1}{2\gamma}(e^{2\gamma} - 1) \right)^{1/2} \|x - y\|_\alpha.$$

This implies that $D^{1/2}$ is continuous on $\mathrm{H}_\alpha^o[0,1]$, and so is D. Using the same arguments, we obtain the continuity of N on $\mathrm{H}_\alpha^o[0,1]$.

Thus, the ratio N/D is continuous as the ratio of two continuous functions except on the subset of $\mathrm{H}_\alpha^o[0,1]$, where $D(x) = 0$, that is, at the null function on $[0,1]$.

As F is linear, it is enough to show its continuity at zero. Recall that

$$\|F(x)\|_\alpha = |F(x)(0)| + \sup_{0 \leqslant t' < t \leqslant 1} \frac{|F(x)(t) - F(x)(t')|}{|t - t'|^\alpha}.$$

Noting $\|x\|_\infty \leqslant \|x\|_\alpha$, we see that

$$\begin{aligned}|F(x)(t) - F(x)(t')| &= \Big|\int_{t'}^{t} \big(x(r) + \gamma \int_0^r e^{(r-s)\gamma} x(s)\, ds\big)\, dr\Big| \\ &\leqslant (1 + \gamma e^\gamma)\, \|x\|_\alpha\, |t - t'|.\end{aligned}$$

Since $F(x)(0) = 0$, we obtain

$$\|F(x)\|_\alpha \leqslant (1 + \gamma e^\gamma)\, \|x\|_\alpha, \tag{3.22}$$

which gives the continuity of F.

The continuity of $\widetilde{T}$ on H_T follows easily from the continuity of N, D and F. Finally, the operator T is continuous on H_T as the difference of two continuous operators.

As the operator T is continuous on H_T and $\mathbb{P}(W = 0) = 0$, also the Hölderian invariance principle holds (see [21]); then we have

$$T(n^{-1/2} W_n) \xrightarrow[n\to\infty]{\mathcal{D}} T(W) = W - A^{-1} B J \quad \text{in } \mathrm{H}^o_\alpha[0,1] \tag{3.23}$$

by the continuous mapping theorem (for details see [3], Theorem 5.1).

By Lemmas 3.3 and 3.4 we have

$$\begin{aligned} n^{-1} \sum_{k=1}^{n} \varepsilon_k y_{n,k-1} &:= N(n^{-1/2} W_n) + R_n, \\ n^{-2} \sum_{k=1}^{n} y^2_{n,k-1} &:= D(n^{-1/2} W_n) + \widetilde{R}_n, \end{aligned}$$

where $R_n = o_{\mathrm{P}}(n^{-\alpha})$ and $\widetilde{R}_n = o_{\mathrm{P}}(n^{-\alpha})$. We have also

$$n^{-3/2} S_n(t) = F(n^{-1/2} W_n)(t) + \widetilde{\widetilde{R}}_n, \quad t \in [0,1],$$

where $\widetilde{\widetilde{R}}_n = o_{\mathrm{P}}(n^{-\alpha})$ (for details see the proof of Theorem 1 in [14]).

Further, we can express $\widehat{W}_n$ as

$$n^{-1/2} \widehat{W}_n = n^{-1/2} W_n - \frac{N(n^{-1/2} W_n) + R_n}{D(n^{-1/2} W_n) + \widetilde{R}_n} \cdot \big(F(n^{-1/2} W_n)(t) + \widetilde{\widetilde{R}}_n\big),$$

and we obtain

$$\begin{aligned} &\|n^{-1/2} \widehat{W}_n - T(n^{-1/2} W_n)\|_\alpha \\ &\quad\leqslant \left| \frac{N(n^{-1/2} W_n) + R_n}{D(n^{-1/2} W_n) + \widetilde{R}_n} - \frac{N(n^{-1/2} W_n)}{D(n^{-1/2} W_n)} \right| \|F(n^{-1/2} W_n) + \widetilde{\widetilde{R}}_n\|_\alpha \\ &\quad\quad + \left| \frac{N(n^{-1/2} W_n)}{D(n^{-1/2} W_n)} \right| \|\widetilde{\widetilde{R}}_n\|_\alpha. \end{aligned}$$

