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ON RECURRENT DIFFERENTIAL REPRESENTATIONS FOR STATIONARY STOCHASTIC PROCESSES

BY

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Abstract. In this paper differential representations for stationary stochastic processes with quotients of analytic functions of minimal type as spectral characteristics are given. Such a process is a limit (in the mean square sense) of stationary stochastic processes $y_n(t)$ ($n = 1, 2, \ldots$) which are solutions of an infinite-dimensional system of stochastic differential equations. There are some recurrent connections between $y_n(t)$ and for that reason we call the differential representations considered in this paper recurrent. The representations are applied to find a necessary and sufficient condition for absolute continuity of measures generated by Gaussian stationary processes with spectral characteristics mentioned above. This condition takes the form

$$\lim_{\lambda \to \infty} \frac{g_y(\lambda)}{g_x(\lambda)} = 1.$$

Thereby the Feldman theorem is generalized.

1. Introduction. On the probability space (Ω, F, P) solutions of many problems of statistics of stochastic processes (for instance filtration, prediction, interpolation, testing of hypothesis) are effective for processes of the form

$$y(t) = y(0) + \int_0^t F(s)\,ds + \int_0^t B(s)\,dW(s), \qquad t \in [0, T], \tag{1}$$

where $F(s)$, $B(s)$ are stochastic processes on (Ω, F, P) and $W(s)$ is the Wiener process. The last integral in (1) is understood as Ito's stochastic integral (see [1], IV).

We may rewrite (1) in the form

$$dy(t) = F(t)\,dt + B(t)\,dW(t)$$

and call it *stochastic differential representation* for $y(t)$.

The particular form of stochastic differential representations are stochastic differential equations (see [1], IV)

$$(2) \qquad y(t) = y(0) + \int_0^t a(s, y)\,ds + \int_0^t b(s, y)\,dW(s), \qquad t \in [0, T].$$

This explains, why it is important to obtain theorems presenting a given stationary process as a solution of stochastic differential equation. The theorem like that for stationary processes with rational spectral densities can be find in [1] (see Theorem 15.4). The purpose of this paper is to find differential representations for more general stationary processes. The representations will be applied to investigate absolute continuity of measures generated by these processes.

2. Recurrent differential representations. We consider the class of functions $g(\lambda) = |h(i\lambda)|^2$ where

$$(3) \quad h(i\lambda) = \frac{P(i\lambda)}{Q(i\lambda)}, \quad P(i\lambda) = \prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{b_k}\right), \quad \text{and} \quad Q(i\lambda) = \prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{a_k}\right).$$

The products $P(i\lambda)$ and $Q(i\lambda)$ are absolutely convergent.

The function $g(\lambda)$ will play a role of spectral density of a stationary stochastic process, for the reason some assumptions will be imposed on $\{a_k\}$ and $\{b_k\}$. These assumptions will turn out necessary (see Remark 1) for existence of recurrent differential representation for stationary process.

We assume

$$(4) \qquad \underset{\delta>0}{\exists} \ \underset{k=1,2,\ldots}{\forall} \ \operatorname{Re} a_k < \delta$$

and the series

$$(5) \qquad \sum_{k=1}^{\infty} |a_{k+1} - b_k|$$

is convergent.

Define $\{h_n(i\lambda)\}$ as

$$h_1(i\lambda) = \frac{1}{1-\dfrac{i\lambda}{a_1}} \quad \text{and} \quad h_n(i\lambda) = \frac{\prod_{k=1}^{n-1}\left(1-\dfrac{i\lambda}{b_k}\right)}{\prod_{k=1}^{n}\left(1-\dfrac{i\lambda}{a_k}\right)} \quad \text{for } n > 1.$$

Therefore, for $n > 1$,

$$h_n(i\lambda) = h_1(i\lambda) \prod_{k=1}^{n-1} \frac{1-\dfrac{i\lambda}{b_k}}{1-\dfrac{i\lambda}{a_{k+1}}}.$$

or

$$h_n(i\lambda) = h_{n-1}(i\lambda)\frac{1-\dfrac{i\lambda}{b_{n-1}}}{1-\dfrac{i\lambda}{a_n}}.$$

