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SOME RESULTS ON CYLINDRICAL MEASURES AND APPLICATION

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Abstract. The aim of this paper which is a continuation of [8] is to study cylindrical measures on a locally convex space which are scalarly concentrated on compact sets. We give some results concerning continuity of the covariance operator and compactness of the reproducing kernel Hilbert space of such a measure. Moreover, we give some applications concerning Gaussian cylindrical measures and the law of the iterated logarithm.

1. Preliminaries. Let E be a locally convex space (l.c.s.) and E' its topological dual. By E'_σ and E'_τ we shall denote the space E' under topologies $\sigma(E', E)$ and $\tau(E', E)$ (weak and Mackey, respectively). Let $\mathfrak{S}$ be a family of subsets of E. We denote by $E'_{\mathfrak{S}}$ the topology $\mathfrak{S}$-convergence on E' (i.e. the topology of uniform convergence on sets from $\mathfrak{S}$). If E and F are two locally convex spaces, then $L(E, F)$ will denote the set of all continuous linear operators from E into F. If $(\Omega, \mathscr{A}, P)$ is a probability space, then the linear map $T\colon E' \to L^0(\Omega, \mathscr{A}, P)$ is called the *cylindrical process*.

We say that a cylindrical process T is *Pettis integrable* if

(1) $T(E') \subset L^1$ and

(2) for each $A \in \mathscr{A}$ there is an $x_A \in E$ such that $\langle x_A, x'\rangle = \int_A Tx'\,dP$ for each $x' \in E'$. We write $\int_A T\,dP = x_A$.

Remark 1. *If T is a $(\tau(E', E), \|\cdot\|_{L^1})$-continuous cylindrical process, then T is Pettis integrable.*

The above remark follows from the fact that the transpose operator T^* to T maps L^∞ into E. More facts concerning connections between Mackey continuity and integrablility can be found in [9].

Now, let $\mathscr{C}(E)$ denote the algebra of cylindrical subsets of E, and μ a cylindrical measure on $\mathscr{C}(E)$. It can be shown (cf. [1], p. 41) that there exist a probability space and a cylindrical process $T_\mu\colon E' \to L^0(\Omega, \mathscr{A}, \mathscr{P})$ such that for each $C \in \mathscr{C}(E)$, $C = \{x \in E: (\langle x, x'_1\rangle, \ldots, \langle x, x'_n\rangle) \in B\}$, where $n \in N$, $x'_1, \ldots, x'_n \in E'$ and $B \in (\mathscr{B}_{R^n})$, we have

$$\mu(C) = P\{(T_\mu x'_1, \ldots, T_\mu x'_n) \in B\}.$$

This correspondence between μ and T_μ is one-to-one. In the sequel $\mu_{x'_1,\ldots,x'_n}$ will denote the distribution of the random vector $(T_\mu x'_1, \ldots, T_\mu x'_n)$. We say that a cylindrical measure μ has a *weak p-order* if

$$\int_\Omega |T_\mu x'|^p \, dP < \infty \quad \text{for each } x' \in E'.$$

If μ has the weak second order then there exist $m_\mu \in E'^*$ (E'^* denotes the algebraic dual to E') and a linear operator R_μ: $E' \to E'^*$ such that

$$\langle m_\mu, x'\rangle = \int_\Omega T_\mu x' \, dP$$

and

$$\langle R_\mu x', y'\rangle = \int_\Omega T_\mu x' \, T_\mu y' \, dP - \langle m_\mu, x'\rangle \langle m_\mu, y'\rangle$$

for each $x', y' \in E'$. m_μ is called the *mean* of μ; R_μ — the *covariance operator* of μ.

On the subspace $R_\mu(E')$ of E'^* we define the inner product. For $h_1, h_2 \in R_\mu(E')$, $h_1 = R_\mu x'_1$, $h_2 = R_\mu x'_2$, we set $(h_1, h_2)_\mu = \langle R_\mu x'_1, x'_2\rangle$. By H_μ we shall denote the completion of $R_\mu(E')$ in the norm $\|\cdot\|_\mu = (\cdot, \cdot)_\mu^{1/2}$. H_μ is called the *reproducing kernel Hilbert space* (RKHS) of μ.