Let us to introduce the random variables Φ_n and $\widetilde{\Phi}_n$ defined by

$$\Phi_n := \begin{cases} \dfrac{N(n^{-1/2}W_n)}{D(n^{-1/2}W_n)} & \text{on } \{D(n^{-1/2}W_n) \neq 0\}, \\ 0 & \text{on } \{D(n^{-1/2}W_n) = 0\}, \end{cases}$$

$$\widetilde{\Phi}_n := \begin{cases} \dfrac{N(n^{-1/2}W_n) + R_n}{D(n^{-1/2}W_n) + \widetilde{R}_n} & \text{on } \{D(n^{-1/2}W_n) + \widetilde{R}_n \neq 0\}, \\ 0 & \text{on } \{D(n^{-1/2}W_n) + \widetilde{R}_n = 0\}. \end{cases}$$

Coming back to the decomposition of $n^{1/2}\widehat{W}_n$ and modifying the definition of $T(n^{-1/2}W_n)$ as

$$T(n^{-1/2}W_n) = n^{-1/2}W_n - \Phi_n F(n^{-1/2}W_n)$$

(for that purpose it suffices to define $T(0) := 0$), we can rewrite the estimate of $\|n^{1/2}\widehat{W}_n - T(n^{-1/2}W_n)\|_\alpha$ as

$$\|n^{1/2}\widehat{W}_n - T(n^{-1/2}W_n)\|_\alpha \leqslant |\Phi_n - \widetilde{\Phi}_n| \big\|F(n^{-1/2}W_n) + \widetilde{\widetilde{R}}_n\big\|_\alpha + |\Phi_n| \, \big\|\widetilde{\widetilde{R}}_n\big\|_\alpha.$$

By continuous mapping, $\big(N(n^{-1/2}W_n), D(n^{-1/2}W_n)\big)$ converges in distribution in $\mathbb{R}^2$ to $\big(N(W), D(W)\big) = (B, A)$. In view of $P\big(D(W) = 0\big) = 0$, Lemma 3.1 gives us the convergence in distribution of Φ_n to B/A, and, in particular, Φ_n is stochastically bounded.

Since $\big\|\widetilde{\widetilde{R}}_n\big\|_\alpha$ converges to zero in probability and $\|F(n^{-1/2}W_n)\|_\alpha$ is stochastically bounded, it remains only to check that $|\Phi_n - \widetilde{\Phi}_n|$ converges to zero in probability.

Note that

$$|\Phi_n - \widetilde{\Phi}_n| \leqslant \frac{|R_n|}{|D(n^{-1/2}W_n) + \widetilde{R}_n|} + \left|\frac{N(n^{-1/2}W_n)}{D(n^{-1/2}W_n)}\right| \cdot \frac{|\widetilde{R}_n|}{|D(n^{-1/2}W_n) + \widetilde{R}_n|}.$$

So the problem reduces to proving that

$$\frac{|R_n|}{|D(n^{-1/2}W_n) + \widetilde{R}_n|} \xrightarrow[n\to\infty]{\mathrm{P}} 0 \quad \text{and} \quad \frac{|\widetilde{R}_n|}{|D(n^{-1/2}W_n) + \widetilde{R}_n|} \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

But these two convergences can be easily checked, so finally the convergence (3.14) is established.

The next step is to prove the necessity. By (3.14), the sequence $(n^{-1/2}\widehat{W}_n)$ is tight on $\mathrm{H}^o_\alpha[0,1]$, and this implies that, for every $\epsilon > 0$,

$$\lim_{\delta\to 0} \sup_{n\geqslant 1} P\big(\omega_\alpha(n^{-1/2}\widehat{W}_n, \delta) > \epsilon\big) = 0,$$

see, e.g., Theorem 13 in [24]. This clearly entails that $\omega_\alpha(n^{-1/2}\widehat{W}_n, 1/n) \xrightarrow[n\to\infty]{\mathrm{P}} 0$. Since

$$\frac{n^{-1/2}\max_{1\leqslant k\leqslant n}|\widehat{\varepsilon}_k|}{n^{-\alpha}} = n^{-1/2+\alpha}\max_{1\leqslant k\leqslant n}\left|\widehat{W}_n\left(\frac{k}{n}\right) - \widehat{W}_n\left(\frac{k-1}{n}\right)\right| \leqslant \omega_\alpha\left(n^{-1/2}\widehat{W}_n, \frac{1}{n}\right),$$

we obtain $n^{-1/2}\max_{1\leqslant k\leqslant n}|\widehat{\varepsilon}_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0$.