Now we prove a technical lemma:

LEMMA. *If* (4) *and* (5) *hold, then*

(a) $h(i\lambda) = \text{l.i.m.}\ h_n(i\lambda)$;

(b) *there exist constants* σ *and* σ_n *such that*

$$i\lambda h(i\lambda) - \sigma = \text{l.i.m.}\ i\lambda h_n(i\lambda) - \sigma_n;$$

where "l.i.m" stands for "limit in mean in the L^2 *sense".*

Proof. (a) From assumption (4) and

$$\prod_{k=1}^{n}(1+|x_k|) \leqslant \exp\sum_{k=1}^{n}|x_k|$$

we obtain

$$\left|\prod_{k=1}^{n-1}\frac{1-\dfrac{i\lambda}{b_k}}{1-\dfrac{i\lambda}{a_{k+1}}}\right| \leqslant \exp\sum_{k=1}^{n-1}\left|\frac{i\lambda}{1-\dfrac{i\lambda}{a_{k+1}}}\left(\frac{1}{a_{k+1}}-\frac{1}{b_k}\right)\right| \tag{6}$$

$$\leqslant \exp C\sum_{k=1}^{n-1}\left|1-\frac{a_{k+1}}{b_k}\right|, \quad C\in\mathbf{R}.$$

But

$$\sum_{k=1}^{n-1}\left|1-\frac{a_{k+1}}{b_k}\right| \leqslant \sum_{k=1}^{n-1}|a_{k+1}-b_k|, \tag{7}$$

therefore, applying (5), we conclude

$$\forall_{n\in N}\ |h_n(i\lambda)| = |h_1(i\lambda)|\left|\prod_{k=1}^{n-1}\frac{1-\dfrac{i\lambda}{b_k}}{1-\dfrac{i\lambda}{a_{k+1}}}\right| \leqslant C_1|h_1(i\lambda)|, \quad C_1\in\mathbf{R}.$$

Hence $\text{l.i.m}\ h_n(i\lambda) = h(i\lambda)$.

(b) Let

$$\sigma_1 = -a_1 \quad \text{and} \quad \sigma_n = -a_1\prod_{k=1}^{n-1}\frac{a_{k+1}}{b_k} \quad \text{for } n>1.$$

By (5) the product

$$\prod_{k=1}^{\infty} \frac{a_{k+1}}{b_k}$$

is absolutely convergent, hence

$$\sigma = \lim \sigma_n = -a_1 \prod_{k=1}^{\infty} \frac{a_{k+1}}{b_k}$$

is well defined.

Write

$$h_n(i\lambda) = \frac{P_{n-1}(i\lambda)}{Q_n(i\lambda)} = \frac{\sum_{k=0}^{n-1} d_k (i\lambda)^k}{\sum_{k=0}^{n} c_k (i\lambda)^k}.$$

Since

$$\lim_{\lambda \to \infty} i\lambda h_n(i\lambda) = \frac{d_{n-1}}{c_n} = \sigma_n,$$

we have

$$i\lambda h_n(i\lambda) - \sigma_n = \frac{d_{n-1}}{c_n} + \frac{r_{n-1}(i\lambda)^{n-1} + \ldots + r_0}{\prod_{k=1}^{n-1}\left(1 - \frac{i\lambda}{a_{k+1}}\right)} h_1(i\lambda) - \sigma_n$$

$$= \frac{r_{n-1}(i\lambda)^{n-1} + \ldots + r_0}{\prod_{k=1}^{n-1}\left(1 - \frac{i\lambda}{a_{k+1}}\right)} h_1(i\lambda),$$

where

$$r_{n-1} = \left(c_n \frac{d_{n-2}}{d_{n-1}} - c_{n-1}\right) \sigma_n.$$

Consider the sequence

$$A_n = \frac{r_{n-1}}{c_n} = \sigma_n \left(\frac{d_{n-2}}{d_{n-1}} - \frac{c_{n-1}}{c_n}\right).$$