We say that a cylindrical measure μ is *Gaussian* if for each $n \in N$ and $x'_1, \ldots, x'_n \in E'$, $(T_\mu x'_1, \ldots, T_\mu x'_n)$ is a Gaussian random vector (or, which is the same, if $\mu_{x'_1,\ldots,x'_n}$ is a Gaussian measure on R^n).

2. Cylindrical measures scalarly concentrated on compact sets. Results of this section are based on the following

LEMMA 1. *Let E and F be l.c.s. and let $\mathfrak{S}$ be the family of all compact and balanced subsets of E. If T is a linear operator from E' into F, then the following conditions are equivalent:*

(a) $T \in L(E'_{\mathfrak{S}}, F)$;

(b) $T \in L(E'_\tau, F)$ *and, for each equicontinuous subset A of E', $T(A)$ is relatively compact in F.*

Proof. (a) $\Rightarrow$ (b). If $T \in L(E'_{\mathfrak{S}}, F)$ then $T \in L(E_\tau, F)$ because the $\mathfrak{S}$-topology on E' is weaker than $\tau(E', E)$. Let A be an equicontinuous subset of E' and let $\bar{A}$ denote the closure of A in $\sigma(E', E)$. Because of [6] (4.3, p. 84) the set $\bar{A}$ is $\sigma(E', E)$-compact. Then on set $\bar{A}$ the topology $\sigma(E', E)$ coincides with $\mathfrak{S}$-topology ([6], 4.5, p. 85). Therefore $\bar{A}$ is compact in $E'_{\mathfrak{S}}$. By assumption $T \in L(E'_{\mathfrak{S}}, F)$, so $T(\bar{A})$ is compact in F and therefore $T(A)$ is relatively compact.

(b) $\Rightarrow$ (a). Let $\mathfrak{S}_1$ denote the family of all compact and balanced subsets of F and let T^* denote the transpose to T. Since $T \in L(E'_\tau, F)$ then T^* maps F' into E. First we show that $T^* \in L(F'_{\mathfrak{S}_1}, E)$. Let U be an absolutely convex neighbourhood of 0 in E'. Then U° ($^\circ$ denotes the polar) is an equicontinuous subset of E', and by the assumption $T(U^\circ)$ is relatively compact in F. If K is the closure of $T(U^\circ)$, it is easy to show that $T^*(K^\circ) \subset U$. Since K° is a neighbourhood of 0 in $F'_{\mathfrak{S}_1}$, $T^* \in L(F'_{\mathfrak{S}_1}, E)$. Therefore, using the implication (a) $\Rightarrow$ (b), we see that $T^* \in L(F'_\tau, E)$ and, for each equicontinuous subset B of F', $T^*(B)$ is relatively compact in E. Now, by repeating the above arguments replacing T by T^*, we have $T = (T^*)^* \in L(E'_{\mathfrak{S}}, F)$.

Let $\mathfrak{S}$ be a family of subsets of E and μ a cylindrical measure on E. We say that μ is *scalarly concentrated* on $\mathfrak{S}$ if for each $\varepsilon > 0$ there is an $A \in \mathfrak{S}$ such that

$$(\mu_{x'})_*(x'(A)) \geqslant 1-\varepsilon \quad \text{for each } x' \in E'$$

($_*$ denotes the inner measure).

The following Theorem 1 is an extension of the similar one (c.f. [8] Theorem 4.2) on the case of an arbitrary locally convex space (without assumption that E is complete). Methods of the proof of this theorem do not use notations of tensor products.

THEOREM 1. *Let E be an l.c.s. and let $\mathfrak{S}$ be a family of all compact and balanced subsets of E. If μ is a Gaussian cylindrical measure scalarly concentrated on $\mathfrak{S}$, then we have*:

(a) $m_\mu \in E$.

(b) R_μ: $E' \to E$ *and* $R_\mu(U^\circ)$ *is a relatively compact subset of E for each neighbourhood U of 0 in E.*

(c) H_μ *is a subspace of E and the canonical injection* θ: $H_\mu \to E$ *is continuous.*

(d) $\{h \in H_\mu \colon \|h\|_\mu \leqslant 1\}$ *is a compact subset of E.*

Proof. (a) By [1], Proposition 2, p. 182, $T_\mu \in L(E'_{\mathfrak{S}}, L^2)$, so $T_\mu \in L(E'_\tau, L^2)$. In particular $T_\mu \in L(E'_\tau, L^1)$ and, therefore, T is Pettis integrable. Hence $m_\mu = \int_\Omega T_\mu d_\mu$ is an element of E.

