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BOUNDARY VALUE PROBLEMS FOR THE DUNKL LAPLACIAN

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Abstract. Let Δ_k be the Dunkl Laplacian on $\mathbb{R}^d$ associated with a reflection group W and a multiplicity function k. The purpose of this paper is to establish the existence and the uniqueness of a positive solution on the unit ball B of $\mathbb{R}^d$ to the following boundary value problem:

$$\Delta_k u = \varphi(u) \ \text{in} \ B \quad \text{and} \quad u = f \ \text{on} \ \partial B.$$

We distinguish two cases of nonnegative perturbation φ: trivial and nontrivial.

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1. INTRODUCTION

The Dunkl Laplacian is the sum of a second order differential operator and a difference term associated with a multiplicity function k and a reflection group W. An important motivation to study the Dunkl Laplacian rises from its relevance for the analysis of certain exactly solvable models of mechanics, namely the Calogero–Moser–Sutherland type (see [5], [13], [19]). Since its introduction by C. F. Dunkl in [6], the analysis of Dunkl theory has been the subject of many articles and it has deep and fruitful interactions with various mathematics fields, namely Fourier analysis and special functions [15], [28], [29], algebra (double affine Hecke algebras [17]) and probability theory (Feller processes with jumps [11], [4]). The Dunkl Laplacian generates a positive strongly continuous contraction semigroup [25]. This fact gives rise to a Hunt process, called a Dunkl process, and so to a corresponding family of harmonic kernels $(H_V)_V$. If the multiplicity function k is identically zero, then the operator Δ_k reduces to the classical Laplace operator Δ, and so the Dunkl process is the Brownian motion and $H_V(x,\cdot)$ is the classical harmonic measure relative to V and x. If k is not trivial, then paths of the Dunkl process are discontinuous (see [11]), and thus it follows from the general theory of balayage spaces [1] that Δ_k generates a balyage space and not a harmonic space. This yields that for every bounded open set V and every $x \in V$ the harmonic measure $H_V(x,\cdot)$ is not necessarily supported by the Euclidean boundary ∂V of V, as in the classical setting $k = 0$, but it may live on the entire complement $V^c := \mathbb{R}^d \backslash V$.

Throughout this paper we assume that k is strictly positive. Our first purpose is to show that, for every bounded open subset V of $\mathbb{R}^d$ and every $x \in V$, the harmonic measure $H_V(x,\cdot)$ is supported by a compact set of V^c and not by the whole V^c. In the particular case where V is invariant under the reflection group W (e.g. V is an open ball of $\mathbb{R}^d$ centered at the origin), we shall prove that the support of $H_V(x,\cdot)$ is contained in ∂V. This fact allows us to investigate, for an open ball B of center zero, the boundary value problem

$$\begin{cases} \Delta_k u = \varphi(u) & \text{in } B, \\ u = f & \text{on } \partial B, \end{cases} \tag{1.1}$$

where f is a nonnegative continuous function on ∂B. We impose that $\varphi : [0,\infty[\to [0,\infty[$ is nondecreasing, continuous and satisfies $\varphi(0) = 0$. Our main goal is to establish the existence and the uniqueness of a positive solution to problem (1.1). We distinguish two cases of perturbation φ (trivial and nontrivial). In the first step, we consider $\varphi = 0$ and we prove that the function $H_B f$ defined on B by

$$H_B f(x) = \int_{\partial B} f(y) H_B(x, dy)$$

is the unique continuous extension u of f on $\overline{B}$ satisfying $\Delta_k u = 0$ in B. That is, $H_B f$ is the unique solution of (1.1) for $\varphi = 0$. Assuming that φ is not trivial, we show that u satisfies (1.1) if and only if

$$u + G_B^k\big(\varphi(u)\big) = H_B f,$$

where G_B^k is the Green operator on B. Then, by a compactness argument of G_B^k, we prove that the map $u \mapsto H_B f - G_B^k\big(\varphi(u)\big)$ admits one and only one fixed point $u \in C(\overline{B})$, and so u is the unique solution of problem (1.1).

2. NOTATION AND PRELIMINARIES

For every subset F of $\mathbb{R}^d$, let $\mathcal{B}(F)$ be the set of all Borel-measurable functions on F and let 1_F be the indicator function of F. Let $C(F)$ be the set of all continuous real-valued functions on F, $C^n(F)$ be the class of all functions that are n times continuously differentiable on F, and $C_0(F)$ be the set of all continuous functions on F such that $u = 0$ on ∂F, which means that $\lim_{x\to z} u(x) = 0$ for all $z \in \partial F$ and $\lim_{x\to\infty} u(x) = 0$ if F is unbounded. We denote by $\mathcal{C}_c^\infty(F)$ the set of all infinitely differentiable functions on F with compact support. If $\mathcal{G}$ is a set of numerical functions, then $\mathcal{G}^+$ (respectively $\mathcal{G}_b$) will denote the class of all functions in $\mathcal{G}$ which are nonnegative (respectively bounded). The uniform norm will be denoted by $\|\cdot\|$.

For every $\alpha \in \mathbb{R}^d \setminus \{0\}$, let H_α be the hyperplane orthogonal to α and let σ_α be the reflection in H_α, i.e.,

$$\sigma_\alpha(x) := x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $\langle \cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^d$ and $|\cdot|$ is the associated norm. A finite subset R of $\mathbb{R}^d \setminus \{0\}$ is called a *root system* if $R \cap \mathbb{R}\cdot\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R, the reflection $\sigma_\alpha,\ \alpha \in R$, generates a finite group W called a *reflection group* associated with R. A function $k : R \to \mathbb{R}_+$ is called a *multiplicity function* if it satisfies $k(\sigma_\alpha\beta) = k(\beta)$ for every $\alpha, \beta \in R$. Throughout this paper we fix a root system R and a multiplicity function k. We consider the differential-difference operators $T_i,\ 1 \leqslant i \leqslant d$, defined in [7] for every $u \in C^1(\mathbb{R}^d)$ by

$$T_i u(x) = \frac{\partial u}{\partial x_i}(x) + \frac{1}{2}\sum_{\alpha\in R} k(\alpha)\alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle},$$

and called *Dunkl operators* in the literature. The Dunkl Laplacian Δ_k is the sum of squares of Dunkl operators:

$$\Delta_k := \sum_{i=1}^{d} T_i^2.$$

It is given explicitly, for $u \in C^2(\mathbb{R}^d)$, by

$$\Delta_k u(x) = \Delta u(x) + \sum_{\alpha\in R} k(\alpha)\left(\frac{\langle \nabla u(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{|\alpha|^2}{2}\frac{u(x) - u\big(\sigma_\alpha(x)\big)}{\langle \alpha, x\rangle^2}\right). \tag{2.1}$$