From Vieta's formula we obtain

$$\frac{d_{n-2}}{d_{n-1}} = -(b_1 + b_2 + \ldots + b_{n-1}) \quad \text{and} \quad \frac{c_{n-1}}{c_n} = -(a_1 + a_2 + \ldots + a_n),$$

therefore

$$A_n = \sigma_n \left[a_1 + \sum_{k=1}^{n-1} (a_{k+1} - b_k)\right]$$

and $\{A_n\}$ is by (5) absolutely convergent, which implies in the similar way as in the first part of the proof

(8) $$\forall_{n \in \boldsymbol{N}} |i\lambda h_n(i\lambda) - \sigma_n| \leqslant C_2 |h_1(i\lambda)|, \quad C_2 \in \boldsymbol{R}.$$

Hence

$$\text{l.i.m. } i\lambda h_n(i\lambda) - \sigma_n = i\lambda h(i\lambda) - \sigma.$$

COROLLARY. *Considering the limits*

$$\lim_{\lambda \to \infty} \lim_{n \to \infty} |i\lambda h_n(i\lambda) - \sigma_n|$$

and applying (8) *we obtain*

(9) $$\lim_{\lambda \to \infty} i\lambda h(i\lambda) = \sigma.$$

Now we can define processes $y(t)$ and $y_n(t)$ as

(10) $$y(t) = \int_{\boldsymbol{R}} e^{it\lambda} h(i\lambda) \Phi(d\lambda), \quad y_n(t) = \int_{\boldsymbol{R}} e^{it\lambda} h_n(i\lambda) \Phi(d\lambda),$$

where $\Phi(d\lambda)$ denotes an orthogonal spectral measure:

$$\mathrm{E}\Phi(d\lambda) = 0, \quad \mathrm{E}|\Phi(d\lambda)|^2 = \frac{d\lambda}{2\pi}, \quad t \in [0, T].$$

THEOREM 1. *Under assumptions* (4) *and* (5) *the process* $y(t)$ *is a limit, in the norm*

$$\|\cdot\| = (\mathrm{E}|\cdot|^2)^{1/2},$$

of processes $y_n(t)$ *which are solutions of the following system of stochastic differential equations*:

(11) $$\begin{cases} dy_1(t) = a_1 y_1(t)\,dt - a_1\,dW(t), \\ dy_n(t) = \dfrac{a_n}{b_{n-1}}\,dy_{n-1}(t) + a_n\big(y_n(t) - y_{n-1}(t)\big)dt, \quad n > 1, \end{cases}$$

where

$$W(t) = \int_{\boldsymbol{R}} \frac{e^{it\lambda} - 1}{i\lambda} \Phi(d\lambda).$$

Remark. It can be proved that $W(t)$ is a Wiener process in wide sense (see [1], XV).

Proof of Theorem 1. We will use the following relations (see Lemma 15.3 in [1]):

If $h(i\lambda)$ is spectral characteristic,

$$\int_R |h(i\lambda)|^2 d\lambda < \infty \quad \text{and} \quad y(t) = \int_R e^{it\lambda} h(i\lambda)\,\Phi(d\lambda),$$

then, with probability 1,

$$\int_0^t |y(s)|\,ds < \infty, \qquad t < \infty,$$

and

$$\int_0^t y(s)\,ds = \int_R \frac{e^{it\lambda}-1}{i\lambda} h(i\lambda)\,\Phi(d\lambda). \tag{12}$$

(a) The first equation of system (11) we may rewrite in the form:

$$y_1(t) - y_1(s) = a_1 \int_0^t y_1(u)\,du - a_1\big(W(t) - W(s)\big). \tag{i}$$