Next decompose $\widehat{\varepsilon}_k = \varepsilon_k - (\widehat{\phi_n} - \phi_n)y_{n,k-1}$. Denote by $y_{n,[n\bullet]}$ the step process $(y_{n,[nt]}, t \in [0,1])$. Recall that, by [18], Lemma 1, part (a), $n^{-1/2}y_{n,[n\bullet]}$ converges in distribution in $\mathrm{D}[0,1]$ to an Ornstein–Uhlenbeck process. As the supremum norm of such a step process is obviously reached at one of the points $t = k/n$, $0 \leqslant k \leqslant n$, this convergence implies the stochastic boundedness of

$$\max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1}| = \|n^{-1/2}y_{n,[n\bullet]}\|_\infty.$$

Notice that

$$n^{\alpha-1/2}\max_{1\leqslant k\leqslant n}|(\widehat{\phi_n} - \phi_n)y_{n,k-1}| \leqslant n^{\alpha-1}|n(\widehat{\phi_n} - \phi_n)|\max_{1\leqslant k\leqslant n}|n^{-1/2}y_{n,k-1}| \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

because by [18] (Theorem 1, part (a)) $|n(\widehat{\phi_n} - \phi_n)|$ is also stochastically bounded. It follows then that

$$n^{\alpha-1/2}\max_{1\leqslant k\leqslant n}|\varepsilon_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0,$$

which gives the condition (2.3) by the independence of (ε_k). ■

4. SECOND TYPE MODEL

For the second type model we obtain the result of the convergence of $n^{-1/2}\widehat{W}_n$ to Wiener process in $\mathrm{H}^o_\beta[0,1]$ for $0 < \beta \leqslant \alpha$, assuming additionally some rate of divergence for γ_n.

THEOREM 4.1. *Suppose* $(y_{n,k})$ *is generated by* (1.1) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of nonnegative numbers*, $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$. *Assume also that the innovations* (ε_k) *are i.i.d. with mean zero*, $\mathbb{E}\varepsilon_1^2 = \sigma^2$, *and satisfy condition* (2.3): $\lim_{t\to\infty} t^p\mathbb{P}(|\varepsilon_0| > t) = 0$ *for some* $p > 2$. *Put* $\alpha = 1/2 - 1/p$. *Then, for* $0 < \beta \leqslant \alpha$,

$$n^{-1/2}\sigma^{-1}\widehat{W}_n \xrightarrow[n\to\infty]{\mathcal{D}} W \quad \text{in } \mathrm{H}^o_\beta[0,1] \tag{4.1}$$

if $y_{n,0} = 0$ *and*

$$\liminf_{n\to\infty} \gamma_n n^{-2\beta/(1+2\alpha)} > 0. \tag{4.2}$$

REMARK 4.1. *If variance is unknown, by Slutsky's theorem it can be replaced in* (4.1) *by its estimator defined by* (3.17) *via Lemma* 4.1 *below.*

Proof of Theorem 4.1. Condition (2.3) (see [21]) gives the weak convergence of W_n, defined by (2.1), to the standard Brownian motion in the space $\mathrm{H}^o_\alpha[0,1]$. By continuous embedding of Hölder spaces, the same convergence remains true in $H^0_\beta[0,1]$ for $0 < \beta \leqslant \alpha$. Therefore, to obtain (4.1) it suffices to prove that

$$\Delta_{n,\beta} := \|n^{-1/2}\widehat{W}_n - n^{-1/2}W_n\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

We first establish the useful inequality

$$\|S_n\|_\beta \leqslant \frac{n}{\gamma_n}[\|W_n\|_\beta + 2n^\beta \max_{1\leqslant k\leqslant n} |y_{n,k}|], \tag{4.3}$$

where S_n is defined by (3.19). We have, for $1 \leqslant j < k \leqslant n$,

$$S_n(k/n) - S_n(j/n) = (1-\phi_n)^{-1}\big(W_n(k/n) - W_n(j/n) - y_{n,k} + y_{n,j}\big).$$