Likewise the classical Laplace operator Δ, the Dunkl Laplacian has the following symmetry property: For $u \in C^2(\mathbb{R}^d)$ and $v \in C_c^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Delta_k u(x) v(x) w_k(x)\, dx = \int_{\mathbb{R}^d} u(x)\Delta_k v(x) w_k(x)\, dx, \tag{2.2}$$

where w_k is the homogeneous weight function defined on $\mathbb{R}^d$ by

$$w_k(x) = \prod_{\alpha\in R} |\langle x, \alpha\rangle|^{k(\alpha)}.$$

A fundamental result in Dunkl theory is the existence of an intertwining operator $V_k : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ between the classical Laplacian Δ and Dunkl Laplacian, i.e., $\Delta_k V_k = V_k \Delta$. We refer to [8], [26], [28] for more details about the intertwining operator. By means of V_k, there exists a counterpart of the usual exponential function, called a *Dunkl kernel* $E_k(\cdot,\cdot)$, which is defined for every $y \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$ by

$$E_k(x,y) = V_k(e^{\langle \cdot, y\rangle})(x).$$

It is clear from (2.1) that if k vanishes identically, then the Dunkl Laplacian reduces to the classical Laplacian Δ. In this case the intertwining operator V_k is the identity operator, and so E_k reduces to the classical exponential function. Notice that E_k is symmetric and positive on $\mathbb{R}^d \times \mathbb{R}^d$ and satisfies $E_k(\lambda y, x) = E_k(y, \lambda x) =$ for every $\lambda \in \mathbb{C}$.

In all this paper we assume that

$$m := d + \sum_{\alpha \in R} k(\alpha) > 2.$$

Let p_t^k be the Dunkl heat kernel, introduced in [25], defined for every $t > 0$ and every $x, y \in \mathbb{R}^d$ by

$$p_t^k(x,y) = \frac{c_k^2}{2^m} \int_{\mathbb{R}^d} e^{-t|\xi|^2} E_k(-ix,\xi) E_k(iy,\xi) w_k(\xi) d\xi, \tag{2.3}$$

where

$$c_k = \Big(\int_{\mathbb{R}^d} e^{-|y|^2} w_k(y)\, dy \Big)^{-1}.$$

For every $x, y \in \mathbb{R}^d$, $p_t^k(x,y) > 0$, $p_t^k(x,y) = p_t^k(y,x)$ and

$$p_t^k(x,y) \leqslant \frac{c_k}{(4t)^{m/2}} \exp\left(-\frac{(|x|-|y|)^2}{4t} \right). \tag{2.4}$$

Also, for every $x \in \mathbb{R}^d$, the function $(t,y) \mapsto p_t^k(x,y)$ solves the generalized heat equation $\partial_t u - \Delta_k u = 0$ on $]0,\infty[\times \mathbb{R}^d$. More precisely, the following holds:

$$\frac{\partial}{\partial t} p_t^k(x,y) = \Delta_k\big(p_t^k(\cdot,y)\big)(x) = \Delta_k\big(p_t^k(x,\cdot)\big)(y). \tag{2.5}$$

For every $f \in C_0(\mathbb{R}^d)$ and $t > 0$ let

$$P_t^k f(x) = \int_{\mathbb{R}^d} p_t^k(x,y) f(y) w_k(y)\, dy, \qquad x \in \mathbb{R}^d.$$

Then $(P_t^k)_{t>0}$ forms a positive strongly continuous contraction semigroup on $C_0(\mathbb{R}^d)$ of generator Δ_k. This fact yields the existence of a Hunt process (X_t, P^x) (see [2], Theorem I.9.4), called the Dunkl process, with state space $\mathbb{R}^d$ and transition kernel

$$P_t^k(x,dy) = p_t^k(x,y) w_k(y)\, dy.$$

3. HARMONIC KERNELS

For every bounded open subset D of $\mathbb{R}^d$, we denote by τ_D the first exit time from D by (X_t), i.e.,

$$\tau_D = \inf\{t > 0; X_t \notin D\}.$$

LEMMA 3.1. *Let D be a bounded open set. Then, for every $x \in D$,*

$$P^x\,(0 < \tau_D < \infty) = 1.$$

P r o o f. Let $x \in D$. Since the Dunkl process has right continuous paths, we immediately conclude that $P^x(0 < \tau_D) = 1$. Let $r > 0$ be such that $D \subset B_r$, the ball of center zero and radius r. Clearly,

$$\begin{aligned} E^x[\tau_D] \leqslant E^x[\tau_{B_r}] &= E^x\Big[\int_0^{\tau_{B_r}} 1_{B_r}(X_t)\,dt\Big] \\ &\leqslant \int_0^\infty E^x[1_{B_r}(X_t)]dt = \int_0^\infty \int_{B_r} p_t^k(x,y)w_k(y)\,dy\,dt. \end{aligned}$$

So, to prove that $P^x(\tau_D < \infty) = 1$, it will be sufficient to show that

$$\int_0^\infty \int_{B_r} p_t^k(x,y)w_k(y)\,dy\,dt < \infty.$$

Using spherical coordinates and applying the fact that the function w_k is homogeneous of degree $m - d$, we infer from the integral representation (2.3) of p_t^k that, for every $y \in \mathbb{R}^d$,

$$p_t^k(x,y) = \frac{c_k^2}{2^m}\int_0^\infty \int_{S^{d-1}} e^{-ts^2}E_k(-ix, s\xi)E_k(iy, s\xi)w_k(\xi)s^{m-1}\sigma(d\xi)ds,$$

where σ denotes the surface area measure on the unit sphere S^{d-1} of $\mathbb{R}^d$. Therefore,

$$\int_0^\infty p_t^k(x,y)dt = \frac{c_k^2}{2^m}\int_0^\infty \int_{S^{d-1}} E_k(-ix, s\xi)E_k(iy, s\xi)w_k(\xi)s^{m-3}\sigma(d\xi)ds.$$

Using again spherical coordinates and then applying Fubini's theorem, we get

$$\begin{aligned} \int_0^\infty \int_{B_r} p_t^k(x,y)w_k(y)\,dy\,dt &= \int_0^r \int_{S^{d-1}} \Big(\int_0^\infty p_t^k(x,uy)dt\Big)w_k(y)u^{m-1}\sigma(dy)du \\ &= \frac{c_k^2}{2^m}\int_0^r\int_0^\infty \int_{S^{d-1}} \Big(\int_{S^{d-1}} E_k(iuy, s\xi)w_k(y)\sigma(dy)\Big) \\ &\quad \times E_k(-ix, s\xi)w_k(\xi)s^{m-3}u^{m-1}\sigma(d\xi)dsdu. \end{aligned}$$

