bm-CENTRAL LIMIT THEOREMS
ASSOCIATED WITH NON-SYMMETRIC POSITIVE CONES*

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Abstract. Analogues of the classical Central Limit Theorem are proved in the noncommutative setting of random variables which are bm-independent and indexed by elements of positive non-symmetric cones, such as the circular cone, sectors in Euclidean spaces and the Vinberg cone. The geometry of the cones is shown to play a crucial role and the related volume characteristics of the cones is shown.

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1. INTRODUCTION

The classical Central Limit Theorem (CLT) establishes the convergence, to the normal law $N(0,1)$, of the normalized partial sums

$$S_N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} X_j$$

of independent identically distributed random variables $\{X_j : j = 1, 2, \ldots\}$ satisfying $\mathbb{E}(X_j) = 0$ and $\mathbb{E}(X_j^2) = 1$. It has been generalized to noncommutative settings of free independence (Voiculescu [8], see also Bożejko [1]), monotone independence (Muraki [5]) and Boolean independence [7]. These generalizations are obtained by replacing the notion of classical independence with the noncommutative ones, and the framework is the noncommutative probability space of the form

$(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital *-algebra and φ is a state on it (which plays the role of the expectation $\mathbb{E}$). In such a framework one considers the convergence of moments $\lim_{N\to\infty} \varphi\big((S_N)^n\big)$ for X_j being (free, monotone or Boolean) independent self-adjoint elements of $\mathcal{A}$ with $\varphi(X_j) = 0$ and $\varphi\big((X_j)^2\big) = 1$ for $j \in \mathbb{N}$. As the result one gets in the limit the (moments of) semicircle (Wigner) law (free CLT), arcsine law (monotone CLT) and Bernoulli law (Boolean CLT).

The notion of *bm-independence* has been introduced in [11] (see also [10]) and it combines the monotone and Boolean independences (the details are provided in Section 2). It is defined for random variables which are indexed by an arbitrary partially ordered set, and this introduces a new factor – the structure of the index set – comparing to the above cases where the index set $\mathbb{N}$ is linearly (totally) ordered. Natural examples of partially ordered sets $(V, \preceq)$ are given by positive cones $\Pi \subset V$ in Euclidean spaces V – then for $u, v \in V$ one defines $u \preceq v$ if $v - u \in \Pi$. Let us recall that a *positive cone* in Euclidean space is a subset closed under addition of vectors and under multiplication by positive scalars.

However, for such general index sets one faces the problem of finding an adequate formulation of the Central Limit Theorem (called bm-CLT). The studies [11] provided the bm-CLT for index sets being discrete subsets $\mathbf{I} \subset \Pi$ (consisting of elements with integer entries) in positive symmetric cones Π (the definition is provided in Section 2), such as $\mathbb{R}_+^d$, the Lorentz cones Λ_1^d ($d \in \mathbb{N}$), the positive definite real symmetric matrices and the positive definite (complex or quaternionic) Hermitian matrices. These examples provide a complete list of positive symmetric cones (except the special cone of 3×3 matrices with octonion entries), according to classification given by Faraut and Korányi [2]. Some of the formulations of the bm-CLT turned out to be not quite satisfactory for later study [3] of Brownian motions (with bm-independent increments), in particular for proving an analogue of the classical Donsker theorem, where one considers dilations of a given interval $\mathbb{R}_+ \supset [s, t] \mapsto [Ns, Nt] \subset \mathbb{R}_+$ by positive integers $N \in \mathbb{N}$ for $s < t \in \mathbb{R}_+$.

A noncommutative version of this invariance principle has been formulated by Speicher in [6] for the free Brownian motions and later by Muraki [4] for the monotone Brownian motions, still with the totally ordered index set of positive reals.

For an arbitrary positive cone Π one can also consider such dilations of intervals $\Pi \supset [\xi, \eta] \mapsto [N\xi, N\eta] \subset \Pi$ for $\xi \preceq \eta \in \Pi$. However, the crucial problem is to control how fast the dilated intervals grow. This has been established for positive symmetric cones in [3] with introducing the notion of *volume characteristic* of a positive symmetric cone.

Let us mention the related formulation of the bm-CLT from [3] for positive symmetric cones in which the normalized partial sums are taken over dilated intervals. The role of positive reals $\mathbb{R}_+$ is played by a positive symmetric cone Π and the role of positive integers $\mathbb{N} \subset \mathbb{R}_+$ by a discrete subset $\mathbf{I} \subset \Pi$ (for details see [3]). We shall use the notation $\mathsf{J}_{\mathbb{N}}(\xi) := [0, N\xi] \cap \mathbf{I}$ for the finite subset of the dilated interval which consists of elements of $\mathbf{I}$.

THEOREM 1.1. *Let* $\mathbf{I} \subset \Pi$ *be as above and let* $\{X_\rho : \rho \in \mathbf{I}\}$ *be bm-independent self-adjoint elements in a noncommutative probability space* $(\mathcal{A}, \varphi)$ *with* $\varphi(X_\rho) = 0$ *and* $\varphi\big((X_\rho)^2\big) = 1$. *For* $\xi \in \Pi$ *let*

$$S_N(\xi) := \frac{1}{|\mathsf{J}_\mathsf{N}(\xi)|} \sum_{\rho \in \mathbf{I},\, \rho \preceq N\xi} X_\rho$$

be the normalized partial sums ($|A|$ *denotes the cardinality of* A). *Then, for every non-negative integer* n *the following limits exist*:

$$\lim_N \varphi\Big(\big(S_N(\xi)\big)^{2n+1}\Big) = 0 \quad and \quad \lim_N \varphi\Big(\big(S_N(\xi)\big)^{2n}\Big) = g_n,$$

where $(g_n)_{n \geqslant 0}$ *is a sequence of* (*even*) *moments of a symmetric probability measure* μ *on* $\mathbb{R}$ (*depending on the positive symmetric cone* Π) *which satisfies the recurrence* $g_0 = g_1 = 1$ *and* $g_n = \sum_{k=1}^n \gamma_k \cdot g_{k-1} g_{n-k}$. *The numbers* $\gamma_k = \gamma_k(\Pi)$, $k \in \mathbb{N}$, *depend on the positive cone* Π *and are given by the formulas*

(1.1) $$\gamma_k = \gamma_k(\Pi) := \frac{1}{v(\xi)^k} \int\limits_{0 \preceq \rho \preceq \xi} v(\rho)^{k-1} d\rho,$$

where $v(\xi)$ *is the volume of the interval* $[0, \xi] \subset \Pi$ *and the right-hand side does not depend on* $\xi \in \Pi$; *thus these numbers are called the* volume characteristics *of the positive symmetric cone* Π (*see* [3], *Theorem* 2).