(b) Since $T_\mu \in L(E'_\tau, L^2)$, it is easy to show that $\int_\Omega \varphi T_\mu dP \in E$ for each function $\varphi \in L^1$. This implies

$$R_\mu x' = \int_\Omega T_\mu x' T_\mu dP - \langle m_\mu, x'\rangle m_\mu \in E \quad \text{for each } x' \in E'.$$

Now we prove the second part of (b). Let $T_1 = T_\mu x' - \langle m_\mu, x'\rangle$. Note that the transpose T_1^* to T_1 is of the form

$$\langle T_1^* \varphi, x'\rangle = \int_\Omega \varphi T_1 x' dP$$

for each $\varphi \in L^2$ and $x' \in E'$, and that $R_\mu = T_1^* \circ T_1$. Since the polar of the neighbourhood U of 0 in E is an equicontinuous subset of E', it follows from Lemma 1 that $R_\mu(U^\circ)$ is a relatively compact subset of E.

(c) Let $\mathscr{H}$ denote the closure of $T_1^*(E')$ in L^2. Since T_1^* is defined on whole L^2, $T_1^*(\mathscr{H}) \subset E$. Moreover, T_1^* is the one-to-one and isometric map from $T_1(E')$ onto $R_\mu(E')$. Therefore, T_1^* can extend to a map from $\mathscr{H}$ onto H_μ, which map possesses the same properties as T_1^*. In particular $H_\mu = T_1^*(\mathscr{H})$ and hence $H_\mu \subset E$. Now we show that the canonical injection $\theta\colon H_\mu \to E$ is continuous. Let U be an equicontinuous neighbourhood of 0 in E and let $p_U(\cdot)$ denote the Minkowski functional of U. For $h \in R_\mu(E')$, $h = R_\mu x'$, we have

$$p_U(h) = \sup_{y' \in U^\circ} |\langle R_\mu x', y'\rangle| \leqslant \|T_1 x'\|_{L^2} \sup_{y' \in U^\circ} \|T_1 y'\|_{L^2}$$
$$= \|h\|_\mu \sup_{y' \in U^\circ} \|T_1 y'\|_{L^2}.$$

Since $T_1 \in L(E'_\tau, L^2)$,

$$\sup_{y' \in U^\circ} \|T_1 y'\|_{L^2} < \infty.$$

Consequently, $\theta\colon R_\mu(E') \to E$ is continuous and hence the injection of H_μ into E is continuous.

(d) Using the similar arguments as in [7] (Theorem 4.2 (d)) and Lemma 1 it can be shown that the unit ball in H_μ is a compact subset of E.

The next theorem extends Theorem 1 on the case of non-Gaussian cylindrical measures.

THEOREM 2. *Let E be an l.c.s. and let $\mathfrak{S}$ be the family of all compact and balanced subsets of E. If μ is a cylindrical measure on E such that $T_\mu \in L(E'_{\mathfrak{S}}, L^2)$, then there are satisfied points* (a), (b), (c) *and* (d) *of Theorem* 1.

The proof of this theorem is similar to the proof of Theorem 1.

3. Applications. Let μ be a cylindrical measure on an l.c.s. E and let T_μ be its cylindrical process. The function $\hat{\mu}\colon E' \to \mathbf{C}$,

$$\hat{\mu}(x') = \int_\Omega \exp(iT_\mu x')\,dP,$$

is called the *characteristic functional* of μ.

Now, let H be a Hilbert space. The cylindrical measure γ_H on H is said to be *canonical* Gaussian measure if $\gamma_H(x) = \exp(-\frac{1}{2}\|x\|^2)$ (here H' is identified with H). Using Bochner Theorem it is easy to show that the function $H \ni x \to \exp(-\frac{1}{2}\|x\|^2)$ is a characteristic functional of some Gaussian measure on H (in the sense of the definition from section 1). This Gaussian measure is scalarly concentrated on family of all balls in H (cf. [7], p. 327), so it is concentrated on $\sigma(H, H')$ compact sets. Theorem 3 shows that a Gaussian measure on an l.c.s. can be obtained as an image of the canonical Gaussian measure if and only if it is scalarly concentrated on weakly compact sets. This theorem extends results of Borell ([2], Theorem 4) and Dudley, Feldman and Le Cam ([3], Theorem 2.1) which were obtained for Radon measures.