Recalling that the Hölder norm of a polygonal line is reached at some pair of vertices (see Lemma A.2 in [14]) and that $S_n(0) = 0$, we have

$$\begin{aligned}
\|S_n\|_\beta &= \max_{1\leqslant j<k\leqslant n} \frac{\big|(1-\phi_n)^{-1}\big(W_n(k/n) - W_n(j/n) - y_{n,k} + y_{n,j}\big)\big|}{|k/n - j/n|^\beta} \\
&\leqslant \frac{n}{\gamma_n}\left[\max_{1\leqslant j<k\leqslant n} \frac{|W_n(k/n) - W_n(j/n)|}{|k/n - j/n|^\beta} + \max_{1\leqslant j<k\leqslant n} \frac{|y_{n,k} - y_{n,j}|}{|k/n - j/n|^\beta}\right] \\
&= \frac{n}{\gamma_n}\left[\|W_n\|_\beta + \max_{1\leqslant j<k\leqslant n} \frac{|y_{n,k} - y_{n,j}|}{|k/n - j/n|^\beta}\right].
\end{aligned}$$

This leads to (4.3) via the elementary estimate

$$\max_{1\leqslant j<k\leqslant n} \frac{|y_{n,k} - y_{n,j}|}{|k/n - j/n|^\beta} \leqslant 2n^\beta \max_{1\leqslant k\leqslant n} |y_{n,k}|. \tag{4.4}$$

Note that $\widehat{W}_n = W_n + (\phi_n - \widehat{\phi}_n)S_n$, see (3.18), thus we have

$$\Delta_{n,\beta} = n^{-1/2}|\phi_n - \widehat{\phi}_n|\,\|S_n\|_\beta.$$

By the results in [6], there is a positive random variable M not depending on n, such that $|\phi_n - \widehat{\phi}_n| \leqslant Mn^{-1}\gamma_n^{1/2}$, so using (4.3), we can bound $\Delta_{n,\beta}$ as follows:

$$\Delta_{n,\beta} \leqslant Mn^{-1/2}\gamma_n^{-1/2}(\|W_n\|_\beta + 2n^\beta \max_{1\leqslant k\leqslant n} |y_{n,k}|).$$

As $n^{-1/2}\left\|W_n\right\|_\beta$ is stochastically bounded, the proof of the theorem is finally reduced to checking that

$$n^{-1/2+\beta}\gamma_n^{-1/2}\max_{1\leqslant k\leqslant n}|y_{n,k}|\xrightarrow[n\to\infty]{\mathrm{P}}0.$$

By Lemma 1 in [14], $\max_{1\leqslant k\leqslant n}|y_{n,k}|=o_P(n^{1/2}\gamma_n^{-\alpha})$, so the above convergence holds provided that

$$\limsup_{n\to\infty}\frac{n^\beta}{\gamma_n^{1/2+\alpha}}<\infty,$$

which is equivalent to assumption (4.2). ∎

Next we show that, for the second type model defined by (1.1), the estimate of variance is consistent.

LEMMA 4.1. *Suppose* $(y_{n,k})$ *is generated by* (1.1) *and* $\phi_n=1-\gamma_n/n$, *where* γ_n *is a sequence of nonnegative numbers*, $\gamma_n/n\to 0$ *and* $\gamma_n\to\infty$ *as* $n\to\infty$. *Assume also that the innovations* (ε_k) *are i.i.d. random variables with* $\mathbb{E}\varepsilon_k=0$, $\mathbb{E}\varepsilon_k^2=\sigma^2$. *The variance estimator* $\widehat{\sigma}^2$ *is defined by* (3.17). *Then*

$$\widehat{\sigma}^2\xrightarrow[n\to\infty]{\mathrm{P}}\sigma^2. \tag{4.5}$$

Proof. We can rearrange (3.17), using (1.3), in the following way:

$$\widehat{\sigma}^2=\frac{1}{n}\sum_{k=1}^n\widehat{\varepsilon_k}^2=\frac{1}{n}\sum_{k=1}^n\varepsilon_k^2-\frac{2}{n}(\widehat{\phi}_n-\phi_n)\sum_{k=1}^n\varepsilon_k y_{n,k-1}$$
$$+\frac{1}{n}(\widehat{\phi_n}-\phi_n)^2\sum_{k=1}^n y_{n,k-1}^2.$$

By the weak law of large numbers we have

$$\frac{1}{n}\sum_{k=1}^n\varepsilon_k^2\xrightarrow[n\to\infty]{\mathrm{P}}\sigma^2. \tag{4.6}$$