On the other hand, we recall from [27] that

$$\int_{S^{d-1}} E_k(iz,y)w_k(y)\sigma(dy) = 2^{m/2}c_k^{-1}\frac{J_{m/2-1}(|z|)}{|z|^{m/2-1}},$$

where $J_{m/2-1}$ is the Bessel function of index $m/2-1$ given by

$$J_{m/2-1}(z) := \left(\frac{z}{2}\right)^{m/2-1}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{4^n n!\Gamma(n+m/2)}.$$

Hence

$$\int_0^\infty\int_{B_r} p_t^k(x,y)w_k(y)\,dy\,dt$$
$$= \int_0^r \frac{u^{m-1}}{(u|x|)^{m/2-1}}\Big(\int_0^\infty J_{m/2-1}(s|x|)J_{m/2-1}(us)s^{-1}ds\Big)du,$$

and so

$$\int_0^\infty\int_{B_r} p_t^k(x,y)w_k(y)\,dy\,dt = \frac{1}{m-2}\int_0^r u^{m-1}\big(\max(u,|x|)\big)^{2-m}\,du \tag{3.1}$$
$$= \frac{1}{m-2}\left(\frac{|x|^2}{m}+\frac{r^2-|x|^2}{2}\right).$$

To get (3.1), one should use a formula from [21], p. 100. ■

For every bounded open set D, we define

$$^W\!D := \bigcup_{w\in W} w(D) \quad\text{and}\quad \Gamma_D := \overline{^W\!D}\setminus D.$$

That is, $^W\!D$ is the smallest open bounded set containing D which is invariant under the reflection group W. In the following theorem, we show that if the process starts from $x \in D$ then, at the first exit time from D, it should be in the compact Γ_D.

THEOREM 3.1. *Let D be a bounded open subset of $\mathbb{R}^d$. Then, for every $x\in D$,*

$$P^x\left(X_{\tau_D}\in\Gamma_D\right) = 1. \tag{3.2}$$

In particular, if D is W-invariant, i.e., $^W\!D = D$, then $\Gamma_D = \partial D$, and therefore

$$P^x\left(X_{\tau_D}\in\partial D\right) = 1.$$

Proof. Let $x \in D$ and consider the function F defined for every $y, z \in \mathbb{R}^d$ by $F(y,z) = 0$ if $z \in \{\sigma_\alpha y; \alpha\in R\}$ and $F(y,z) = 1$ otherwise. Let

$$Y_t := \sum_{s<t} 1_{\{X_{s-}\neq X_s\}}F(X_{s-},X_s), \quad t>0.$$

It follows from Proposition 3.2 in [11] that for every $t > 0$, $P^x(Y_t = 0) = 1$, and consequently

$$P^x\big(1_{\{X_{s^-}\neq X_s\}}F(X_{s^-}, X_s) = 0;\ \forall s > 0\big) = 1.$$

Then, since $P^x(0 < \tau_D < \infty) = 1$, we deduce that

$$P^x\big(1_{\{X_{\tau_D^-}\neq X_{\tau_D}\}}F(X_{\tau_D^-}, X_{\tau_D}) = 0\big) = 1.$$

On the other hand, since $X_{\tau_D^-} \in \overline{D}$ on $\{0 < \tau_D < \infty\}$, we have

$$\{X_{\tau_D} \notin \Gamma_D, 0 < \tau_D < \infty\} \subset \{1_{\{X_{\tau_D^-}\neq X_{\tau_D}\}}F(X_{\tau_D^-}, X_{\tau_D}) = 1\}.$$

This completes the proof. ∎

For every bounded open set D and every $x \in \mathbb{R}^d$, let $H_D(x, \cdot)$ be the harmonic measure relative to x and D, i.e., for every Borel set A,

$$H_D(x, A) := P^x(X_{\tau_D} \in A).$$

For every $f \in \mathcal{B}_b(\mathbb{R}^d)$, let $H_D f$ be the function defined on $\mathbb{R}^d$ by

$$H_D f(x) = \int f(y) H_D(x, dy).$$

Since, for $x \in D$, the harmonic measure $H_D(x, \cdot)$ is supported by the compact set Γ_D, it will be convenient to put again

(3.3) $$H_D f(x) = \int f(y) H_D(x, dy), \quad x \in D,$$

for every $f \in \mathcal{B}_b(\Gamma_D)$.

Let ${}^*\mathcal{H}^+(\mathbb{R}^d)$ denote the set of all nonnegative lower semicontinuous functions f on $\mathbb{R}^d$ such that

$$H_D f \leqslant f \quad \text{for every bounded open set } D.$$

Because $(\mathbb{R}^d, P^x)$ is a Hunt process, it follows from Theorem IV.8.1 in [1] that $\big(\mathbb{R}^d, {}^*\mathcal{H}^+(\mathbb{R}^d)\big)$ is a balayage space. Hence, it follows from the general theory of balayage spaces that for every $f \in \mathcal{B}_b(\Gamma_D)$

(3.4) $$H_D f \in C(D)$$

and

(3.5) $$H_V H_D f = H_D f \quad \text{on } V \text{ for all open sets } V \text{ such that } \overline{V} \subset D.$$

Furthermore, a function $f \in \mathcal{B}^+(\mathbb{R}^d)$ belongs to ${}^*\mathcal{H}^+(\mathbb{R}^d)$ if and only if

$$\sup_{t>0} P_t^k f = f.$$

Let us now introduce the Green function G^k of the Dunkl Laplacian which will play an important role in our approach. It is defined for every $x, y \in \mathbb{R}^d$ by

$$G^k(x,y) = \int_0^\infty p_t^k(x,y)dt.$$

For every $y \in \mathbb{R}^d$, the function $G_y^k := G^k(\cdot, y) \in {}^*\mathcal{H}^+(\mathbb{R}^d)$. Indeed, by the semi-group property,

$$P_t^k G_y^k(x) = \int_t^\infty p_s^k(x,y)ds \leqslant G^k(x,y).$$

This implies that the map $t \mapsto P_t^k G_y^k$ is decreasing on $]0, \infty[$, and so

$$\sup_{t>0} P_t^k G_y^k = \lim_{t\to 0} P_t^k G_y^k = G_y^k.$$

Hence $G_y^k \in {}^*\mathcal{H}^+(\mathbb{R}^d)$, which means that for every bounded open set D,

$$\int G^k(z,y) H_D(x,dz) \leqslant G^k(x,y). \tag{3.6}$$

Furthermore, it is obvious that G^k is positive and symmetric on $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, it follows from Theorem VI.1.16 in [2] that for every bounded open set D and every $x, y \in \mathbb{R}^d$,

$$\int G^k(x,z) H_D(y,dz) = \int G^k(y,z) H_D(x,dz). \tag{3.7}$$

4. DIRICHLET PROBLEM

Let B be an open ball of $\mathbb{R}^d$ of center zero and radius $r > 0$. We first introduce the following three kinds of harmonicity on B:

A continuous function $h : B \to \mathbb{R}$ is said to be

(i) Δ_k-*harmonic* on B if $h \in C^2(B)$ and $\Delta_k h(x) = 0$ for every $x \in B$.