Applying (12) we obtain

$$\begin{aligned} \int_0^t y_1(u)\,du &= \int_0^t \int_R e^{iu\lambda} h_1(i\lambda)\,\Phi(d\lambda)\,du \\ &= \int_R \frac{e^{it\lambda}-1}{i\lambda} h_1(i\lambda)\,\Phi(d\lambda). \end{aligned} \tag{ii}$$

From (ii) we obtain

$$a_1 \int_R \frac{e^{it\lambda}-1}{i\lambda} h_1(i\lambda)\,\Phi(d\lambda) - a_1 \int_R \frac{e^{it\lambda}-1}{i\lambda}\,\Phi(d\lambda)$$

$$= \int_R (e^{it\lambda}-1)\frac{a_1\big(h_1(i\lambda)-1\big)}{i\lambda}\,\Phi(d\lambda) = \int_R (e^{it\lambda}-1)\,h_1(i\lambda)\,\Phi(d\lambda)$$

$$= y_1(t) - y_1(0),$$

hence $y_1(t)$ satisfies equation (i).

(b) The n-th equation of system (11) we may rewrite in the following form:

(iii) $$\frac{1}{a_n}(y_n(t)-y_n(s))-\frac{1}{b_{n-1}}(y_{n-1}(t)-y_{n-1}(s)) = \int_s^t y_n(u)-y_{n-1}(u)\,du.$$

For the sake of simplicity we put $s = 0$.

$$\frac{y_n(t)}{a_n}-\frac{y_{n-1}(t)}{b_{n-1}}-\left[\frac{y_n(0)}{a_n}-\frac{y_{n-1}(0)}{b_{n-1}}\right] = \int_{\mathbb{R}}(e^{it\lambda}-1)\left[\frac{h_n(i\lambda)}{a_n}-\frac{h_{n-1}(i\lambda)}{b_{n-1}}\right]\Phi(d\lambda)$$

$$= \int_{\mathbb{R}}(e^{it\lambda}-1)\,h_{n-1}(i\lambda)\left[\frac{1}{a_n}\,\frac{1-\dfrac{i\lambda}{b_{n-1}}}{1-\dfrac{i\lambda}{a_n}}-\frac{1}{b_{n-1}}\right]\Phi(d\lambda)$$

$$= \int_{\mathbb{R}}\frac{e^{it\lambda}-1}{i\lambda}\,h_{n-1}(i\lambda)\frac{\dfrac{i\lambda}{a_n}-\dfrac{i\lambda}{b_{n-1}}}{1-\dfrac{i\lambda}{a_n}}\,\Phi(d\lambda) = \int_0^t\int_{\mathbb{R}} e^{iu\lambda}\,h_{n-1}(i\lambda)\frac{\dfrac{i\lambda}{a_n}-\dfrac{i\lambda}{b_{n-1}}}{1-\dfrac{i\lambda}{a_n}}\,\Phi(d\lambda)\,du$$

$$= \int_0^t\int_{\mathbb{R}} e^{iu\lambda}\,[h_n(i\lambda)-h_{n-1}(i\lambda)]\,\Phi(d\lambda)\,du = \int_0^t y_n(u)-y_{n-1}(u)\,du,$$

therefore $y_n(t)$ satisfies equation (iii).

(c) Since

$$\|y_n(t)-y(t)\|^2$$
$$= \mathrm{E}\int_{\mathbb{R}} e^{it\lambda}(h_n(i\lambda)-h(i\lambda))\,\Phi(d\lambda)\,\overline{\int_{\mathbb{R}} e^{it\lambda}(h_n(i\lambda)-h(i\lambda))\,\Phi(d\lambda)}$$
$$= \int_{\mathbb{R}}|h_n(i\lambda)-h(i\lambda)|^2\,d\lambda,$$

we conclude (see Lemma) that $y_n(t)$ tends to $y(t)$ in the mean square norm.

In order to take into account applications we describe system (11) in another form. The recurrence with reference to stochastic differential of the n-th equation will be replaced by a recurrence with reference to coefficients of the n-th equation of the system.

