REMARKS ON BANACH SPACES OF STABLE TYPE

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Abstract. In this note we give a new characterization of Banach spaces of stable type.

1. Introduction. Throughout this paper, \( E \) stands for a separable real Banach space. A Banach space \( E \) is said to be of Rademacher type \( p \) (R-type \( p \), for short) if for every sequence \( (x_n) \subset E \) the convergence of \( \sum \|x_n\|^p \) implies the a.e. convergence of \( \sum r_n x_n \), where \( (r_n) \) is the Rademacher sequence. If \( E \) is of R-type \( p \), then there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left\| \sum_{i=1}^n X_i \right\|^p \leq C \sum_{i=1}^n \mathbb{E} \|X_i\|^p
\]

for all \( E \)-valued independent random vectors \( X_1, \ldots, X_n \) satisfying conditions \( \mathbb{E} \|X_i\|^p < \infty \) and \( \mathbb{E} X_i = 0 \) for \( i = 1, \ldots, n, n \geq 1 \) (see [5]). A Banach space \( E \) is said to be of stable type \( p \) if for every sequence \( (x_n) \subset E \) the convergence of \( \sum \|x_n\|^p \) implies the a.e. convergence of \( \sum g_n x_n \), where the \( g_n \)'s are independent stable random variables with characteristic functions \( \mathbb{E} \exp (itg_n) = \exp (-|t|^p) \). It is known (see [4], [8] and [10]) that every Banach space is of stable type \( p \) for \( p < 1 \). Moreover, \( E \) is of stable type \( p \) for \( p < 2 \) if and only if there exists a number \( p' > p \) such that \( E \) is of R-type \( p' \). A Banach space is of stable type 2 if and only if it is of R-type 2. A space \( L^q(S, \Sigma, m) \), where \( m \) is \( \sigma \)-finite, is of stable type \( p \) for \( p < q \). Finite-dimensional normed spaces and Hilbert spaces are of stable type \( p \) for every \( 0 < p \leq 2 \).

2. A characterization of Banach spaces of stable type. Let \( (\Omega, \mathcal{F}, P) \) be a probability space. By \( L^p(E) = L^p(\Omega, \mathcal{F}, P; E) \), \( 0 \leq p \leq \infty \), we denote a standard Fréchet space of random vectors. For each \( 0 < p < \infty \) let \( \Lambda_p \)
be a function defined on \( L^0(E) \) by

\[
A_p(X) = \sup_{t \geq 0} t^p P\{ \| X \| > t \}.
\]

It is easy to note that \( A_p \) is a \( p \)-homogeneous metrizable modular and, consequently,

\[
A_p(E) = A_p(\Omega, \mathcal{F}, P; E) = \{ X \in L^0(E) : A_p(X) < \infty \}
\]

forms a Fréchet space with the topology of convergence in \( A_p \) (for details see [11], p. 17). Moreover, for every \( 0 < q < p \),

\[
L^p(E) \subset A_p(E) \subset L^q(E),
\]

and the natural imbeddings are continuous.

A symmetric random vector \( X \) (or probability measure \( \mathcal{L}(X) \)) is said to be stable of order \( p \) if \( \mathcal{L}(aX_1 + bX_2) = \mathcal{L}\left((a^p + b^p)^{1/p} X\right) \) for all \( a, b > 0 \), where \( X_1, X_2 \) are independent copies of \( X \), and \( \mathcal{L}(X) \) denotes the distribution of \( X \). It is well known that if \( X \) is a non-degenerate stable random vector of order \( p \) for \( 0 < p < 2 \), then \( E \| X \|^p = \infty \). However, in this case, \( A_p(X) < \infty \) as shown in [1].

The following theorem was inspired by the weak law of large numbers in the spaces of stable type \( p \) for \( 0 < p < 2 \) established by Marcus and Woyczyński in [7]:

**Theorem 1.** A Banach space \( E \) is of stable type \( p \) for \( 0 < p < 2 \) if and only if there exists a constant \( C > 0 \) such that

\[
A_p\left( \sum_{i=1}^{n} X_i \right) \leq C \sum_{i=1}^{n} A_p(X_i)
\]

for all symmetric independent \( E \)-valued random vectors \( X_1, \ldots, X_n \) such that \( A_p(X_i) < \infty \), \( i = 1, \ldots, n \), \( n \geq 1 \).

**Proof.** As in the Introduction, let \( g_j \) \((j = 1, \ldots, n)\) denote independent random variables with characteristic functions \( E \exp(itg_j) = \exp(-|t|^p) \). Let \( X_j = g_jx_j \), where \( x_j \in E \), \( j = 1, \ldots, n \). If (2) holds, then

\[
A_p\left( \sum_{j=1}^{n} g_jx_j \right) \leq C \sum_{j=1}^{n} A_p(g_jx_j) = C A_p(g_1) \sum_{j=1}^{n} \| x_j \|^p.
\]

Since \( A_p(g_1) < \infty \), the convergence of \( \sum \| x_j \|^p \) implies the a.e. convergence of \( \sum g_jx_j \) for every sequence \( (x_j) \subset E \).