Using (5) and (10) of Giraitis and Phillips [6] for $\frac{1}{n}(\widehat{\phi_n}-\phi_n)^2\sum_{k=1}^n y_{n,k-1}^2$, we obtain

$$\frac{1}{n}(\widehat{\phi_n}-\phi_n)^2\sum_{k=1}^n y_{n,k-1}^2\xrightarrow[n\to\infty]{\mathrm{P}}0. \tag{4.7}$$

Also, for $\frac{2}{n}(\widehat{\phi}_n-\phi_n)\sum_{k=1}^n\varepsilon_k y_{n,k-1}$, by (5) and (9) in Giraitis and Phillips [6], we find

$$\frac{2}{n}(\widehat{\phi}_n-\phi_n)\sum_{k=1}^n\varepsilon_k y_{n,k-1}\xrightarrow[n\to\infty]{\mathrm{P}}0. \tag{4.8}$$

Thus (4.6)–(4.8) give us (4.5). ∎

5. APPLICATIONS

The Hölderian limit theorems provide applications via continuous mapping. To be more precise, we will formulate two general corollaries.

COROLLARY 5.1. *Under the conditions of Theorem* 3.1, *for any continuous function* $F : \mathrm{H}^o_\alpha[0,1] \mapsto \mathbb{R}$, $\alpha \in (0, 1/2)$,

$$F(n^{-1/2}\sigma^{-1}\widehat{W}_n) \xrightarrow[n\to\infty]{\mathcal{D}} F(W - A^{-1}BJ).$$

COROLLARY 5.2. *Under the conditions of Theorem* 4.1, *for any continuous function* $\widetilde{F} : \mathrm{H}^o_\beta[0,1] \mapsto \mathbb{R}$, $\beta \in (0, 1/2 - 1/p)$, $p > 2$,

$$\widetilde{F}(n^{-1/2}\sigma^{-1}\widehat{W}_n) \xrightarrow[n\to\infty]{\mathcal{D}} \widetilde{F}(W).$$

5.1. Epidemic change detection. In this section we give an application of functional limit theorems to the epidemic type change problem. The epidemic change in parameter θ is described as the change at some unknown time or location such that $\theta_1 = \ldots = \theta_{k^*} = \theta_{m^*+1} = \theta_n = \theta_0$ and $\theta_{k^*+1} = \ldots = \theta_{m^*} = \theta$. That is, at some point of time or location the value of the unknown parameter changes, but after a certain period it returns to an initial value. Epidemic change detection is a widely investigated question. We refer to Levin and Kline [13], Commenges et al. [5], Broemeling and Tsurumu [4], Gombay [7], Avery and Henderson [1], etc., for more information. The Hölderian framework for epidemic change was introduced by Račkauskas and Suquet [21] for i.i.d. random variables. They showed that the Hölderian weighting allows us to detect epidemics shorter than $\sqrt{n}$.

We implement this setting for the innovations of the first order nearly nonstationary autoregressive process. Assume we are given a sample $y_{n,1}, \ldots, y_{n,n}$ for a fixed n, generated from the first order autoregressive process

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k + a_{n,k}, \quad k = 1, \ldots, n,\ n \geqslant 1, \quad y_{n,0} = 0, \tag{5.1}$$

where $\phi_n = e^{\gamma/n}$, $\gamma < 0$ is a constant, or $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of nonnegative numbers, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$. The innovations $(\varepsilon_k, k \leqslant n)$ are unobservable, centered, square integrable random variables.

The aim is to propose tests for the null hypothesis

$$H_0 : a_{n,1} = \ldots = a_{n,n} = 0$$

against the epidemic alternative

$$\begin{aligned} H_A : \ & \text{there exist } 0 \leqslant k_n^* < n, \text{ and } 1 \leqslant m_n^* \leqslant n \text{ such that} \\ & a_{n,k} = a_n \neq 0 \text{ for } k \in \mathbb{I}_n^*, \text{ whereas } a_{n,k} = 0 \text{ for } k \notin \mathbb{I}_n^*, \end{aligned}$$

where $\mathbb{I}_n^* = \{k_n^* + 1, \ldots, m_n^*\}$. The value a_n during the period $\mathbb{I}_n^*$ is interpreted as an epidemic deviation from the usual (zero) mean of innovations, and $\ell_n^* = m_n^* - k_n^*$ is the duration of the epidemic state. Note that Markevičiūtė et al. [15] investigated uniform increments statistics built on observations for the nearly nonstationary process. Since innovations are not observed, we may build a uniform increments statistics also on least squares residuals. Set for $\alpha \in [0, 1)$