THEOREM 2. *System* (11) *is equivalent to the system of stochastic differential equations*

(13) $$dy_n(t) = F_n(t)\,dt+\sigma_n\,dW(t), \qquad n = 1, 2, \ldots,$$

where

$$F_n(t) = \begin{cases} a_1 y_1(t) & \text{for } n = 1, \\ \sum_{i=1}^{n-1} (a_i - b_i) \prod_{k=i}^{n-1} \frac{a_{k+1}}{b_k} y_i(t) + a_n y_n(t) & \text{for } n > 1, \end{cases} \tag{14}$$

and

$$\sigma_n = \begin{cases} -a_1 & \text{for } n = 1, \\ -a_1 \prod_{k=1}^{n-1} \frac{a_{k+1}}{b_k} & \text{for } n > 1. \end{cases} \tag{15}$$

Moreover,

$$\mathrm{E}\, y_n(0)\, W(t) = \mathrm{E}\, y(0)\, W(t) = 0 \quad \text{for } n = 1, 2, \ldots$$

Proof. For $n = 1$ (11) is equivalent to (13). For $n = 2$

$$\begin{aligned} dy_2(t) &= \frac{a_2}{b_1} dy_1(t) + a_2\big(y_2(t) - y_1(t)\big) dt \\ &= \frac{a_2}{b_1}\big(a_1 y_1(t)\, dt - a_1\, dW(t)\big) + a_2\big(y_2(t) - y_1(t)\big) dt \\ &= \left[(a_1 - b_1)\frac{a_2}{b_1} y_1(t) + a_2 y_2(t)\right] dt - a_1 \frac{a_2}{b_1} dW(t) \\ &= F_2(t)\, dt + \sigma_2\, dW(t). \end{aligned}$$

Assume that for the $(n-1)$-th equation systems (11) and (13) are equivalent. Then from (11), (14) and (15) we obtain

$$\begin{aligned} dy_n(t) &= \frac{a_n}{b_{n-1}} dy_{n-1}(t) + a_n\big(y_n(t) - y_{n-1}(t)\big) dt \\ &= \frac{a_n}{b_{n-1}} \left[\sum_{i=1}^{n-2} (a_i - b_i) \prod_{k=i}^{n-2} \frac{a_{k+1}}{b_k} y_i(t) + a_{n-1} y_{n-1}(t)\right] dt - \\ &\qquad - a_1 \frac{a_n}{b_{n-1}} \prod_{k=1}^{n-2} \frac{a_{k+1}}{b_k} dW(t) + a_n\big(y_n(t) - y_{n-1}(t)\big) dt \\ &= \left[\sum_{i=1}^{n-2} (a_i - b_i) \prod_{k=i}^{n-1} \frac{a_{k+1}}{b_k} y_i(t) + a_n\left(\frac{a_{n-1}}{b_{n-1}} - 1\right) y_{n-1}(t) + a_n y_n(t)\right] dt - \\ &\qquad - a_1 \prod_{k=1}^{n-1} \frac{a_{k+1}}{b_k} dW(t) \end{aligned}$$

$$= \left[\sum_{i=1}^{n-2} (a_i - b_i) \prod_{k=i}^{n-1} \frac{a_{k+1}}{b_k} y_i(t) + a_n \left(\frac{a_{n-1}}{b_{n-1}} - 1 \right) y_{n-1}(t) + a_n y_n(t) \right] dt -$$

$$- a_1 \prod_{k=1}^{n-1} \frac{a_{k+1}}{b_k} dW(t)$$

$$= \left[\sum_{k=1}^{n-1} (a_i - b_i) \prod_{k=i}^{n-1} \frac{a_{k+1}}{b_k} y_i(t) + a_n y_n(t) \right] dt - a_1 \prod_{k=1}^{n-1} \frac{a_{k+1}}{b_k} dW(t)$$

$$= F_n(t)\, dt + \sigma_n\, dW(t).$$

Hence systems (11) and (13) are equivalent.