(ii) X-*harmonic* on B if $H_D h(x) = h(x)$ for every bounded open set D such that $\overline{D} \subset B$ and every $x \in D$.

(iii) Δ_k-*harmonic on* B *in the distributional sense* if

$$\langle h, \Delta_k \varphi \rangle_k := \int_B h(x) \Delta_k \varphi(x) w_k(x) dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B).$$

LEMMA 4.1. *Let $f \in C_c^2(\mathbb{R}^d)$. For every $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} G^k(x,y)\Delta_k f(y) w_k(y) dy = -f(x). \tag{4.1}$$

In particular, for every bounded open set D and every $x \in D$,

$$H_D f(x) - f(x) = E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s) ds\Big]. \tag{4.2}$$

P r o o f. Let $x \in \mathbb{R}^d$. Using Fubini's theorem and formulas (2.2) and (2.5), we have

$$\begin{aligned}\int_{\mathbb{R}^d} G^k(x,y)\Delta_k f(y) w_k(y)\, dy &= \int_0^\infty \int_{\mathbb{R}^d} p_t^k(x,y)\Delta_k f(y) w_k(y)\, dy\, dt\\ &= \int_0^\infty \int_{\mathbb{R}^d} \Delta_k\big(p_t^k(x,\cdot)\big)(y) f(y) w_k(y)\, dy\, dt\\ &= \int_0^\infty \int_{\mathbb{R}^d} \Delta_k\big(p_t^k(\cdot,y)\big)(x) f(y) w_k(y)\, dy\, dt\\ &= \lim_{t\to\infty} P_t^k f(x) - \lim_{t\to 0} P_t^k f(x) = -f(x).\end{aligned}$$

To get $\lim_{t\to\infty} P_t^k f(x) = 0$, we only use (2.4) and the fact that f has compact support. Formula (4.2) follows from (4.1) and the strong Markov property. In fact, let D be a bounded open set and let $x \in D$. Then

$$\begin{aligned}&-f(x) = \int G^k(x,y)\Delta_k f(y) w_k(y) dy = \int_0^\infty \int p_t^k(x,y)\Delta_k f(y) w_k(y) dy dt\\ &= E^x\Big[\int_0^\infty \Delta_k f(X_s) ds\Big] = E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s) ds\Big] + E^x\Big[\int_{\tau_D}^\infty \Delta_k f(X_s) ds\Big]\\ &= E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s) ds\Big] + E^x\Big[E^{X_{\tau_D}}\Big[\int_0^\infty \Delta_k f(X_s) ds\Big]\Big]\\ &= E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s) ds\Big] + E^x\left[-f(X_{\tau_D})\right] = E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s) ds\Big] - H_D f(x). \quad ■\end{aligned}$$

LEMMA 4.2. *For every bounded open set D and for every $\varphi, \psi \in C_c^2(\mathbb{R}^d)$,*

$$\langle H_D\psi, \Delta_k\varphi\rangle_k = \langle \Delta_k\psi, H_D\varphi\rangle_k. \tag{4.3}$$

P r o o f. Applying formula (4.1) to ψ, we get

$$\langle H_D\psi, \Delta_k\varphi\rangle_k = -\int\int\int G^k(z,y)\Delta_k\psi(y) w_k(y) dy H_D(x,dz)\Delta_k\varphi(x) w_k(x) dx.$$

Then (4.3) is obtained by Fubini's theorem by using formula (3.7) and formula (4.1) applied to φ. ■

Now, we show that the three kinds of harmonicity on B introduced at the beginning of this section are equivalent.

THEOREM 4.1. *Let $h \in C(B)$. The following three assertions are equivalent*:
(i) *h is Δ_k-harmonic on B.*
(ii) *h is X-harmonic on B.*
(iii) *h is Δ_k-harmonic on B in the distributional sense.*

Proof. (i) Assume that h is Δ_k-harmonic on B. Let D be a bounded open set such that $\overline{D} \subset B$ and let $x \in D$. We claim that

$$H_D h(x) - h(x) = E^x\Big[\int_0^{\tau_D} \Delta_k h(X_s)ds\Big]. \tag{4.4}$$

Let V be a bounded open set such that $\overline{D} \subset V \subset \overline{V} \subset B$. By C^∞-Uryshon's lemma, there exists $\theta \in C_c^\infty(B)$ such that $\theta = 1$ on V. Let $f := h\theta$ and $\psi := h - f$. Obviously, $h = f$ on V, $\psi = 0$ on V and $f \in C_c^2(B)$. Then, using (4.2), we obtain

$$H_D h(x) - h(x) = E^x\Big[\int_0^{\tau_D} \Delta_k f(X_s)ds\Big] + H_D\psi(x). \tag{4.5}$$

For every $y \in \mathbb{R}^d$, let $N(y, dz)$ be the Lévy kernel of the Dunkl process X which is given in [11] by the following formula:

$$N(y, dz) = \sum_{\alpha \in \mathbb{R}_+, \langle y,\alpha\rangle \neq 0} \frac{k(\alpha)}{\langle \alpha, y\rangle^2} \delta_{\sigma_\alpha y}(dz). \tag{4.6}$$

Since $\psi = 0$ on V, it follows from Theorem 1 in [14] that

$$H_D\psi(x) = E^x\Big[\int_0^{\tau_D} \int \psi(z) N(X_s, dz)ds\Big]. \tag{4.7}$$

On the other hand, by (2.1) and (4.6) we easily see that for every $y \in D$,

$$\Delta_k f(y) = \Delta_k h(y) - \int \psi(z)N(y, dz). \tag{4.8}$$

Thus formula (4.4) is obtained by combining (4.5), (4.7) and (4.8). Hence, by (4.4), $H_D h(x) = h(x)$, and so h is X-harmonic on B.