$$\widehat{T}_{\alpha,n} = T_{\alpha,n}(\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n) = \max_{1 \leqslant \ell \leqslant n} \ell^{-\alpha} \max_{1 \leqslant k \leqslant n-\ell} \left| \sum_{j=k+1}^{k+\ell} \widehat{\varepsilon}_j - \frac{\ell}{n} \sum_{j=1}^{n} \widehat{\varepsilon}_j \right|, \tag{5.2}$$

where $(\widehat{\varepsilon}_k)$ are residuals of the model (5.1) defined by (1.3) and (1.4).

Using the Hölderian limit theorems, it is easy to find the limit of test statistics (5.2) under the null hypothesis. Note that the test statistics for the second type model with different approach is investigated in detail in Markevičiūtė et al. [16], so we find the limit under the null hypothesis only for the first type model, and we assume that all $a_{n,k} = 0$.

THEOREM 5.1. *In the first type model defined by* (1.1) *and* $\phi_n = e^{\gamma/n}$, γ *is a negative constant. Assume that innovations satisfy* $\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0$ *for some* $p > 2$*. Then under* H_0 *for any* $\alpha \in (0, 1/2 - 1/p)$

$$n^{-1/2+\alpha} \sigma^{-1} \widehat{T}_{\alpha,n} \xrightarrow[n\to\infty]{\mathcal{D}} T_{\alpha,\infty}(Z), \tag{5.3}$$

where $\sigma^2 = \mathbb{E}\varepsilon_1^2$, *and*

$$Z(t) = W(t) - A^{-1} B J(t). \tag{5.4}$$

P r o o f. Using Lemma A.1 in [15] and Theorem 3.1, we immediately obtain the result. ∎

5.1.1. *Test power analysis.* In this section we perform the test power analysis. The results are presented in Tables 1 and 2. We compute empirical power on the size-adjusted (not nominal size) basis, i.e., we replace the nominal value of significance level by the value of empirical distribution function for p-values under the null hypothesis.

Here we compute $N = 1000$ realizations of test statistics with the sample size n for different values of parameters γ, γ_n, α, k^*, ℓ^* and a_n. Innovations (ε_j) are generated as standard normally distributed random variables. For the limit distribution we compute $N = 5000$ realizations of test statistics with the sample size $n = 5000$. We approximate the values of the standard Wiener process by

$$W\left(\frac{k}{5000}\right) = 5000^{-1/2} \sum_{j=1}^{k} \varepsilon(j), \quad k = 1, \ldots, 5000.$$

TABLE 1. Empirical power at the size-adjusted significance level 0.05 for the first type model with Gaussian innovations

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 12.5/32$
$\ell^*/n = 0.035$	0.462	0.715	0.968
$\ell^*/n = 0.050$	0.879	0.981	0.998
$\ell^*/n = 0.065$	0.988	1.000	1.000
$k^*/n = 0.2$	0.903	0.981	1.000
$k^*/n = 0.4$	0.879	0.981	0.998
$k^*/n = 0.8$	0.784	0.967	0.997
$a_n = 0.8$	0.574	0.793	0.957
$a_n = 1$	0.879	0.981	0.998
$a_n = 1.2$	0.989	1.000	1.000
$n = 500$	0.498	0.700	0.884
$n = 1000$	0.879	0.981	0.998
$n = 2000$	1.000	1.000	1.000
$\gamma = -2$	0.879	0.981	0.998
$\gamma = -12$	0.831	0.976	0.998
$\gamma = -100$	0.010	0.267	0.975

TABLE 2. Empirical power at the size-adjusted significance level 0.05 for the second type model with Gaussian innovations