In this paper we shall study such bm-CLT for the following specific *non-symmetric* positive cones:

1. the circular cones $\mathcal{C}_\theta^d := \{(t; x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} : \|x\| \leqslant t \cdot \tan \theta\}$ defined by parameters $\theta \in (0, \pi/2)$;

2. the sectors $\Omega_{\mathbf{u}}^d := \big\{ \sum_{i=1}^d \alpha_i \mathbf{u}_i : \alpha_i \geqslant 0, \mathbf{u}_i \in \mathbb{R}^d, i = 1, \ldots, d \big\}$ spanned by d-tuples $\mathbf{u} := (\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of linearly independent unit vectors;

3. the Vinberg cone $\Pi_\mathcal{V} \subset \mathbb{M}_3(\mathbb{R})$.

The circular cones are generalizations of the Lorentz cones: $\Lambda_1^d = \mathcal{C}_\theta^d$ for $\theta = \pi/4$. The sectors $\Omega_{\mathbf{u}}^d$ generalize the positive symmetric cones $\mathbb{R}_+^d \subset \mathbb{R}^d$ which are obtained for orthonormal bases $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\} \subset \mathbb{R}^d$. The Vinberg cone has already been considered in [11], however, the formulation of the bm-CLT was different there, not related to dilations of intervals (hence neither to the Donsker theorem), and not related to the volume characteristics of the cone. In this study we shall show the volume characteristics for the first two cases and the related property for the Vinberg cone.

2. PRELIMINARIES

In this section we shall present basic objects of our study.

2.1. bm-independence. The general formulation of *bm-independence* was given in [11] for families of algebras indexed by partially ordered sets. If $(\mathcal{X}, \preceq)$

is a partially ordered set with partial order $\preceq$, then, for $\xi, \eta \in \mathcal{X}$, we shall write $\xi \prec \eta$ if $\xi \preceq \eta$ and $\xi \neq \eta$; we shall also write $\xi \sim \eta$ if ξ and η are comparable (i.e. $\xi \preceq \eta$ or $\eta \preceq \xi$) and $\xi \nsim \eta$ if ξ and η are incomparable.

DEFINITION 2.1. Let $\mathcal{A}$ be an algebra and let φ be a functional on $\mathcal{A}$. We say that a family $\{\mathcal{A}_\eta : \eta \in \mathcal{X}\}$ of subalgebras of $\mathcal{A}$, indexed by a partially ordered set $(\mathcal{X}, \preceq)$, is *bm-independent* if the following two conditions hold:

BM1: If $\xi, \eta, \rho \in \mathcal{X}$ are such that $\xi \prec \rho \succ \eta$ or $\xi \nsim \rho \succ \eta$ or $\xi \prec \rho \nsim \eta$, then for any $a_1 \in \mathcal{A}_\xi$, $a_2 \in \mathcal{A}_\rho$, $a_3 \in \mathcal{A}_\eta$ we have

$$a_1 a_2 a_3 = \varphi(a_2) \cdot a_1 a_3.$$

BM2: If $\xi_1 \succ \ldots \succ \xi_m \nsim \ldots \nsim \xi_k \prec \ldots \prec \xi_n$ for some $1 \leqslant m \leqslant k \leqslant n$ and $\xi_1, \ldots, \xi_n \in \mathcal{X}$, with $a_j \in \mathcal{A}_{\xi_j}$ for $1 \leqslant j \leqslant n$, then

$$\varphi(a_1 \ldots a_n) = \prod_{j=1}^{n} \varphi(a_j).$$

Noncommutative random variables $\{a_\xi \in \mathcal{A} : \xi \in \mathcal{X}\}$ are called *bm-independent* if the subalgebras $\mathcal{A}_\xi \subset \mathcal{A}$ they generate are bm-independent. Let us recall that in this definition one gets the monotone independence if $\mathcal{X}$ is totally ordered, and the Boolean independence if $\mathcal{X}$ is totally disordered.

2.2. Positive symmetric cones. For the reader's convenience we recall the notion of a *positive symmetric cone*. Let $(V, \langle \cdot, \cdot \rangle)$ be a real vector space with an inner product. A subset $\Omega \subset V$ is called a *positive cone* if it is closed under addition of vectors and under multiplication by positive scalars (in particular, it is a convex set). Let $\overline{\Omega}$ be the closure; then the dual cone of Ω is $\Omega^* := \{v \in V : \langle v, u \rangle > 0 \text{ for all } u \in \overline{\Omega}\}$ (observe that $0 \notin \Omega^*$). A positive cone Ω is called *self-dual* if $\Omega = \Omega^*$. A cone $\Omega \subset V$ is called *homogeneous* if its group of automorphisms acts transitively, i.e. for any $u, v \in \Omega$ there exists a linear mapping φ on V such that φ is bijection on Ω and $\varphi(u) = v$. The cone Ω is called *regular* if the only element $v \in \overline{\Omega}$ for which also $-v \in \overline{\Omega}$ is $v = 0$.

DEFINITION 2.2. A positive convex cone $\Omega \subset V$ is called a *positive symmetric cone* if it is open, regular, self-dual and homogeneous. Otherwise, the cone is said to be *non-symmetric*.

For details and classification of positive symmetric cones we refer to [2]. In this paper we study the bm-CLT associated with some specific non-symmetric positive cones. The positive non-symmetric cones we consider are the following non-self-dual ones.