(ii) Assume that h is X-harmonic on B. Let $\varphi \in C_c^\infty(B)$ and let $D \subset \overline{D} \subset B$ be a W-invariant bounded open set which contains the support of φ. Let $(h_n)_{n\geqslant 1} \subset C_c^2(B)$ be a sequence which converges uniformly to h on ∂D. Since $H_D\varphi = 0$ on D, applying (4.3), we obtain

$$\langle H_D h_n, \Delta_k\varphi\rangle_k = 0, \quad n \geqslant 1. \tag{4.9}$$

On the other hand,

$$\sup_{x\in D}|H_D h_n(x) - H_D h(x)| = \sup_{x\in D}\Big|\int_{\partial D}\big(h_n(y)-h(y)\big)H_D(x,dy)\Big|$$
$$\leqslant \sup_{y\in\partial D}|h_n(y)-h(y)| \to 0 \quad \text{as } n\to\infty.$$

Hence, by letting n tend to infinity in (4.9), we get $\langle H_D h, \Delta_k\varphi\rangle_k = 0$, and therefore $\langle h, \Delta_k\varphi\rangle_k = 0$ since $h = H_D h$ on D.

(iii) Assume that h is Δ_k-harmonic on B in the distributional sense. The hypoellipticity of the Dunkl Laplacian Δ_k on W-invariant open sets [12], [22] yields $h \in C^\infty(B)$. Thus, by (2.2), it follows that, for every $\varphi \in C_c^\infty(B)$,

$$\int_B \Delta_k h(x)\varphi(x)w_k(x)\,dx = 0.$$

Hence $\Delta_k h(x) = 0$ for every $x \in B$, which means that h is Δ_k-harmonic on B. ■

It is worth noting that the equivalence established in the above theorem remains valid if we replace the ball B by any W-invariant open set, for example, the whole space $\mathbb{R}^d$.

THEOREM 4.2. *For every $f \in C^+(\partial B)$, the problem*

$$\begin{cases}\Delta_k h = 0 & \text{on } B,\\ h = f & \text{on } \partial B\end{cases} \tag{4.10}$$

admits one and only one solution in $C^+(\overline{B})$ which is given by $H_B f$.

Proof. Let $f \in C^+(\partial B)$. By (3.4) and (3.5), the function $H_B f$ is continuous and X-harmonic on B. We shall show that $H_B f$ is a continuous extension of f on $\overline{B}$. Let $z \in \partial B$ and consider $V = \mathbb{R}^d\backslash\{0\}$, and let u be the function defined on V by

$$u(x) = G^k(x,0) - G^k(z,0).$$

Since

$$p_t^k(x,0) = \frac{c_k}{(4t)^{m/2}}e^{-|x|^2/(4t)}, \quad x\in\mathbb{R}^d,$$

it follows that

$$G^k(x,0) = \frac{c_k}{4}\frac{\Gamma(m/2-1)}{|x|^{m-2}}. \tag{4.11}$$

Then, using (3.6) and (4.11), it is easy to verify that u is a barrier of z (with respect to B), i.e.,

(i) u is hyperharmonic on $V \cap B$,

(ii) u is positive on $V \cap B$,

(iii) $\lim_{x \in V \cap B, x \to z} u(x) = 0$.

Hence, by Propositions VII.3.1 and VII.3.3 in [1], we obtain $H_B(z, \cdot) = \delta_z$ and $\lim_{x \in B, x \to z} H_B f(x) = f(z)$. Since z is arbitrary in ∂B, $H_B f$ is a continuous extension of f on $\overline{B}$. So, it remains to prove the uniqueness of the solution. Let h be another continuous extension of f on $\overline{B}$ which is the solution to the problem (4.10). Let $x \in B$ and let $(D_n)_{n \geqslant 1}$ be a sequence of nonempty bounded open sets such that $x \in D_n \subset \overline{D_n} \subset D_{n+1}$ and $B = \bigcup_n D_n$. Then $(\tau_{D_n})_n$ converges to τ_B almost surely. Hence, the continuity of h on $\overline{B}$ together with the quasi-left-continuity of the Dunkl process yield $H_B h(x) = \lim_n H_{D_n} h(x)$, and consequently $H_B h(x) = h(x)$, since $H_{D_n} h(x) = h(x)$ for every $n \geqslant 1$. Thus $h(x) - H_B f(x) = H_B(h - f)(x) = 0$ since $h = f$ on ∂B. So, $h = H_B f$ on B and the uniqueness is proved. ■

5. GREEN OPERATORS

The Green operator G^k on the whole space $\mathbb{R}^d$ is defined, for every $f \in \mathcal{B}^+(\mathbb{R}^d)$, by the formula

$$G^k f(x) := \int_{\mathbb{R}^d} G^k(x, y) f(y) w_k(y)\, dy, \qquad x \in \mathbb{R}^d.$$

By Fatou's lemma, for each $y \in \mathbb{R}^d$, $G^k(\cdot, y)$ is lower semicontinuous on $\mathbb{R}^d$, and so $G^k f$ is lower semicontinuous on $\mathbb{R}^d$.

In the sequel, B_r denotes the ball of $\mathbb{R}^d$ of center zero and radius $r > 0$, and $A_{t,s}$ denotes the annulus of $\mathbb{R}^d$ of center zero and radius $0 < t < s < \infty$.

LEMMA 5.1. (i) *For every* $0 < r < \infty$,

$$G^k 1_{B_r}(x) = \begin{cases} \frac{1}{m-2}\left(\frac{|x|^2}{m} + \frac{r^2 - |x|^2}{2}\right) & \text{if } |x| \leqslant r, \\ \frac{1}{m(m-2)} r^m |x|^{2-m} & \text{if } |x| \geqslant r. \end{cases} \tag{5.1}$$

(ii) *For every* $0 \leqslant t < s < \infty$,

$$0 \leqslant \sup_{x \in A_{t,s}} G^k 1_{A_{t,s}}(x) \leqslant \frac{2}{m-2} s(s-t). \tag{5.2}$$

