ON THE CONSTRUCTION OF THE WOLD DECOMPOSITION
FOR NON-STATIONARY STOCHASTIC PROCESSES

BY

HANNU NIEMI (HELSINKI)

Abstract. It is presented a relation between the Wold decomposition for a second order stochastic process $x(t)$, $t \in \mathbb{R}$, having a spectral representation and the Lebesgue decomposition, with respect to the Lebesgue measure, for the spectral measure of $x(t)$, $t \in \mathbb{R}$.

Introduction. We are concerned with the construction of the Wold decomposition for non-stationary quadratic mean (q.m.) continuous stochastic processes $x: \mathbb{R} \to L^2(\Omega, A, P)$ having a spectral representation in the form

$$(*) \quad x(t) = \int e^{it\lambda} d\mu(\lambda), \quad t \in \mathbb{R},$$

where $\mu$ is a bounded vector measure on $\mathbb{R}$ with values in $L^2(\Omega, A, P)$.

It is well known that the Wold decomposition for a q.m. continuous stationary stochastic process $x: \mathbb{R} \to L^2(\Omega, A, P)$ can be stated in terms of the Lebesgue decomposition for its spectral measure $\mu$ with respect to the Lebesgue measure $m$ on $\mathbb{R}$ (cf. [13], p. 115 and 116).

In this paper we present a relation between the Wold decomposition for a stochastic process $x: \mathbb{R} \to L^2(\Omega, A, P)$ of the form $(*)$ and the Lebesgue decomposition for its spectral measure $\mu$ with respect to $m$ (cf. Theorem 3). It appears that, in general, the $m$-singular and $m$-continuous parts of $\mu$ do not fully characterize the deterministic and purely nondeterministic parts of $x: \mathbb{R} \to L^2(\Omega, A, P)$, respectively (cf. Example 2). However, it is shown that the Wold decomposition for a stochastic process $x: \mathbb{R} \to L^2(\Omega, A, P)$ of the form $(*)$ can be explicitly stated in terms of the Lebesgue decomposition for its spectral measure $\mu$ with respect to $m$ provided that $x: \mathbb{R} \to L^2(\Omega, A, P)$ is in addition in the class of uniformly bounded linearly stationary stochastic processes introduced by Tjøstheim and Thomas [16] (cf. Theorem 5).
1. A Lebesgue decomposition theorem for bounded vector measures and their orthogonally scattered dilations. In this section we present a relation between the Lebesgue decomposition for a bounded vector measure \( \mu \) with values in a Hilbert space and the Lebesgue decompositions for the so-called orthogonally scattered dilations of \( \mu \). The result is merely a reformulation of results obtained in [10], Theorems 7-9.

Let \( T \) be a locally compact Hausdorff space. By \( C_0(T) \) we denote the linear space of all continuous functions \( f: T \to C \) vanishing at infinity, carrying the supremum norm topology.

Let \( \mu: C_0(T) \to B \) be a bounded vector measure with values in a (complex) Banach space \( B \), i.e., \( \mu \) is a bounded linear mapping. By \( \mathcal{L}^p(\mu) \) we denote the linear space of all the functions \( u: T \to C \) for which \( |u|^p \) is \( \mu \)-integrable, \( p = 1, 2 \). By \( \overline{\mathcal{S}}p \{ \mu \} \) we denote the closure of \( \mu(C_0(T)) \) in \( B \). Recall that

\[
\int ud\mu \in \overline{\mathcal{S}}p \{ \mu \} \quad \text{for all } u \in \mathcal{L}^1(\mu).
\]

(In this paper we apply the integration technique of vector measures introduced by Thomas [15].)

Let \( \mu: C_0(T) \to B \) be a bounded weakly compact vector measure with values in a (complex) Banach space \( B \) and let \( \beta \) be a positive Radon measure on \( T \). Recall that there exist bounded weakly compact vector measures \( \mu_\beta: C_0(T) \to \overline{\mathcal{S}}p \{ \mu \} \), \( \mu_\mu: C_0(T) \to \overline{\mathcal{S}}p \{ \mu \} \) and a Borel set \( E^* \subset T \) such that \( \beta(E^*) = 0 \), \( \mu = \mu_\beta + \mu_\mu \), \( \mu_\beta \) is \( \beta \)-singular, \( \mu_\mu \) is \( \beta \)-continuous, \( \mathcal{L}^1(\mu) = \mathcal{L}^1(\mu_\beta) \), \( \mathcal{L}^1(\mu) = \mathcal{L}^1(\mu_\mu) \) and

\[
\int ud\mu_\beta = \int u\chi_{E^*} d\mu, \quad \int ud\mu_\mu = \int u\chi_{T\setminus E^*} d\mu, \quad u \in \mathcal{L}^1(\mu),
\]

where \( \chi_E \) stands for the characteristic function of a Borel set \( E \subset T \) (cf. [12], Theorem 4.5, [3], [10], Theorem 3, and the references given therein).