2.3. The circular cones $\mathcal{C}_\theta^d$. Circular cones [12] are defined for $d \in \mathbb{N}$ and $\theta \in (0, \pi/2)$ (called a *rotation angle*) as

$$\mathcal{C}_\theta^d := \{x = (x_1, x_{2:d}) \in V = \mathbb{R} \times \mathbb{R}^{d-1} : (\cos\theta)\|x\| \leqslant x_1\}$$
$$= \{x = (x_1, x_{2:d}) \in V = \mathbb{R} \times \mathbb{R}^{d-1} : \|x_{2:d}\| \leqslant x_1 \tan\theta\},$$

where $\|\cdot\|$ denotes the Euclidean norm. The dual cone $(\mathcal{C}_\theta^d)^*$ of $\mathcal{C}_\theta^d$ is given by

$$(\mathcal{C}_\theta^d)^* = \{(x_1, x_{2:d}) \in V : (\sin\theta)\|x\| \leqslant x_1\} = \mathcal{C}_{\pi/2-\theta}^d,$$

so the circular cone $\mathcal{C}_\theta^d$ is non-symmetric, unless the rotation angle is $\theta = \pi/4$, in which case one gets the Lorentz cone

$$\Lambda_1^d := \{x = (x_1, x_{2:d}) \in V = \mathbb{R}_+ \times \mathbb{R}^{d-1} : \|x_{2:d}\| \leqslant x_1\}.$$

The relation between $\mathcal{C}_\theta^d$ and Λ_1^d is of the form

$$(2.1) \qquad \mathcal{C}_\theta^d = M_\theta \Lambda_1^d, \quad \text{where } M_\theta = \begin{bmatrix} \cot\theta & 0 \\ 0 & I_{d-1} \end{bmatrix},$$

and I_{d-1} is the $(d-1)$-dimensional unit matrix (see [12]).

2.4. The sectorial cones $\Omega_{\mathbf{u}}^d \subset \mathbb{R}^d$. For $d \in \mathbb{N}$ let $\mathbf{u}_1, \ldots, \mathbf{u}_d \in \mathbb{R}^d$ be arbitrary linearly independent unit vectors (a linear basis). For $\mathbf{u} := (\mathbf{u}_1, \ldots, \mathbf{u}_d)$ define

$$\Omega_{\mathbf{u}}^d := \Big\{ \sum_{i=1}^d \alpha_i \mathbf{u}_i : \alpha_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant d \Big\}$$

as the set of all linear combinations of these vectors with non-negative coefficients. Then $\Omega_{\mathbf{u}}^d$ is a positive non-symmetric cone (unless the basis $\mathbf{u}$ is orthogonal), and its dual cone is $(\Omega_{\mathbf{u}}^d)^* = \Omega_{\mathbf{v}}^d$, where $\mathbf{v} := (\mathbf{v}_1, \ldots, \mathbf{v}_d)$ is the dual basis of $\mathbf{u}$.

2.5. Vinberg's cone $\Pi_{\mathcal{V}}$. Let V be the five-dimensional real vector space of all real three-dimensional matrices of the form

$$V := \left\{ a = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & 0 \\ a_5 & 0 & a_3 \end{pmatrix} \in \mathbb{M}_3(\mathbb{R}) : a_1, \ldots, a_5 \in \mathbb{R} \right\},$$

with the inner product $\langle a, b \rangle = \sum_{i=1}^3 a_i b_i + 2 \sum_{j=4}^5 a_j b_j$ for $a, b \in V$. Then the *Vinberg cone* $\Pi_{\mathcal{V}} \subset V$ is defined by the following three positivity conditions:

$$a_1 \geqslant 0, \quad a_1 a_2 \geqslant a_4^2, \quad a_1 a_3 \geqslant a_5^2.$$

3. FORMULATION AND COMBINATORIAL REDUCTION OF THE bm-CLT

The formulation of the bm-CLT requires considering a proper discrete subset $\mathbf{I} \subset \Pi$, which would play the role of positive integers $\mathbb{N} \subset \mathbb{R}$, as an index set for random variables. As mentioned in the Introduction, the idea is to consider elements in Π with integer entries:

1. For the sectors $\Omega_{\mathbf{u}}^d \subset \mathbb{R}^d$ we put $\mathbf{I} := \Omega_{\mathbf{u}}^d \cap \mathbb{Z}^d$.

2. For the circular cones $\mathcal{C}_\theta^d$ we put $\mathbf{I} := \mathcal{C}_\theta^d \cap (\mathbb{N} \times \mathbb{Z}^{d-1})$.

3. For the Vinberg cone $\Pi_\mathcal{V} \subset \mathbb{M}_3(\mathbb{R})$ we put $\mathbf{I} := \Pi_\mathcal{V} \cap \mathbb{M}_3(\mathbb{Z})$.

In our bm-CLT we consider a sequence $\{X_\rho : \rho \in \mathbf{I} \subset \Pi\}$ of bm-independent self-adjoint elements in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(X_\rho) = 0$ and $\varphi\big((X_\rho)^2\big) = 1$. For $\xi \in \Pi$ and $N \in \mathbb{N}$ we define the normalized partial sums

$$(3.1) \qquad S_N(\xi) := \frac{1}{|\mathsf{J}_\mathbb{N}(\xi)|} \sum_{\rho \in \mathbf{I},\, \rho \preceq N\xi} X_\rho.$$

Our goal is to show, for each $n \in \mathbb{N}$, the existence of the limits

$$(3.2) \qquad g_n := \lim_{N \to \infty} \varphi\Big(\big(S_N(\xi)\big)^n\Big).$$

Proving this existence, we shall conclude that the sequence $(g_n)_{n \geqslant 0}$ is the moment sequence of probability measure μ^Π on $\mathbb{R}$, as it is the limit of positive definite sequences $\big\{\varphi\big((S_N(\xi))^n\big) : n \in \mathbb{N}\big\}$. Moreover, as we shall see, $g_n = 0$ for odd $n \in \mathbb{N}$, so we shall conclude that the measure μ^Π is symmetric on $\mathbb{R}$. This measure will be called the *bm-Central Limit measure* for the positive cone Π.