Proof. Formula (5.1) follows immediately from (3.1) because

$$G^k 1_{B_r}(x) = \int_0^\infty \int_{B_r} p_t^k(x, y) w_k(y)\, dy\, dt.$$

Let $0 \leqslant t < s < \infty$. It is clear that $0 \leqslant G^k 1_{A_{t,s}}$ and that

$$G^k 1_{A_{t,s}} = G^k 1_{B_s} - G^k 1_{B_t}.$$

Then, by (5.1), it follows that for every $x \in A_{t,s}$,

$$\begin{aligned} G^k 1_{A_{t,s}}(x) &= \frac{1}{m-2}\left[\frac{|x|^2}{m} + \frac{s^2-|x|^2}{2}\right] - \frac{1}{m(m-2)} t^m |x|^{2-m} \\ &= \frac{1}{m-2}\left[\frac{|x|^2}{m}\left(1-\left(\frac{t}{|x|}\right)^m\right) + \frac{s^2-|x|^2}{2}\right] \\ &\leqslant \frac{1}{m-2}\left[\frac{s^2}{m}\left(1-\left(\frac{t}{s}\right)^m\right) + \frac{s^2-t^2}{2}\right] \\ &\leqslant \frac{1}{m-2}\left[s^2\left(1-\frac{t}{s}\right) + \frac{s^2-t^2}{2}\right] \\ &\leqslant \frac{2}{m-2} s(s-t). \quad \blacksquare \end{aligned}$$

An immediate consequence of the above lemma is that for each $x \in \mathbb{R}^d$ the function $G^k(\cdot, x)w_k$ is locally Lebesgue-integrable on $\mathbb{R}^d$. Thus, by Fubini's theorem, for every $f \in \mathcal{B}_b(\mathbb{R}^d)$ with compact support, we have

$$\begin{aligned} G^k f(x) = \int_{\mathbb{R}^d} G^k(x,y) f(y) w_k(y) dy &= \int_0^\infty \int_{\mathbb{R}^d} p_t^k(x,y) f(y) w_k(y) dy dt \\ &= \int_0^\infty E^x\left[f(X_t)\right] dt = E^x\Big[\int_0^\infty f(X_t) dt\Big]. \end{aligned}$$

PROPOSITION 5.1. *Let $f \in \mathcal{B}_b(\mathbb{R}^d)$ with compact support. Then $G^k f \in C_0(\mathbb{R}^d)$ and*

$$\Delta_k G^k f = -f \quad \text{in } \mathbb{R}^d \tag{5.3}$$

in the distributional sense, i.e., for every $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} G^k f(x) \Delta_k \psi(x) w_k(x)\, dx = -\int_{\mathbb{R}^d} f(x)\psi(x) w_k(x)\, dx.$$

Moreover, $G^k f$ is radially symmetric whenever f is.

P r o o f. Let $r > 0$ be such that the support of f is contained in B_r. Let us assume first that $f \geqslant 0$ and put $g = \|f\|\, 1_{B_r} - f$. Then, applying the Green operator G^k, we obtain

$$G^k f + G^k g = \|f\|\, G^k 1_{B_r}. \tag{5.4}$$

Since $G^k f$ and $G^k g$ are lower semicontinuous on $\mathbb{R}^d$ and $G^k 1_{B_r} \in C_0(\mathbb{R}^d)$ (see (5.1)), we immediately deduce from (5.4) that $G^k f \in C_0(\mathbb{R}^d)$. For f of arbitrary sign, we write $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then the same reasoning shows that $G^k f^+$ and $G^k f^-$ are in $C_0(\mathbb{R}^d)$. Hence $G^k f = G^k f^+ - G^k f^-$ is in $C_0(\mathbb{R}^d)$, as desired. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then, by (4.1), for every $y \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} G^k(x,y)\Delta_k\psi(x)w_k(x) = -\psi(y).$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^d} G^k f(x)\Delta_k\psi(x)w_k(x)\,dx &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} G^k(x,y)f(y)w_k(y)\,dy\Big)\Delta_k\psi(x)w_k(x)\,dx\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} G^k(x,y)\Delta_k\psi(x)w_k(x)\,dx\Big)f(y)w_k(y)\,dy\\ &= -\int_{\mathbb{R}^d} f(y)\psi(y)w_k(y)\,dy.\end{aligned}$$

Formula (5.1) justifies the transformation of the above integrals by Fubini's theorem. Now, assume that f is radially symmetric. Let $(f_n)_n$ be an increasing sequence of functions of the form

$$f_n = \sum_{i=1}^{n} \alpha_i 1_{B_{r_i}},$$

which converges pointwise to f on $\mathbb{R}^d$. Clearly, by formula (5.1), $G^k f_n$ is radially symmetric. On the other hand, using the dominated convergence theorem, we get for every $x \in \mathbb{R}^d$, $\lim_{n\to\infty} G^k f_n(x) = G^k f(x)$. Thus $G^k f$ is radially symmetric. ∎

For every open set D, we define the Green operator G_D^k on $\mathcal{B}_b(D)$ by

$$G_D^k f(x) := E^x\Big[\int_0^{\tau_D} f(X_s)\,ds\Big], \quad x \in D.$$

For every $f \in \mathcal{B}_b(D)$, we denote by $\widetilde{f}$ the extension of f on $\mathbb{R}^d$ such that $\widetilde{f} = 0$ on $\mathbb{R}^d \setminus D$. Since the Dunkl process satisfies the strong Markov property, for every $x \in D$ we have

$$\begin{aligned}G^k\widetilde{f}(x) &= E^x\Big[\int_0^{\infty}\widetilde{f}(X_s)\,ds\Big]\\ &= E^x\Big[\int_0^{\tau_D}\widetilde{f}(X_s)\,ds\Big] + E^x\Big[\int_{\tau_D}^{\infty}\widetilde{f}(X_s)\,ds\Big]\\ &= E^x\Big[\int_0^{\tau_D} f(X_s)\,ds\Big] + E^x\Big[E^{X_{\tau_D}}\Big[\int_0^{\infty}\widetilde{f}(X_s)ds\Big]\Big]\\ &= E^x\Big[\int_0^{\tau_D} f(X_s)\,ds\Big] + H_D G^k\widetilde{f}(x).\end{aligned}$$

Therefore,

$$G_D^k f = G^k \widetilde{f} - H_D G^k \widetilde{f} \quad \text{on } D. \tag{5.5}$$

Let B be an open ball of $\mathbb{R}^d$ of center zero and radius $r > 0$. Then it follows from (5.5) that, for every $f \in \mathcal{B}_b(B)$, $G_B^k f$ can be represented by

$$G_B^k f(x) = \int_B G_B^k(x,y) f(y) w_k(y)\, dy,$$

where, for every $x, y \in B$,

$$G_B^k(x,y) := G^k(x,y) - \int_{\partial B} G^k(y,z) H_B(x,dz). \tag{5.6}$$

Since, by (2.4), for every $y, z \in \mathbb{R}^d$, we have

$$G^k(y,z) \leqslant \frac{c_k \Gamma(m/2-1)}{4\,(|y|-|z|)^{m-2}}, \tag{5.7}$$

it is immediate to see that, for every $x, y \in B$,

$$\int_{\partial B} G^k(y,z) H_B(x,dz) \leqslant \frac{c_k \Gamma(m/2-1)}{4\,(|y|-r)^{m-2}} < \infty.$$

Therefore, $G_B^k(x,y)$ introduced in (5.6) exists, and so the Green function $G_B^k(\cdot,\cdot)$ is well defined from $B \times B$ into $]0,\infty]$. In the following corollary, we collect some properties of the Green operator G_B^k.