Now for each n we may rewrite a part of system (13) in the form

$$dY_n(t) = R_n Y_n(t)\, dt + S_n\, dW(t), \qquad n = 1, 2, \ldots,$$

where

$$Y_n(t) = \big(y_1(t), y_2(t), \ldots, y_n(t)\big)^*, \qquad S_n = (\sigma_1, \sigma_2, \ldots, \sigma_n)^*$$

and $R_n = (r_{ij})$ where $i, j = 1, 2, \ldots, n$, and

$$r_{ij} = \begin{cases} (a_j - b_j) \prod_{k=j}^{i-1} \frac{a_{k+1}}{b_k} & \text{for } j < i, \\ a_i & \text{for } j = i, \\ 0 & \text{for } j > i. \end{cases}$$

Since eigenvalues of the matrix R_n lie in the left half-plane, the proof of $\mathrm{E}\, y_n(0) W(t) = 0$ is the same as in [1].

The equation $\mathrm{E}\, y(0) W(t) = 0$ follows from inequality

$$\forall_n \; |h_n(i\lambda)| \left| \frac{e^{it\lambda} - 1}{i\lambda} \right| \leqslant t C_1 |h_1(i\lambda)|, \qquad C_1 \in \mathbf{R},$$

because

$$\mathrm{E}\, y(0) W(t) = \int_{\mathbf{R}} h(i\lambda) \frac{e^{it\lambda} - 1}{i\lambda} d\lambda = \int_{\mathbf{R}} \lim_n h_n(i\lambda) \frac{e^{it\lambda} - 1}{i\lambda} d\lambda$$

$$= \lim_n \int_{\mathbf{R}} h_n(i\lambda) \frac{e^{it\lambda} - 1}{i\lambda} d\lambda = \lim_n \mathrm{E}\, y_n(0) W(t) = 0.$$

COROLLARY. *There are the following connections between the processes $F_n(t)$ and the constants σ_n:*

$$F_n(t) = \begin{cases} a_1 y_1(t) & \textit{for } n = 1, \\ \dfrac{a_n}{b_{n-1}} F_{n-1}(t) + a_n \big(y_n(t) - y_{n-1}(t)\big) & \textit{for } n > 1, \end{cases} \tag{16}$$

$$(17)\qquad \sigma_n = \begin{cases} -a_1 & \text{for } n = 1 \\ \dfrac{a_n}{b_{n-1}}\sigma_{n-1} & \text{for } n > 1. \end{cases}$$

These formulas follow from (14), (15) and the proof of Theorem 2.

THEOREM 3. *The process $y(t)$, defined by* (10), *has the differential representation*

$$(18)\qquad dy(t) = F(t)\,dt + \sigma dW(t),$$

where

$$y(t) = \text{l.i.m}\; y_n(t), \qquad F(t) = \text{l.i.m}\; F_n(t) \qquad \text{and} \qquad \sigma = \lim \sigma_n.$$

Moreover, for $y_n(t)$, $F_n(t)$ and σ_n formulas (11), (16), (17) *respectively hold.*

Proof. From Theorem 2 we have $y(t) = \text{l.i.m}\; y_n(t)$ and from (5) we get $\sigma = \lim \sigma_n$. System (13) implies

$$F_n(t) = \int_{\mathbf{R}} e^{it\lambda}\big(i\lambda h_n(i\lambda) - \sigma_n\big)\Phi(d\lambda),$$

therefore the convergence of $F(t) = \text{l.i.m}\; F_n(t)$ is a consequence of part (b) of Lemma.