Moreover, we shall show that the sequence of even moments satisfies the recurrence of the form $g_0 = g_2 = 1$ and

$$(3.3) \qquad g_{2n} = \sum_{k=1}^{n} \gamma_k g_{2k-2} g_{2n-2k}, \quad n \geqslant 1,$$

with some additional numbers $(\gamma_n)_{n \geqslant 1}$ which are specific to each cone. For the sectorial cones and the circular cones these numbers are the *volume characteristics* of the cone. For the Vinberg cone this is not the case, but the numbers depend on two parameters related to the element ξ.

3.1. Combinatorial reduction of the proof of the bm-CLT. Now we describe how the proof of the bm-CLT is reduced by some combinatorial considerations. The details can be found in Section 6 of [11]. We shall use the notation $[0, \xi]_\mathbf{I} := [0, \xi] \cap \mathbf{I}$.

For fixed $n \in \mathbb{N}$ the quantity $\varphi\big((S_N(\xi))^n\big)$ can be represented as the sum

$$(3.4) \qquad \varphi\Big(\big(S_N(\xi)\big)^n\Big) = \frac{1}{|\mathsf{J}_\mathbb{N}(\xi)|^n} \sum_{\rho_1,\ldots,\rho_n \in \mathsf{J}_\mathbb{N}(\xi)} \varphi(X_{\rho_1} \ldots X_{\rho_n}).$$

As in [11], using quantitative arguments, one can show that the limit, as $N \to \infty$, of the above is the same as if the summation was taken just over the sequences $(\rho_1, \ldots, \rho_n)$ which are associated with noncrossing pair partitions. Recall that this means that each element in the sequence appears exactly twice and the pairings

connecting the same elements have no crossings. From this observation it follows that the limit $\lim_{N\to\infty} \varphi\big((S_N(\xi))^n \big) = 0$ if n is odd (there is no pair partition of an odd number of elements). Hence we can restrict ourselves to even n only; thus, in what follows we shall write $2n$ instead of n and consider the limit $\lim_{N\to\infty} \varphi\big((S_N(\xi))^{2n} \big)$.

In the next step one shows that, in fact, the limit is the same if one restricts the summation to sequences which are not only associated with noncrossing pair partitions but are also bm-ordered. This means that the sequence $(\rho_1, \ldots, \rho_{2n})$ is such that if $1 \leqslant i < j < k < l \leqslant 2n$ and $\rho_i = \rho_l$, $\rho_j = \rho_k$, then it follows that $\rho_l = \rho_i \prec \rho_j = \rho_k$. In other words, the blocks of a pair partition are labelled by the elements of the sequence in such a way that a block which is inside another one has a bigger label. We shall denote by $\mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big)$ the set of all such sequences $(\rho_1, \ldots, \rho_{2n})$ which, in addition, satisfy $\rho_1, \ldots, \rho_{2n} \in \mathtt{J}_{\mathtt{N}}(\xi)$. Then, it turns out that for $(\rho_1, \ldots, \rho_{2n}) \in \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big)$ one gets $\varphi(X_{\rho_1} \ldots X_{\rho_n}) = 1$. Hence one concludes that computing the limit (as $N \to \infty$) in (3.4) becomes equivalent to proving the existence of the following

$$(3.5) \qquad \lim_{N\to\infty} \varphi\Big(\big(S_N(\xi) \big)^{2n} \Big) = \lim_{N\to\infty} \frac{\big| \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big) \big|}{|\mathtt{J}_{\mathtt{N}}(\xi)|^n}.$$

On the right-hand side of this equality we have cardinalities of purely combinatorial objects, so the proof of the bm-CLT depends on estimates of the ratio of the quantities $\big| \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big) \big|$ and $|\mathtt{J}_{\mathtt{N}}(\xi)|^n$.

The idea of dealing with this ratio is to consider the place at which the element ρ_1 in a sequence $(\rho_1, \ldots, \rho_{2n}) \in \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big)$ appears again, as a consequence of the fact that the sequence is associated with a pair partition. Since the pair partition has no crossings, it follows that $\rho_1 = \rho_j$ if and only if j is even. Thus, we put $j = 2k$ and write $\rho_1 = \rho_{2k}$ for some $1 \leqslant k \leqslant n$ and this splits the set $\mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big)$ into the disjoint sum

$$(3.6) \qquad \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big) = \bigcup_{k=1}^{n} \mathtt{bmNC}_2^{\mathtt{n},\mathtt{k}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big).$$

Now, given $\rho_1 = \rho_{2k} \in \mathtt{J}_{\mathtt{N}}(\xi)$ and $(\rho_1, \ldots, \rho_{2n}) \in \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big)$, the elements $\rho_2, \ldots, \rho_{2k-1}$ must satisfy $\rho_j \in (\rho_1, N\xi]$ for $2 \leqslant j \leqslant 2k - 1$, and the other elements $\rho_i \in \mathtt{J}_{\mathtt{N}}(\xi)$, with $2k + 1 \leqslant i \leqslant 2n$, can be arbitrary. The number of such subsequences $(\rho_1, \ldots, \rho_{2k})$ will be (approximately, as $N \to \infty$) the same as if the elements $\rho_2, \ldots, \rho_{2k-1}$ satisfied $\rho_2, \ldots, \rho_{2k-1} \in (0, N\xi - \rho_1]$. This number, however, is the cardinality $|\mathtt{bmNC}_2^{\mathtt{k}-1}([0, N\xi - \rho_1])|$ (in particular, it does not depend on translation), hence we can write

(3.7)
$$\frac{\big| \mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big) \big|}{|\mathtt{J}_{\mathtt{N}}(\xi)|^n} \approx \sum_{k=1}^{n} \sum_{\rho \in \mathtt{J}_{\mathtt{N}}(\xi)} \frac{\big| \mathtt{bmNC}_2^{\mathtt{k}-1}([0, N\xi - \rho]_{\mathbf{I}}) \big|}{|\mathtt{J}_{\mathtt{N}}(\xi)|^k} \cdot \frac{\big| \mathtt{bmNC}_2^{\mathtt{n}-\mathtt{k}}\big(\mathtt{J}_{\mathtt{N}}(\xi) \big) \big|}{|\mathtt{J}_{\mathtt{N}}(\xi)|^{n-k}}.$$