THEOREM 3. *Let E be an l.c.s. and $\mathfrak{S}$ a family of all $\sigma(E, E')$ compact and balanced subsets of E. If μ is a Gaussian cylindrical measure on E with $m_\mu = 0$, then the following conditions are equivalent*:

(a) *there exists a Hilbert space H and weakly continuous linear operator $S\colon H \to E$ such that $S(\gamma_H) = \mu$.*

(b) *μ is scalarly concentrated on $\mathfrak{S}$.*

Proof. Suppose μ is scalarly concentrated on $\mathfrak{S}$. Then, applying Theorem 1 to the space $(E, \sigma(E, E'))$, we obtain that $H_\mu \subset E$ and that $\theta\colon H_\mu \to (E, \sigma(E, E'))$ is continuous. Hence, θ is $(\sigma(H_\mu, H'_\mu), \sigma(E, E'))$ continuous and therefore θ^* maps E' into H_μ ([6], 7.4, p. 158). Now let γ_H be canonical Gaussian measure on H. Since

$$\theta(\gamma_H)(x') = \hat{\gamma}_H(\theta^* x') = \exp\left(-\tfrac{1}{2}\|\theta^* x'\|_\mu^2\right)$$
$$= \exp\left(-\tfrac{1}{2}\langle R_\mu x', x'\rangle\right) = \hat{\mu}(x')$$

for each $x' \in E'$, we have $\theta(\gamma_H) = \mu$.

Conversely, let us suppose that there exist a Hilbert space and a weakly continuous linear operator $S\colon H \to E$ such that $S(\gamma_H) = \mu$. Then

$$\hat{\mu}(x') = \hat{\mu}_H(S^* x') = \exp\left(-\tfrac{1}{2}\|S^* x'\|_H^2\right).$$

On the other hand, $\hat{\mu}(x') = \exp\left(-\tfrac{1}{2}\|T_\mu x'\|_{L^2}^2\right)$. Since the linear operator $S^*\colon E' \to H$ is $(\tau(E', E), \|\cdot\|_H)$ continuous ([6], 7.8, p. 158) and $E'_{\mathfrak{S}} = E'_\tau$, it follows that $T_\mu \in L(E'_\tau, L^2) = L(E'_{\mathfrak{S}}, L^2)$. Therefore, by [1], Proposition 2, p. 182, μ is scalarly concentrated on $\mathfrak{S}$.

In the sequel we shall use the following

LEMMA 2. *Let E be a separable and complete l.c.s. and $\mathfrak{S}$ a family of all compact and balanced subsets of E. Assume that T is a linear operator from E' into a Banach space F such that $\|Tx'_n\|_F \to 0$ for each sequence $\{x'_n\} \subset E'$, where $x'_n \to 0$ in $\sigma(E', E)$. Then $T \in L(E'_{\mathfrak{S}}, F)$.*

Proof. Since $y' \circ T$ is, for each $y' \in F'$, a $*$-sequentially continuous linear functional on E', then $y' \circ T$ is $(\tau(E', E), \|\cdot\|_F)$ continuous ([6], 7.4, p. 158).

Now, let A be an equicontinuous subset of E'. We show that $T(A)$ is relatively compact in F. Let $\{y_n\}$ be a sequence from $T(A)$, $y_n = Tx'_n$, where $x'_n \in A$, and let $\bar{A}$ denote the closure of A in $\sigma(E', E)$. Since E is separable, $\bar{A}$ equiped with $\sigma(E', E)$ topology is compact metric space ([6], 1.7, p. 128). Then there exists a subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ which weakly converges to some $x'_0 \in \bar{A}$. Therefore $\|Tx'_{n_k} - Tx'_0\| \to 0$ and by Lemma 1 we obtain that $T \in L(E'_{\mathfrak{S}}, F)$.