Parameters	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 10/32$
$\ell^*/n = 0.035$	0.049	0.190	0.763
$\ell^*/n = 0.050$	0.093	0.573	0.965
$\ell^*/n = 0.065$	0.216	0.880	0.998
$k^*/n = 0.2$	0.077	0.589	0.974
$k^*/n = 0.4$	0.093	0.573	0.965
$k^*/n = 0.8$	0.105	0.615	0.974
$a_n = 0.8$	0.102	0.328	0.791
$a_n = 1$	0.093	0.573	0.965
$a_n = 1.2$	0.080	0.810	1.000
$n = 500$	0.062	0.171	0.552
$n = 1000$	0.093	0.573	0.965
$n = 2000$	0.660	0.997	1.000
$\gamma_n = n/\ln(n)$	0.035	0.416	0.950
$\gamma_n = \ln^{2.5}(n)$	0.020	0.353	0.935
$\gamma_n = n^{3/4}$	0.093	0.573	0.965

The Ornstein–Uhlenbeck process has been approximated by the following discretization:

$$S(j) = S(j-1)e^{\gamma/n} + \sqrt{\frac{1-e^{2\gamma/n}}{-2\gamma}} \cdot \varepsilon(j), \quad \varepsilon(j) \sim \mathfrak{N}(0,1). \tag{5.5}$$

Using values generated by (5.5), we approximate the integrated Ornstein–Uhlenbeck process by

$$J\left(\frac{k}{5000}\right) = 5000^{-1}\sum_{j=1}^{k} S(j), \quad k = 1, \dots, 5000,$$

and values

$$A = 5000^{-1}\sum_{j=1}^{n} S^2(j), \quad B = \sum_{j=1}^{n} S(j)\left(W\left(\frac{j}{5000}\right) - W\left(\frac{j-1}{5000}\right)\right).$$

For the first type model ($\phi_n = e^{\gamma/n}$) with innovations that satisfy the integrability condition (2.3) the basic parameters are

$$\gamma = -2, \quad a_n = 1, \quad n = 1000, \quad \frac{\ell^*}{n} = 0.05, \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

We modify them separately and compute the empirical size-power. We keep all the parameters fixed except one (indicated in the first column in both tables), which is allowed to vary.

As one can see in Table 1 the test power increases with the α. The test statistics has a quite big power in detecting short epidemics with α closer to $1/2$. Naturally, increasing n increases the test power. In general, the test has a quite big power for all chosen parameters.

For the second type model ($\phi_n = 1 - \gamma_n/n$) with innovations that satisfy the integrability condition (2.3), the basic parameters are

$$\gamma_n = n^{3/4}, \quad a_n = 1, \quad n = 1000, \quad \frac{\ell^*}{n} = 0.05, \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

For the second type model (Table 2), the test power is very low for the small α. The test power increases with n, ℓ^* and the rate of divergence of γ_n.

5.2. Comparison with other test statistics. Table 3 shows size-adjusted test power for statistics $\widetilde{T}_{\alpha,n} = T_{\alpha,n}(y_{n,1}, \dots, y_{n,n})$ (see [15]) and statistics $\widehat{T}_{\alpha,n} = T_{\alpha,n}(\widehat{\varepsilon}_1, \dots, \widehat{\varepsilon}_n)$ (see also [16]), where innovations satisfy the integrability condition (2.3). The result shows that with different parameters for the second type model, both statistics give opposite results. In this example, statistics $\widehat{T}_{\alpha,n}$ with $\gamma_n = n^{0.45}$ detects epidemics better, while statistics $\widetilde{T}_{\alpha,n}$ performs better with $\gamma_n = n^{0.8}$.

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TABLE 3. Comparison of the statistics $\widetilde{T}_{\alpha,n}$ and $\widehat{T}_{\alpha,n}$ for the first type model (model I) and for the second type model (model II)

$a_n = 1, \ell^* = 30, k^* = 400, n = 1000, \gamma = -2, \gamma_n = n^{0.45}$				
		$\alpha_1 = 0.0625$	$\alpha_2 = 0.1875$	$\alpha_3 = 0.39$ (model I) $\alpha_3 = 0.31$ (model II)
$\widetilde{T}_{\alpha,n}$	model I	0.318	0.327	0.306
	model II	0.276	0.330	0.429
$\widehat{T}_{\alpha,n}$	model I	0.335	0.526	0.914
	model II	0.061	0.452	0.836
$a_n = 1, \ell^* = 30, k^* = 400, n = 1000, \gamma = -20, \gamma_n = n^{0.8}$				
$\widetilde{T}_{\alpha,n}$	model I	0.280	0.322	0.467
	model II	0.314	0.505	0.796
$\widehat{T}_{\alpha,n}$	model I	0.088	0.502	0.913
	model II	0.073	0.213	0.682

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