At this point we shall start the induction on n, assuming that for all $1 \leqslant j \leqslant n-1$ and for all $\rho \in \mathbf{I}$ the following limits exist:

$$(3.8) \qquad g_j := \lim_{N \to \infty} \frac{|\mathrm{bmNC}_2^{\mathrm{j}} \mathrm{J_N}(\rho)|}{|\mathrm{J_N}(\rho)|^j}.$$

Hence the right-hand side of (3.7) can be simplified, so that we get

$$(3.9) \qquad \frac{|\mathrm{bmNC}_2^{\mathrm{n}}(\mathrm{J_N}(\xi))|}{|\mathrm{J_N}(\xi)|^n} \approx \sum_{k=1}^{n} g_{n-k} \sum_{\rho \in \mathrm{J_N}(\xi)} \frac{|\mathrm{bmNC}_2^{\mathrm{k-1}}([0, N\xi - \rho]_{\mathbf{I}})|}{|\mathrm{J_N}(\xi)|^k}$$
$$\approx \sum_{k=1}^{n} g_{n-k} \sum_{\rho \in \mathrm{J_N}(\xi)} \frac{|\mathrm{bmNC}_2^{\mathrm{k-1}}([0, \rho]_{\mathbf{I}})|}{|\mathrm{J_N}(\xi)|^k},$$

where the second approximation results from the substitution $N\xi - \rho \mapsto \rho$. Then, by similar estimates to those in Section 3 of [3] and in Proposition 7 of [11] one can show that

$$(3.10) \qquad \sum_{\rho \in \mathrm{J_N}(\xi)} \frac{|\mathrm{bmNC}_2^{\mathrm{k-1}}([0, \rho]_{\mathbf{I}})|}{|\mathrm{J_N}(\xi)|^k} \approx \frac{g_{k-1}}{|[0, N\xi]|^k} \sum_{\rho \in [0, N\xi]} |[0, \rho]_{\mathbf{I}}|^{k-1}.$$

The next step is to use the approximation of the cardinality of an interval by the volume. Let $v(\rho)$ denote the Euclidean volume of the interval $[0, \rho]$ for $\rho \in \Pi$ (each of the cones Π is considered to be embedded in $\mathbb{R}^d$ for the minimal dimension $d \in \mathbb{N}$). Then we shall use the approximation $v(\rho) \approx |[0, \rho]_{\mathbf{I}}|$ so that

$$\lim_{N \to \infty} \frac{g_{k-1}}{|[0, N\xi]_{\mathbf{I}}|^k} \sum_{\rho \in [0, N\xi]_{\mathbf{I}}} |[0, \rho]_{\mathbf{I}}|^{k-1} = \lim_{N \to \infty} \frac{g_{k-1}}{v(N\xi)^k} \sum_{\rho \in [0, N\xi]_{\mathbf{I}}} v(\rho)^{k-1},$$

and (3.10) can be written as

$$(3.11) \qquad \sum_{\rho \in \mathrm{J_N}(\xi)} \frac{|\mathrm{bmNC}_2^{\mathrm{k-1}}([0, \rho]_{\mathbf{I}})|}{|\mathrm{J_N}(\xi)|^k} \approx \frac{g_{k-1}}{v(N\xi)^k} \sum_{\rho \in [0, N\xi]_{\mathbf{I}}} v(\rho)^{k-1}.$$

Since the right-hand side of (3.11) can be written by using the integral form, that is,

$$\lim_{N \to \infty} \frac{g_{k-1}}{v(N\xi)^k} \sum_{\rho \in [0, N\xi]_{\mathbf{I}}} v(\rho)^{k-1} = \lim_{N \to \infty} \frac{g_{k-1}}{v(N\xi)^k} \int_0^{N\xi} v(\rho)^{k-1} \, d\rho,$$

the computation of the limit (3.5) is equivalent to

$$(3.12) \qquad \lim_{N \to \infty} \frac{|\mathrm{bmNC}_2^{\mathrm{n}}(\mathrm{J_N}(\xi))|}{|\mathrm{J_N}(\xi)|^n} = \sum_{k=1}^{n} g_{n-k} g_{k-1} \lim_{N \to \infty} \frac{1}{v(N\xi)^k} \int_0^{N\xi} v(\rho)^{k-1} \, d\rho.$$

Therefore, if we show that for each $k \geqslant 1$ the limit

$$(3.13) \qquad \gamma_k = \lim_{N \to \infty} \frac{1}{v(N\xi)^k} \int_{\rho \in [0, N\xi]} v(\rho)^{k-1}$$

exists, then (3.5) would become

$$(3.14) \qquad \lim_{N \to \infty} \varphi\Big(\big(S_N(\xi)\big)^{2n}\Big) = \lim_{N \to \infty} \frac{\big|\mathtt{bmNC}_2^{\mathtt{n}}\big(\mathtt{J}_{\mathtt{N}}(\xi)\big)\big|}{|\mathtt{J}_{\mathtt{N}}(\xi)|^n} = \sum_{k=1}^{n} g_{n-k} g_{k-1} \gamma_k,$$

proving, by induction on n, the existence of the limit g_n on the left-hand side and, in addition, the recurrence

$$(3.15) \qquad g_n = \sum_{k=1}^{n} g_{n-k} g_{k-1} \gamma_k.$$

These problems will be discussed in the next sections where we shall prove that for sectorial and circular cones the sequence $(\gamma_k)_{k \geqslant 1}$ exists and represents the volume characteristics of the cone (i.e. it does not depend on ξ and N) and exists but depends on two parameters related to ξ for the Vinberg cone.

4. PROOFS OF THE bm-CLT ASSOCIATED WITH SECTORIAL AND CIRCULAR CONES

In this section we shall prove the existence of the volume characteristic for the sectorial and circular non-symmetric cones. This means that we shall show the formula (3.13) for these cones where the sequence $(\gamma_k)_{k \geqslant 1}$ depends on the cone Π but does not depend on a given $\xi \in \Pi$. By $v(\xi)$ we denote the Euclidean volume v of the interval $[0, \xi] \subset \Pi$ (recall that, by definition, $\Pi \supset [0, \xi] := \{\rho \in \Pi : 0 \preceq \rho \preceq \xi\}$).