Now, as the first application of Theorem 2 we shall give some generalization of Theorem 2.1' from [4].

THEOREM 4. *Let E be a metrisable and complete l.c.s. If μ is a Radon measure on E such that*

$$\int_E \|x\|^2 \mu(dx) < \infty$$

for each seminorm on E, then points (a)-(b) *of Theorem* 1 *are satisfied.*

Proof. Since μ is a Radon measure, it may be assumed that E is separable. Let $\{x'_n\}$ be a sequence from E' which converges to 0 in $\sigma(E', E)$. By assumptions on E, it follows that $\{x'_n\}$ is equicontinuous and therefore $\{x_n\}^\circ$ (the polar with respect to $\langle E', E\rangle$) is a neighbourhood of 0 in E. Let $\|\cdot\|$ denote the Minkowski functional of $\{x_n\}^\circ$. Then there exists a constant C such that $|\langle x, x'_n\rangle| \leqslant C\|x\|$ for each $n \in N$. Note, that $T_\mu x' = \langle \cdot, x'\rangle$ is the associated with μ cylindrical process on the probability space $(E, \mathscr{B}_E, \mu)$. Since $|T_\mu x'_n| \leqslant C\|\cdot\| \in L^2(E, \mathscr{B}_E, \mu)$, it follows from Lebesgue theorem that $\|Tx'_n\|_{L^2} \to 0$. Therefore, using Lemma 2, we obtain that μ satisfies the assumptions of Theorem 2.

Now we give some applications to the law of the iterated logarithm (LIL). Let E be a metrizable space and $(\Omega, \mathscr{A}, P)$ a probability space. Let $X: \Omega \to E$ be a random variable (i.e. weakly measurable an separably valued function) and let X_n, $n = 1, 2, \ldots$, be independent copies of X. Let $S_n = X_1 + \ldots + X_n$ and $Q_n = \sqrt{2n \ln \ln n}$ for $n \geqslant 3$, $a_1 = a_2 = 1$. We say that a random variable X satifies the LIL if there exists a compact subset K of E such that

$$P\{\omega \in \Omega: a_n^{-1} S_n(\omega) \in K\} = 1.$$

The below theorem about compactness of the unit ball in RKHS of a random variable X which satisfies LIL, was first shown by Kuelbs ([4], Theorem 3.1) under assumption that E is a Banach space and that X has the strong second order. Then Pisier ([4], Theorem 1.1) has shown this theorem without assumption that X has the strong second order. Thanks to the Theorem 2 we can give an extension of this result on the case of Fréchet spaces and, moreover, the proof is now simpler.

THEOREM *Let $(\Omega, \mathscr{A}, P)$ be a probability space, E a metrizable and complete l.c.s., and $X: \Omega \to E$ a random variable which satisfies LIL. Denote by H_X the RKHS of X and by B_X the unit ball in H_X. Then B_X is a compact subset of E.*

Proof. Denote by T the cylindrical process associated with X, i.e. $Tx' = \langle X, x'\rangle$. By the LIL for real random variable we have:

(1) $\int_{\Omega} Tx'\,dP = 0$ for each $x' \in E'$;

(2) $\overline{\lim_n} \langle a_n^{-1} S_n, x' \rangle = \left(\int_{\Omega} \langle X, x' \rangle^2\,dP\right)^{1/2} = \|Tx'\|_{L^2}$.

Let $\Omega_1 = \{\omega\colon a_n^{-1} S_n(\omega) \in K\}$, where K is a compact subset of E. Let $\{x_k'\}$ be a sequence from E' such that $x_k' \to 0$ in $\sigma(E', E)$, and let $\omega_k \in \Omega_1$ satisfy

$$\overline{\lim_n} \langle a_n^{-1} S_n(\omega_k), x_k' \rangle = \|Tx_k'\|_{L^2}.$$

Then

$$\|Tx_k'\|_{L^2} = \overline{\lim_n} \langle a_n^{-1} S_n(\omega_k), x_k' \rangle \leqslant \sup_{x \in K} \langle x, x_k' \rangle \to 0,$$

if $k \to \infty$. Therefore, using Lemma 2 and Theorem 2, we obtain that B_X is compact in E.

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