Since $i\lambda h(i\lambda) - \sigma \in L^2(\mathbf{R})$, we can apply (12). In view of

$$y(t) - y(0) - \sigma W(t) = \int_{\mathbf{R}} \frac{e^{it\lambda} - 1}{i\lambda}\big(i\lambda h(i\lambda) - \sigma\big)\Phi(d\lambda)$$

and

$$\int_{\mathbf{R}} e^{is\lambda}\big(i\lambda h(i\lambda) - \sigma\big)\Phi(d\lambda) = F(s),$$

we obtain

$$y(t) - y(0) = \int_0^t F(s)\,ds + \sigma W(t).$$

Remark 1. If we assume that there exists a constant σ such that

$$(19)\qquad i\lambda h(i\lambda) - \sigma \in L^2(\mathbf{R}),$$

then instead of assumption (5) we can assume that

$$(20)\qquad \prod_{k=1}^{\infty} \frac{a_{k+1}}{b_k}$$

is absolutely convergent.

Since

$$-a_1 \prod_{k=1}^{n-1} \frac{a_{k+1}}{b_k} = \sigma_n,$$

expression (20) is equivalent to the convergence of the sequence $\{\sigma_n\}$.

Under assumptions (4), (19) and (20) we obtain Theorems 1 and 2 and also representation (18).

Note that instead of the convergence $\operatorname{l.i.m} F_n(t) = F(t)$ we obtain in this case the convergence of the integrals

$$\operatorname*{l.i.m}_{n} \int_0^t F_n(s)\,ds = \int_0^t F(s)\,ds.$$

Indeed, since (20) is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \left|1 - \frac{a_{k+1}}{b_k}\right|,$$

inequality (6) holds. Therefore we obtain part (a) of Lemma and also Theorems 1 and 2. Under assumption (19) we can apply (12) and, in the similar way as in the proof of Theorem 3, we get

$$\int_{R} \frac{e^{it\lambda}-1}{i\lambda} \big(i\lambda h(i\lambda) - \sigma\big) \Phi(d\lambda) = \int_0^t F(s)\,ds.$$

Hence

$$dy(t) = F(t)\,dt + \sigma dW(t), \quad \text{where} \quad \operatorname{l.i.m.} \int_0^t F_n(s)\,ds = \int_0^t F(s)\,ds.$$

Remark 2. If $y(t)$ is a Gaussian process, then σ_n and σ are real positive constants (see [2]).

Remark 3. If spectral density of process $y(t)$ is a rational function

$$g(\lambda) = \left| \frac{\prod_{k=1}^{m-1} 1 - \frac{i\lambda}{b_k}}{\prod_{k=1}^{m} 1 - \frac{i\lambda}{a_k}} \right|^2, \tag{21}$$

then the problem of the differential representation for $y(t)$ is reduced to considering finite systems of stochastic differential equations. In this case from Theorem 2 we obtain a theorem analogous to Theorem 15.4 from [1]. The difference is that our stochastic differentials for $y_n(t)$ $(n = 1, 2, \ldots, m)$, the processes $F_n(t)$ and the constants σ_n satisfy some recurrent formulas.

Therefore we obtain:

THEOREM 4. *Let* $y(t)$ *and* $y_n(t)$ *be stationary processes defined by* (10). *Let* $g(\lambda)$, *defined by* (21), *be the spectral density of process* $y(t) = y_m(t)$. *Then* $y_m(t)$ *is a component of the* m-*dimensional stationary stochastic process* $Y_m(t) = (y_1(t), \ldots, y_m(t))^*$. *The processes* $y_n(t)$ *satisfy the system of stochastic differential equations*

$$dy_n(t) = F_n(t)\,dt + \sigma_n\,dW(t), \qquad n = 1, 2, \ldots, m,$$

where $F_n(t)$ *and* σ_n *are defined by* (14) *and* (15). $W(t)$ *is defined in Theorem* 1,

$$\mathrm{E}\, y_n(0)\, W(t) = 0, \qquad n = 1, 2, \ldots, m.$$

Moreover, for $y_n(t)$, $F_n(t)$ *and* σ_n *recurrent relations* (11), (16), *and* (17) *respectively hold.*

3. Absolute continuity of Gaussian stationary processes. The purpose of this section is to give a necessary and sufficient condition for absolute continuity of measures generated by Gaussian stationary processes with spectral densities of the form $f(\lambda) = |h(i\lambda)|^2$, where $h(i\lambda)$ is the spectral characteristic given by (3).