THEOREM 4.1 (Volume characteristic for sectorial and circular cones). *Let* $\Pi = \Omega_{\mathbf{u}}^d$ *or* $\Pi = \mathcal{C}_\theta^d$. *Then there exists a sequence* $(\gamma_k)_{k \geqslant 1}$ *depending on* Π *and such that for every* $\xi \in \Pi$

$$\gamma_k = \gamma_k(\Pi) := \frac{1}{\big(v(\xi)\big)^k} \int_{\rho \in [0, \xi]_\Pi} \big(v(\rho)\big)^{k-1} d(\rho).$$

The sequences are given by the formulas related to the specific symmetric cones:
  1. *For the sectors* $\Omega_{\mathbf{u}}^d \subset \mathbb{R}^d$ *we have*

$$\gamma_k = \gamma_k^{\mathbf{u}} = \frac{1}{|D|} \gamma_k',$$

where $D := \det(\mathbf{u})$ *is the determinant of the vectors* $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_d)$ *and the sequence* $\gamma_k' = 1/k^d$ *corresponds to* $\Pi = \mathbb{R}_+^d$.
  2. *For the circular cone* $\mathcal{C}_\theta^d$ *we have* $\gamma_k = \gamma_k^\theta = \gamma_k'$ *and the sequence* γ_k' *corresponds to the Lorentz cone* $\Pi = \Lambda_1^d$.

4.1. Proof of volume characteristic and the bm-CLT for sectorial cones $\Omega_{\mathbf{u}}^d$.
For a d-tuple $\mathbf{u} := (\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of linearly independent vectors $\mathbf{u}_i \in \mathbb{R}^d$ ($1 \leqslant i \leqslant d$)
consider the sectorial cone $\Omega_{\mathbf{u}}^d \subset \mathbb{R}^d$ and an element $\xi = \sum_{i=1}^d a_i \mathbf{u}_i \in \Omega_{\mathbf{u}}^d$. Let
$\mathbf{D} := \det(\mathbf{u})$ be the determinant of these vectors (i.e. of the matrix whose columns
are these vectors). Then the Euclidean volume $v(\xi)$ of the interval $[0, \xi] \subset \Omega_{\mathbf{u}}^d$ is
given by the formula

$$(4.1) \qquad v(\xi) = |\mathbf{D}| \cdot \prod_{i=1}^d a_i.$$

Therefore, for $k \in \mathbb{N}$, with the notation $\rho := (b_1, \ldots, b_d)$ we have
$$(4.2)$$
$$\int_{[0,\xi]} v(\rho)^{k-1}\, d\rho = |\mathbf{D}|^{k-1} \prod_{i=1}^d \int_0^{a_i} (b_i)^{k-1}\, db_i = |\mathbf{D}|^{k-1} \prod_{i=1}^d \frac{a_i^k}{k} = \frac{v(\xi)^k}{|\mathbf{D}|} \cdot \frac{1}{k^d}.$$

Since, for the symmetric cone $\mathbb{R}_+^d$, $1/k^d = \gamma_k'$ is the volume characteristic se-
quence, we get

$$(4.3) \qquad \frac{1}{v(\xi)^k} \int_{[0,\xi]} v(\rho)^{k-1}\, d\rho = \frac{\gamma_k'}{|\mathbf{D}|} =: \gamma_k,$$

which does not depend on ξ. Hence, putting $N\xi$ instead of ξ, one can see that the
formula (3.13) holds true.

4.2. Proof of volume characteristic and of the bm-CLT for circular cones $\mathcal{C}_\theta^d$.
For M_θ defined by (2.1) consider the transformation

$$(4.4) \quad \Lambda_1^d \ni \xi = (t; x_2, \ldots, x_d) \longmapsto M_\theta(\xi) := (t \cot \theta; x_2, \ldots, x_d) \in \mathcal{C}_\theta^d.$$

Recall that the Euclidean volume $v(\xi)$ of the interval $[0, \xi] \subset \Lambda_1^d$ in the Lorentz
cones is given by

$$(4.5) \qquad v(\xi) = B_\Lambda\left(\frac{d}{2}, \frac{d}{2}\right) \cdot \left(\det_\Lambda(\xi)\right)^{d/2},$$

where $B_\Lambda(p, q)$ is the Euler beta function for the Lorentz cone and for $\Lambda_1^d \ni \xi = (t; x_2, \ldots, x_d)$ one defines $\det_\Lambda(\xi) := t^2 - x_2^2 - \ldots - x_d^2$ as the associated gener-
alized determinant (for details we refer to [2] and [3]).

Since $\rho \preceq_{\mathcal{C}} \xi$ if and only if $M_\theta^{-1}(\rho) \preceq_\Lambda M_\theta^{-1}(\xi)$, it follows that

$$(4.6) \qquad [0, \xi]_{\mathcal{C}} = M_\theta([0, M_\theta^{-1}\xi]_\Lambda).$$

The transformations M_θ and M_θ^{-1} are linear in $\mathbb{R}^d$, so they change the volumes
by their determinants. If $v([0, \xi]_{\mathcal{C}})$ denotes the Euclidean volume of the interval

$[0, \xi] \subset \mathcal{C}_\theta^d$ in the circular cone and $v([0, \xi]_\Lambda)$ denotes the Euclidean volume of the interval $[0, \xi] \subset \Lambda_1^d$ in the Lorentz cone, then one gets the following relation between the volumes:

$$(4.7) \qquad v([0, \xi]_{\mathcal{C}}) = v\big(M_\theta([0, M_\theta^{-1}\xi]_\Lambda)\big) = \cot(\theta) \cdot v\big([0, M_\theta^{-1}(\xi)]_\Lambda\big).$$