COROLLARY 5.1. *Let $f \in \mathcal{B}_b(B)$. Then $G_B^k f \in C_0(B)$ and*

$$\Delta_k G_B^k f = -f \quad \text{in } B \tag{5.8}$$

in the distributional sense.

Proof. Clearly, $G_B^k f$ is continuous on B since $G^k \widetilde{f}$ and $H_B G^k \widetilde{f}$ are. For every $z \in \partial B$,

$$\lim_{x \to z} G_B^k f(x) = 0$$

since $\lim_{x\to z} H_B G^k \widetilde{f}(x) = G^k \widetilde{f}(z)$. Thus $G_B^k f \in C_0(B)$. Formula (5.8) follows immediately from (5.3) and (5.5). ■

PROPOSITION 5.2. *For every $M > 0$, the family $\{G_B^k f,\ \|f\| \leqslant M\}$ is relatively compact in $C_0(B)$ endowed with the uniform norm.*

P r o o f. In virtue of the Arzelà–Ascoli theorem, we need to show that $\{G_B^k f, \|f\| \leqslant M\}$ is uniformly bounded and equicontinuous on B. Let r be the radius of the ball B. Let $f \in \mathcal{B}_b(B)$ be such that $\|f\| \leqslant M$. Obviously, $\|G_B^k f\| \leqslant M\|G_B^k 1\| \leqslant M\|G^k 1_B\|$. Thus, using (5.1), we obtain

$$\|G_B^k f\| \leqslant \frac{r^2 M}{2(m-2)}.$$

This means that the family $\{G_B^k f,\ \|f\| \leqslant M\}$ is uniformly bounded. Next, we claim that the family $\{G_B^k(x,\cdot),\ x \in B\}$ is uniformly integrable with respect to the measure $w_k(y)\,dy$. Let $x \in B$ and $\epsilon > 0$ be small enough. Let $A_{t,s}$ be the annulus of $\mathbb{R}^d$ of center zero and radius $t = \max(0, |x| - \epsilon)$ and $s = |x| + \epsilon$. Then, for every Borel subset D of B, we have

$$\begin{aligned}\int_D G_B^k(x,y)w_k(y)dy &\leqslant \int_D G^k(x,y)w_k(y)dy \\ &= \int_{D\cap A_{t,s}} G^k(x,y)w_k(y)\,dy + \int_{D\setminus A_{t,s}} G^k(x,y)w_k(y)dy \\ &\leqslant G^k 1_{A_{t,s}}(x) + \Big(\sup_{y\in D\setminus A_{t,s}} G^k(x,y)\Big)\int_D w_k(y)dy.\end{aligned}$$

Hence, it follows from (5.7) and (5.2) that

$$\int_D G_B^k(x,y)w_k(y)\,dy \leqslant \frac{4r}{m-2}\epsilon + \frac{c_k\Gamma(m/2-1)}{4\epsilon^{m-2}}\int_D w_k(y)\,dy.$$

Put $\eta = \epsilon^{m-1}$. Then for every Borel subset D of B such that $\int_D w_k(y)\,dy < \eta$, we have

$$\int_D G_B^k(x,y)w_k(y)\,dy \leqslant \left(\frac{4r}{m-2} + \frac{c_k\Gamma(m/2-1)}{4}\right)\epsilon.$$

Thus, the uniform integrability of the family $\{G_B^k(x,\cdot),\ x \in B\}$ is shown. Therefore, in virtue of Vitali's convergence theorem, for $z \in B$,

$$\lim_{x\to z}\int_B |G_B^k(x,y) - G_B^k(z,y)|w_k(y)\,dy = 0.$$

Hence, the family $\{G_B^k f,\ \|f\| \leqslant M\}$ is equicontinuous on B since

$$\begin{aligned}&\lim_{x\to z}\sup_{\|f\|\leqslant M} |G_B^k f(x) - G_B^k f(z)| \\ &\qquad \leqslant M\lim_{x\to z}\int_B |G_B^k(x,y) - G_B^k(z,y)|w_k(y)\,dy = 0. \quad \blacksquare\end{aligned}$$

6. SEMILINEAR DIRICHLET PROBLEM

Let B be an open ball of $\mathbb{R}^d$ of center zero. Let $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing continuous function such that $\varphi(0) = 0$. By a solution of

$$\Delta_k u = \varphi(u) \quad \text{in } B \tag{6.1}$$

we shall mean every function $u \in C(B)$ such that

$$\int_B u(x)\,\Delta_k\psi(x)\,w_k(x)\,dx = \int_B \varphi\big(u(x)\big)\,\psi(x)\,w_k(x)\,dx$$

holds for every $\psi \in \mathcal{C}_c^\infty(B)$. We recall from Theorem 4.2 that if $\varphi \equiv 0$, then $H_B f$ is the unique solution of (6.1) satisfying $u = f$ on ∂B. In all the following, we assume that φ is not identically zero.

LEMMA 6.1. *Let $u \in C^+(\overline{B})$. Then u is a solution of equation* (6.1) *if and only if $u + G_B^k\big(\varphi(u)\big) = H_B u$ on B.*

Proof. Let us note first that $G_B^k\big(\varphi(u)\big) \in C_0(B)$ since the function $\varphi(u)$ is bounded on B. Put $h := u + G_B^k\big(\varphi(u)\big)$. Clearly, $h \in C(\overline{B})$ and $h = u$ on ∂B. On the other hand, using Fubini's theorem and formula (5.8), we obtain for every $\psi \in C_c^\infty(B)$,

$$\begin{aligned}
&\int_B h(x)\Delta_k\psi(x)w_k(x)\,dx\\
&\qquad= \int_B u(x)\Delta_k\psi(x)w_k(x)\,dx + \int_B G_B^k\big(\varphi(u)\big)(x)\Delta_k\psi(x)w_k(x)\,dx\\
&\qquad= \int_B u(x)\Delta_k\psi(x)w_k(x)\,dx - \int_B \varphi\big(u(x)\big)\psi(x)w_k(x)\,dx.
\end{aligned}$$

So, $\Delta_k u = \varphi(u)$ in B if and only if $\Delta_k h = 0$ in B. In this case, since $h = u$ on ∂B, the uniqueness of the solution to problem (4.10) yields $h = H_B u$ on B. This completes the proof. ∎

LEMMA 6.2. *Let $u, v \in C^+(\overline{B})$ be two solutions of equation* (6.1). *If $u \geqslant v$ on ∂B, then $u \geqslant v$ on B.*

Proof. Define $w := u - v$ and $\rho := \varphi(u) - \varphi(v)$. By Lemma 6.1, we have

$$w + G_B^k\rho = H_B w \quad \text{on } \overline{B}. \tag{6.2}$$

Suppose that the open set $D := \{x \in B;\ w(x) < 0\}$ is not empty. Since φ is nondecreasing, it follows that $\rho \leqslant 0$ on D, and hence $G_D^k\rho \leqslant 0$ on D. Let $x \in D$.