A well-known necessary and sufficient condition for absolute continuity and singularity of measures for Gaussian stationary processes with rational spectral densities is given below.

Let $g_i(\lambda)$ $(i = 1, 2)$ denote spectral densities for stochastic processes $x_i(t)$, and μ_{x_i} — measures generated by $x_i(t)$. Then

$$\mu_{x_1} \sim \mu_{x_2} \quad \text{if and only if} \quad \lim_{\lambda\to\infty} \frac{g_1(\lambda)}{g_2(\lambda)} = 1$$

for Gaussian stationary processes $x_i(t)$ with rational spectral densities $g_i(\lambda)$.

The necessity of

$$\lim_{\lambda\to\infty} \frac{g_1(\lambda)}{g_2(\lambda)} = 1 \tag{22}$$

for absolute continuity of measures μ_{x_1} and μ_{x_2} follows from Baxter's Theorem (see [6], [7]). The necessity of condition (22) can be proved for a class of spectral densities larger than the class of rational spectral densities (see [3]-[6]). However, sufficiency of (22) was proved, as far as we know, only for the class of rational spectral densities.

Now we apply the results of previous section (the differential representation for stationary process) to generalize in a simple way Feldman's Theorem (concerning sufficiency of (22)) for the class of spectral densities given by (3).

THEOREM 5. *Let $y(t)$ and $x(t)$ be Gaussian stationary processes with spectral densities $f(\lambda)$ and $g(\lambda)$, respectively, such that*

$$f(\lambda) = \left| \frac{\prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{b_k}\right)}{\prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{a_k}\right)} \right|^2, \quad g(\lambda) = \left| \frac{\prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{d_k}\right)}{\prod_{k=1}^{\infty}\left(1-\frac{i\lambda}{c_k}\right)} \right|^2.$$

Assume that for $\{a_k\}$ and $\{b_k\}$ as well as $\{c_k\}$ and $\{d_k\}$ expressions (4) *and* (5) *hold. Then*

$$\mu_y \sim \mu_x \quad \textit{if and only if} \quad \lim_{\lambda\to\infty} \frac{f(\lambda)}{g(\lambda)} = 1, \tag{23}$$

where μ_y and μ_x are measures generated by the processes $y(t)$ and $x(t)$, respectively.

Proof. From Theorem 3 we have

$$dy(t) = F(t)\,dt + \sigma_y\,dW(t),$$
$$dx(t) = G(t)\,dt + \sigma_x\,dW(t).$$

Note, if $\Phi(d\lambda)$ from (10) is Gaussian, then $W(t)$ (from Theorem 1) is a Wiener process (in usual sense).

From well-known theorems concerning the absolute continuity of stochastic processes having differential representation (see [1], VII, or [6]), we obtain

$$\mu_y \sim \mu_x \quad \text{if and only if } \sigma_y = \sigma_x. \tag{24}$$

But, on the other hand, (9) yields

$$\lim_{\lambda\to\infty} |i\lambda h_y(i\lambda)| = |\sigma_y| \quad \text{and} \quad \lim_{\lambda\to\infty} |i\lambda h_x(i\lambda)| = |\sigma_x|,$$

where $h_y(i\lambda)$ and $h_x(i\lambda)$ are spectral characteristics for $y(t)$ and $x(t)$, respectively, so

$$\lim_{\lambda\to\infty} \lambda^2 f(\lambda) = \sigma_y^2 \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda^2 g(\lambda) = \sigma_x^2.$$

Since σ_y and σ_x are positive constants (see [2]),

$$\sigma_y = \sigma_x \quad \text{if and only if} \quad \lim_{\lambda\to\infty} \frac{\lambda^2 f(\lambda)}{\lambda^2 g(\lambda)} = 1.$$

This fact and (24) imply (23).

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