Consequently, we can compute the integrals (for $k \geqslant 1$ and $\xi \in \mathcal{C}_\theta^d$):

$$\int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \big(v([0, \rho]_{\mathcal{C}})\big)^{k-1} d\rho = \int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \Big(\cot(\theta) \cdot v\big([0, M_\theta^{-1}(\rho)]_\Lambda\big) \Big)^{k-1} d\rho$$

$$= \cot^{k-1}(\theta) \int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \Big(v\big([0, M_\theta^{-1}(\rho)]_\Lambda\big) \Big)^{k-1} d\rho$$

$$= \cot^{k-1}(\theta) \int\limits_{\eta \in [0, M_\theta^{-1}(\xi)]_\Lambda} \big(v([0, \eta]_\Lambda)\big)^{k-1} \cot(\theta) d\eta$$

$$= \cot^k(\theta) \int\limits_{\eta \in [0, M_\theta^{-1}(\xi)]_\Lambda} \big(v([0, \eta]_\Lambda)\big)^{k-1} d\eta.$$

By the volume characteristic theorem for the Lorentz cone Λ_1^d there exists a sequence $(\gamma_k')_{k \geqslant 1}$ such that the last integral is equal to

$$(4.8) \qquad \int\limits_{\eta \in [0, M_\theta^{-1}(\xi)]_\Lambda} \big(v([0, \eta]_\Lambda)\big)^{k-1} d\eta = \gamma_k' \cdot \Big(v\big([0, M_\theta^{-1}(\xi)]_\Lambda\big) \Big)^k.$$

Hence

$$(4.9) \qquad \int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \big(v([0, \rho]_{\mathcal{C}})\big)^{k-1} d\rho = \gamma_k' \cdot \big[\cot(\theta) \cdot v\big([0, M_\theta^{-1}(\xi)]_\Lambda\big) \big]^k,$$

so

$$(4.10) \qquad \int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \big(v([0, \rho]_{\mathcal{C}})\big)^{k-1} d\rho = \gamma_k' \cdot v([0, \xi]_{\mathcal{C}})^k.$$

This shows that for the circular cone $\mathcal{C}_\theta^d$ we have the same volume characteristic as for the Lorentz cone:

$$(4.11) \qquad \gamma_k' := \frac{1}{v([0, \xi]_{\mathcal{C}})^k} \int\limits_{\rho \in [0, \xi]_{\mathcal{C}}} \big(v([0, \rho]_{\mathcal{C}})\big)^{k-1} d\rho = \gamma_k.$$

4.3. The bm-CLT for the Vinberg cone. For the Vinberg cone $\Pi_\mathcal{V} \subset \mathbb{M}_3(\mathbb{R})$ the sequence $(\gamma_k)_{k \geqslant 1}$ is not independent of $\xi \in \Pi_\mathcal{V}$. To show its existence and description, we first introduce the following transformation:

$$(4.12) \quad \mathbb{M}_3(\mathbb{R}) \ni A = (a_{ij})_{i,j=1}^3 \mapsto T(A) = \left(\frac{a_{ij}}{\sqrt{a_{ii} a_{jj}}} \right)_{i,j=1}^3 \in \mathbb{M}_3(\mathbb{R}),$$

defined only for matrices with nonzero entries on diagonal. Since in the Vinberg cone a matrix $\xi \in \Pi_{\mathcal{V}}$ enjoys this property, the transformation $\xi \mapsto T(\xi)$ gives

$$\Pi_{\mathcal{V}} \ni \xi = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & 0 \\ a_5 & 0 & a_3 \end{pmatrix} \longmapsto T(\xi) := \begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} =: \xi(\alpha, \beta) \in \Pi_{\mathcal{V}},$$

where

$$(4.13) \qquad \alpha := \frac{a_4}{\sqrt{a_1 a_2}}, \quad \beta := \frac{a_5}{\sqrt{a_1 a_3}}$$

with $-1 \leqslant \alpha, \beta \leqslant 1$.

Now, we shall compute the volume of an interval $(0, \xi] \subset \Pi_{\mathcal{V}}$. If we put

$$(4.14) \qquad \xi = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & 0 \\ a_5 & 0 & a_3 \end{pmatrix}, \quad \rho = \begin{pmatrix} b_1 & b_4 & b_5 \\ b_4 & b_2 & 0 \\ b_5 & 0 & b_3 \end{pmatrix},$$

then $\rho \in (0, \xi]$ is equivalent to $\rho \in \Pi_{\mathcal{V}}$ and $\xi - \rho \in \Pi_{\mathcal{V}}$, which is satisfied if and only if $0 < b_i \leqslant a_i$ for $i = 1, 2, 3$ and

$$(4.15) \qquad (a_j - b_j)^2 \leqslant (a_1 - b_1)(a_{j-2} - b_{j-2}), \quad b_j^2 \leqslant b_1 b_{j-2} \quad \text{for } j = 4, 5.$$

Let $E_j := \{b_j \in \mathbb{R} : (a_j - b_j)^2 \leqslant (a_1 - b_1)(a_{j-2} - b_{j-2}), \ b_j^2 \leqslant b_1 b_{j-2}\}$ for $j = 4, 5$. Then the volume $v(\xi)$ of the interval $(0, \xi] \subset \Pi_{\mathcal{V}}$ can be represented as the multiple integral:

$$(4.16) \qquad v(\xi) = \int\limits_0^{a_1} \int\limits_0^{a_2} \int\limits_0^{a_3} \left(\int\limits_{E_4} \int\limits_{E_5} db_5 \, db_4 \right) db_3 \, db_2 \, db_1.$$

Dividing both inequalities in (4.15) by $a_1 a_{j-2}$ and putting

$$(4.17) \qquad \begin{aligned} x_i &:= b_i/a_i & \text{for } i = 1, 2, 3, \\ x_j &:= b_j/\sqrt{a_1 a_{j-2}} & \text{for } j = 4, 5, \end{aligned}$$

one gets $x_1, x_2, x_3 \in (0, 1]$ and

$$(4.18) \qquad \begin{aligned} (\alpha - x_4)^2 &\leqslant (1 - x_1)(1 - x_2), & x_4^2 &\leqslant x_1 x_2, \\ (\beta - x_5)^2 &\leqslant (1 - x_1)(1 - x_3), & x_5^2 &\leqslant x_1 x_3, \end{aligned}$$

with α, β given by (4.13). Observe that these conditions are equivalent to $\zeta \in [0, \xi(\alpha, \beta)] \subset \Pi_{\mathcal{V}}$ for

$$\zeta := \begin{pmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{pmatrix}.$$