It is clear that $\overline{B}$ contains the support of the measure $H_D(x,\cdot)$. Now integrate (6.2) with respect to $H_D(x,\cdot)$ to obtain

$$H_D w(x) + H_D(G_B^k\rho)(x) = H_D H_B w(x) = H_B w(x).$$

Consequently,

$$\text{(6.3)}\quad H_D w(x) = H_B w(x) - H_D(G_B^k\rho)(x) = w(x) + \big(G_B^k\rho(x) - H_D G_B^k\rho(x)\big).$$

On the other hand, using the strong Markov property, we obtain

$$\text{(6.4)}\qquad G_B^k\rho(x) - G_D^k\rho(x) = E^x\Big[\int_{\tau_D}^{\tau_B}\rho(X_s)\,ds\Big] = E^x\Big[E^{X_{\tau_D}}\Big[\int_0^{\tau_B}\rho(X_s)\,ds\Big]\Big] = H_D G_B^k\rho(x).$$

Thus, it follows from (6.3) and (6.4) that $w(x) + G_D^k\rho(x) = H_D w(x)$. But this is absurd since $w(x) + G_D^k\rho(x) < 0$ and $H_D w(x) \geqslant 0$. Therefore, D is empty, and consequently $u \geqslant v$ on B. ∎

THEOREM 6.1. *For every $f \in C^+(\partial B)$, the semilinear Dirichlet problem*

$$\text{(6.5)}\qquad \begin{cases} \Delta_k u = \varphi(u) & \text{in } B,\\ u = f & \text{on } \partial B \end{cases}$$

admits one and only one solution $u \in C^+(\overline{B})$.

P r o o f. It follows from Lemma 6.2 that problem (6.5) admits at most one solution. To prove the existence, in virtue of Lemma 6.1, it will be sufficient to establish the existence of $u \in C^+(\overline{B})$ such that

$$\text{(6.6)}\qquad u + G_B^k\big(\varphi(u)\big) = H_B f \quad \text{on } B.$$

Since $G_B^k 1 \leqslant G^k 1_B$, we immediately deduce by (5.1) that $\sup_{x\in B} G_B^k 1(x) < \infty$. Let $f \in C^+(\partial B)$, $a = \|f\|$ and $M = a + \varphi(a)\|G_B^k 1\|$. Let ϕ be the function defined on $\mathbb{R}$ by

$$\phi(t) = \begin{cases} 0 & \text{if } t \leqslant 0,\\ \varphi(t) & \text{if } 0 \leqslant t \leqslant a,\\ \varphi(a) & \text{if } t \geqslant a. \end{cases}$$

Let $\Lambda := \{u \in C(\overline{B});\ \|u\| \leqslant M\}$ and consider the operator $T : \Lambda \to C(\overline{B})$ defined by

$$Tu(x) = H_B f(x) - G_B^k\big(\phi(u)\big)(x), \qquad x \in \overline{B}.$$

Since $\sup_{x\in B}\phi\big(u(x)\big)\leqslant\varphi(a)$, we easily deduce that

$$\|Tu\|\leqslant M$$

for every $u\in\Lambda$. This implies that $T(\Lambda)\subset\Lambda$. Now, let $(u_n)_n$ be a sequence in Λ converging uniformly to $u\in\Lambda$. Let $\varepsilon>0$. Since ϕ is uniformly continuous on the interval $[-M,M]$, we immediately deduce that there exists $n_0\in\mathbb{N}$ such that, for every $n\geqslant n_0$,

$$\|\phi(u_n)-\phi(u)\|\leqslant\varepsilon.$$

Then, for every $n\geqslant n_0$ and every $x\in B$,

$$|Tu_n(x)-Tu(x)|\leqslant G_B^k\big(|\phi(u_n)-\phi(u)|\big)(x)\leqslant\varepsilon\sup_{x\in B}G_B^k1(x).$$

This show that $(Tu_n)_n$ converges uniformly to Tu, and therefore T is continuous. On the other hand, Λ is a closed bounded convex subset of $C(\overline{B})$ and, in virtue of Proposition 5.2, $T(\Lambda)$ is relatively compact. Thus, the Schauder fixed point theorem ensures the existence of a function $u\in\Lambda$ such that

$$u+G_B^k\big(\phi(u)\big)=H_Bf\quad\text{ on } B.$$

Clearly, $u\in C(\overline{B})$ and $u(x)\leqslant H_Bf(x)\leqslant a$ for every $x\in B$. So, to obtain (6.6), we need to show that $\phi(u)=\varphi(u)$ on B, or equivalently, $u\geqslant 0$ on B. Assume that the open set $D:=\{x\in B,\ u(x)<0\}$ is not empty. Let $x\in D$. Then,

$$H_Du(x)=H_D\Big(H_Bu-G_B^k\big(\phi(u)\big)\Big)(x)=H_Bu(x)-H_DG_B^k\big(\phi(u)\big)(x).$$

The same reasoning as in (6.4), based on the strong Markov property, shows that

$$H_DG_B^k\big(\phi(u)\big)(x)=G_B^k\big(\phi(u)\big)(x)-G_D^k\big(\phi(u)\big)(x).$$

Thus, because $\phi(u)=0$ on D, we get

$$\begin{aligned}H_Du(x)&=H_Bu(x)-G_B^k\big(\phi(u)\big)(x)+G_D^k\big(\phi(u)\big)(x)\\&=u(x)+G_D^k\big(\phi(u)\big)(x)=u(x)<0.\end{aligned}$$

But, $H_Du(x)\geqslant 0$ since $u\geqslant 0$ on $\overline{B}\setminus D$, which contains the support of $H_D(x,\cdot)$. So D must be empty, and consequently $u\geqslant 0$ on B. This completes the proof. ∎

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