Now, let \( E \) be of stable type \( p \) for \( 0 < p < 2 \). Then there exists a number \( p' > p \) such that \( E \) is of \( R \)-type \( p' \). Let \( C' \) be the constant appearing in (1) for \( p = p' \). Now let \( X_1, \ldots, X_n \) be independent symmetric random vectors...
such that $\Lambda_p(X_i) < \infty$ for $i = 1, \ldots, n$. Let $Y_i = X_i I_{\|X_i\| \leq 1}$. We have

$$P \left\{ \sum_{i=1}^{n} X_i \neq 1 \right\} \leq P \left\{ \sum_{i=1}^{n} X_i \neq 1, \max_{1 \leq i \leq n} \|X_i\| \leq 1 \right\} + P \left\{ \max_{1 \leq i \leq n} \|X_i\| > 1 \right\}$$

$$\leq P \left\{ \sum_{i=1}^{n} Y_i \neq 1 \right\} + \sum_{i=1}^{n} \Lambda_p(X_i)$$

and

$$E \left\{ \sum_{i=1}^{n} Y_i \right\}^{p'} + \sum_{i=1}^{n} \Lambda_p(X_i) \leq C \sum_{i=1}^{n} E \|Y_i\|^{p'} + \sum_{i=1}^{n} \Lambda_p(X_i)$$

Finally, replacing $X_i$ by $t^{-1} X_i$, $t > 0$, we get

$$P \left\{ \sum_{i=1}^{n} X_i \neq 1 \right\} \leq C \sum_{i=1}^{n} \Lambda_p(t^{-1} X_i) = t^{-p} C \sum_{i=1}^{n} \Lambda_p(X_i).$$

Thus (2) is proved.

**Proposition 1.** Let $E$ be a Banach space of stable type $p$ for $0 < p < 2$ and let $C$ be the corresponding constant in (2). If $F$ is a closed subspace of $E$, then inequality (2) holds with the same constant $C$ for independent and symmetric random vectors taking values in the quotient space $E/F$.

**Proof.** It is enough to observe that if $E$ is of $R$-type $p'$, then $E/F$ is of $R$-type $p'$ with the same constant $C$.

**3. Normal domains of attractions of stable measures.** A symmetric random vector $X$ is said to belong to the normal domain of attraction of a stable measure $\mu$ of order $p$ if

$$\mathcal{L} \left( n^{-1/p} \sum_{i=1}^{n} X_i \right) \Rightarrow \mu \quad \text{as } n \to \infty$$

for any sequence $(X_n)$ of independent copies of $X$.

For a symmetric random vector $X$ the following theorem may be easily deduced from Theorem 3.1 established by Araujo and Giné in [2]:

**Theorem 2.** Let $E$ be a Banach space of stable type $p$ for $0 < p < 2$. 


If $X$ is a symmetric $E$-valued random vector such that

\[ \lim_{t \to \infty} t^p P \{ |x^* X| > t \} \text{ exists for every } x^* \in E^* \]

and if

\[ \text{for every } \varepsilon > 0 \text{ there exists a finite-dimensional subspace } F \text{ of } E \text{ such that} \]

\[ \sup_{t > 0} t^p P \{ \text{dist} (X, F) > t \} \leq \varepsilon, \]

then $X$ belongs to the normal domain of attraction of a stable measure of order $p$.

Theorem 2 has been proved independently by Marcus and Woyczyński in [7], but their conditions differ from (3) and (4). Our proof of Theorem 2, by using Theorem 1, is simpler than that given in [2].

Proof. First we notice that condition (3) is equivalent to the following (see Theorem 5, VII, 35 in [3]):

The weak limit

\[ \mathcal{L} \left( n^{-1/p} \sum_{i=1}^{n} x^* X_i \right) \]

exists for every $x^* \in E^*$.

Thus it is sufficient to show that for every $\delta > 0$ there exists a finite-dimensional subspace $F$ of $E$ such that

\[ \sup_n P \{ \text{dist} \left( n^{-1/p} \sum_{i=1}^{n} X_i, F \right) > \delta \} \leq \delta. \]

Let $\delta > 0$ be fixed and let $C$ be the constant appearing in (2). It follows from (4) that for $\varepsilon = \delta^{1+p} C^{-1}$ there exists a finite-dimensional subspace $F$ of $E$ such that

\[ \sup_{t > 0} t^p P \{ \text{dist} (X, F) > t \} \leq \delta^{1+p} C^{-1}. \]

Let $\pi_f: E \to E/F$ denote the canonical surjection and $\| \cdot \|_F$ the standard norm in $E/F$. Using Proposition 1 we obtain

\[ P \{ \text{dist} \left( n^{-1/p} \sum_{i=1}^{n} X_i, F \right) > \delta \} = P \{ \| n^{-1/p} \sum_{i=1}^{n} \pi_f(X_i) \|_F > \delta \} \]

\[ \leq \delta^{-p} A_p \left( n^{-1/p} \sum_{i=1}^{n} \pi_f(X_i) \right) \]

\[ \leq \delta^{-p} CA_p(\pi_f(X)) \leq \delta. \]

This completes the proof.
Finally, we note that if $X, X_1, X_2, \ldots$ are symmetric independent and identically distributed real random variables, then the stochastical boundedness of $\{n^{-1/2} S_n\}$, where $S_n = \sum_{i=1}^{n} X_i$, implies the weak convergence in law. However, for $0 < p < 2$ we may construct a symmetric real random variable $X$ such that $\{n^{-1/p} S_n\}$ is stochastically bounded and that $\mathcal{L}(n^{-1/p} S_n)$ diverges at the same time. Indeed, by virtue of Theorem 1 the sequence $\{n^{-1/p} S_n\}$ is stochastically bounded if and only if $\Lambda_p(X) < \infty$. Therefore, it suffices to take a symmetric random variable $X$ such that $\Lambda_p(X) < \infty$ and $\lim_{t \to \infty} t^p P\{|X| > t\}$ does not exist. Such a random variable may easily be constructed.

REFERENCES


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