Therefore, putting for $j = 4, 5$

(4.19)
$$D_j(s) := \{x_j \in [-1, 1] \: : \: x_j^2 \leqslant x_1 x_{j-2}, \: (s - x_j)^2 \leqslant (1 - x_1)(1 - x_{j-2})\}$$

and using the above change of variables (4.13) in the integrals (4.16), for which the Jacobian is $a_1^2(a_2 a_3)^{3/2}$, we can write the volume $v(\xi)$ of the interval $(0, \xi] \subset \Pi_{\mathcal{V}}$ as the integral

(4.20)
$$v(\xi) = a_1^2(a_2 a_3)^{3/2} \cdot \int_0^1 \int_0^1 \int_0^1 \Big(\int_{D_4(\alpha)} \int_{D_5(\beta)} dx_5 \, dx_4 \Big) dx_1 \, dx_2 \, dx_3$$

$$= a_1^2(a_2 a_3)^{3/2} \cdot v\big(\xi(\alpha, \beta)\big) = a_1^2(a_2 a_3)^{3/2} \cdot \Upsilon(\alpha, \beta),$$

where the function $\Upsilon(\alpha, \beta) = v\big(\xi(\alpha, \beta)\big) = v\big(T(\xi)\big)$ represents the volume of the interval $[0, \xi(\alpha, \beta)] \subset \Pi_{\mathcal{V}}$.

PROPOSITION 4.1. *For the Vinberg cone $\Pi_{\mathcal{V}}$ and an element $\xi \in \Pi_{\mathcal{V}}$, with $T(\xi) = \xi(\alpha, \beta)$, there exists a sequence $\big(\gamma_k(\alpha, \beta)\big)_{k \geqslant 1}$ such that*

(4.21)
$$\gamma_k(\alpha, \beta) = \frac{1}{v(\xi)^k} \int_0^\xi v(\rho)^{m-1} d\rho.$$

Given $\alpha, \beta \in [-1, 1]$, the sequence $\gamma_k(\alpha, \beta)$ can be expressed as

(4.22)
$$\gamma_k(\alpha, \beta) = \frac{1}{\Upsilon(\alpha, \beta)^k} \int_0^1 \int_0^1 \int_0^1 \int_{D_4(\alpha)} \int_{D_5(\beta)} \left[x_1^2 (x_2 x_3)^{3/2} \Upsilon\left(\frac{x_4}{\sqrt{x_1 x_2}}, \frac{x_5}{\sqrt{x_1 x_3}} \right) \right]^{k-1} d\mathbf{x},$$

where we used the abbreviation $d\mathbf{x} := dx_5 dx_4 dx_3 dx_2 dx_1$.

P r o o f. The formula (4.20) can be written as

$$v(\xi) = a_1^2(a_2 a_3)^{3/2} \cdot \Upsilon\left(\frac{a_4}{\sqrt{a_1 a_2}}, \frac{a_5}{\sqrt{a_1 a_3}} \right)$$

and in the proof we shall use this form for other elements of the Vinberg cone $\Pi_{\mathcal{V}}$. Using the substitution (4.17) and the notation $d\mathbf{x} := dx_5 dx_4 \, dx_3 dx_2 dx_1$, $d\mathbf{b} := db_5 db_4 \, db_3 db_2 db_1$ and (4.14), we can write

$$\int_0^\xi v(\rho)^{k-1} d\rho = \int_0^{a_1} \int_0^{a_2} \int_0^{a_3} \int_{E_4} \int_{E_5} \left[b_1^2 (b_2 b_3)^{3/2} \Upsilon\left(\frac{b_4}{\sqrt{b_1 b_2}}, \frac{b_5}{\sqrt{b_1 b_3}} \right) \right]^{k-1} d\mathbf{b}$$

$$= [a_1^2 (a_2 a_3)^{3/2}]^k \int_0^1 \int_0^1 \int_0^1 \int_{D_4(\alpha)} \int_{D_5(\beta)} \left[x_1^2 (x_2 x_3)^{3/2} \Upsilon\left(\frac{x_4}{\sqrt{x_1 x_2}}, \frac{x_5}{\sqrt{x_1 x_3}} \right) \right]^{k-1} d\mathbf{x}$$

$$= \left[\frac{v(\xi)}{\Upsilon(\alpha, \beta)} \right]^k \int_0^1 \int_0^1 \int_0^1 \int_{D_4(\alpha)} \int_{D_5(\beta)} \left[x_1^2 (x_2 x_3)^{3/2} \Upsilon\left(\frac{x_4}{\sqrt{x_1 x_2}}, \frac{x_5}{\sqrt{x_1 x_3}} \right) \right]^{k-1} d\mathbf{x}$$

(observe that $b_j/\sqrt{b_1 b_{j-2}} = x_j/\sqrt{x_1 x_{j-2}}$ for $j = 4, 5$). Consequently, we get

$$\frac{1}{v(\xi)^k} \int_0^\xi v(\rho)^{k-1}\, d\rho$$

$$= \frac{1}{\Upsilon(\alpha, \beta)^k} \int_0^1 \int_0^1 \int_0^1 \int_{D_4(\alpha)} \int_{D_5(\beta)} \left[x_1^2 (x_2 x_3)^{3/2} \Upsilon \left(\frac{x_4}{\sqrt{x_1 x_2}}, \frac{x_5}{\sqrt{x_1 x_3}} \right) \right]^{k-1} d\mathbf{x},$$

where the right-hand side depends only on $k \in \mathbb{N}$ and $\alpha, \beta \in [-1, 1]$ and means $\gamma_k(\alpha, \beta)$. This proves the proposition. ∎

REMARK 4.1. *Since for $N \in \mathbb{N}$ and $\xi \in \Pi_\mathcal{V}$ we have $T(N\xi) = T(\xi)$, it follows that*

$$(4.23) \qquad\qquad \gamma_k(\alpha, \beta) = \frac{1}{v(N\xi)^k} \int_0^{N\xi} v(\rho)^{m-1} d\rho.$$

This shows that the limit in (3.13) *exists, which completes the proof of the bm-CLT for the Vinberg cone.*

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