Example (Pop-Stojanovic [11]). Let \( \mu: C_0(T) \to H \) be a bounded vector measure with values in a Hilbert space \( H \). Suppose, in addition, that \( \mu \) is orthogonally scattered, i.e., there exists a (uniquely determined) bounded positive Radon measure \( v: C_0(T) \to C \) such that \( \mathcal{L}^1(\mu) = \mathcal{L}^2(v) \) and

\[
(\int ud\mu)(\int vd\mu) = \int u\bar{v} d\nu \quad \text{for all } u, v \in \mathcal{L}^1(\mu)
\]

(cf. [5], Theorem 5.9). Let \( \beta \) be a positive Radon measure on \( T \) and let \( \mu = \mu_\beta + \mu_\mu \) and \( v = v_\beta + v_\mu \) be the Lebesgue decompositions with respect to \( \beta \) for \( \mu \) and \( v \), respectively. Then:

(i) \( \mu_\beta \) and \( \mu_\mu \) are orthogonally scattered;

(ii) furthermore

\[
(\int ud\mu_\beta)(\int vd\mu_\beta) = \int u\bar{v} d\nu_\beta, \quad u, v \in \mathcal{L}^1(\mu_\beta),
\]

\[
(\int ud\mu_\mu)(\int vd\mu_\mu) = \int u\bar{v} d\nu_\mu, \quad u, v \in \mathcal{L}^1(\mu_\mu);
\]

(iii) \( \overline{\mathcal{S}}p \{ \mu \} = \overline{\mathcal{S}}p \{ \mu_\beta \} \oplus \overline{\mathcal{S}}p \{ \mu_\mu \}. \)
The following lemma is obvious:

**Lemma 1.** Let $B$ and $B'$ be two (complex) Banach spaces. Suppose that $\mu: C_0(T) \rightarrow B$ is a bounded weakly compact vector measure, $\beta$ is a positive Radon measure on $T$ and $\mu = \mu_\beta + \mu_c$ is the Lebesgue decomposition for $\mu$ with respect to $\beta$. If $A: \overline{sp} \{\mu\} \rightarrow B'$ is a bounded linear operator, then

(i) $\mu' = A \circ \mu$ is a bounded weakly compact vector measure;

(ii) $\mu'_\beta = A \circ \mu_\beta$ and $\mu'_c = A \circ \mu_c$ are the $\beta$-singular and $\beta$-continuous parts of $\mu'$, respectively;

(iii) if, in addition, $A: \overline{sp} \{\mu\} \rightarrow \overline{sp} \{\mu'\}$ has a bounded inverse

$$A^{-1}: \overline{sp} \{\mu'\} \rightarrow \overline{sp} \{\mu\},$$

then

$$A(\overline{sp} \{\mu_\beta\}) = \overline{sp} \{\mu'_\beta\}, \quad A^{-1}(\overline{sp} \{\mu'_\beta\}) = \overline{sp} \{\mu_\beta\},$$

$$A(\overline{sp} \{\mu_c\}) = \overline{sp} \{\mu'_c\}, \quad A^{-1}(\overline{sp} \{\mu'_c\}) = \overline{sp} \{\mu_c\}.$$

In what follows, by $P_K$ we denote the orthogonal projection of a Hilbert space $H$ onto its given closed linear subspace $K$.

The following theorem can now be proved as Theorem 13 in [7] by applying Theorems 7-9 in [10] (cf. [1], [2], Theorem 3.1, and [6], Corollary 6).

**Theorem 1.** Let $\mu: C_0(T) \rightarrow H$ be a bounded vector measure with values in a (complex) Hilbert space $H$, let $\beta$ be a positive Radon measure on $T$ and let $E^* \subset T$ be a Borel set in $T$ such that $\beta(E^*) = 0$ and

$$\mu_\beta(f) = \int f \chi_{E^*} \, d\mu, \quad \mu_c(f) = \int f \chi_{T \setminus E^*} \, d\mu, \quad f \in C_0(T),$$

are the $\beta$-singular and $\beta$-continuous parts of $\mu$, respectively. Then there exist a (complex) Hilbert space $H'$ and a bounded orthogonally scattered vector measure $\mu': C_0(T) \rightarrow H'$ satisfying the following conditions:

(i) $\mathcal{L}^1(\mu') \subset \mathcal{L}^1(\mu)$.

(ii) The $\beta$-singular and $\beta$-continuous parts of $\mu$ are

$$\mu_\beta'(f) = \int f \chi_{E^*} \, d\mu', \quad \mu_c'(f) = \int f \chi_{T \setminus E^*} \, d\mu', \quad f \in C_0(T),$$

respectively.

(iii) There exists a linear mapping $j: \overline{sp} \{\mu\} \rightarrow H'$ such that $j: \overline{sp} \{\mu\} \rightarrow j(\overline{sp} \{\mu\})$ is an inner product preserving isomorphism and, for all $u \in \mathcal{L}^1(\mu')$,

(1a) \quad $j(\int u \, d\mu) = P_{j(\overline{sp} \{\mu\})}(\int u \, d\mu')$,

(1b) \quad $j(\int u \, d\mu_\beta) = P_{j(\overline{sp} \{\mu_\beta\})}(\int u \, d\mu'_\beta)$,

(1c) \quad $j(\int u \, d\mu_c) = P_{j(\overline{sp} \{\mu_c\})}(\int u \, d\mu'_c)$.
(iv) The bounded vector measure $\mu: C_0(T) \to H$ is $\beta$-singular (respectively, $\beta$-continuous) if and only if there exists a $\beta$-singular (respectively, $\beta$-continuous) bounded orthogonally scattered vector measure $\mu': C_0(T) \to H'$ satisfying (1a).

Remark. Statement (iii) in Theorem 1 can also be formulated as follows: The diagram

$$
\begin{array}{c}
C_0(T) \xleftarrow{\mu} \xrightarrow{\tilde{\mu}} H' \\
\xrightarrow{s_p \{ \tilde{\mu} \}} \xrightarrow{j(s_p \{ \tilde{\mu} \})} \end{array}
$$

is commuting for all pairs:
(a) $\tilde{\mu} = \mu$, $\tilde{\mu}' = \mu'$;
(b) $\tilde{\mu} = \mu_\sigma$, $\tilde{\mu}' = \mu'_\sigma$;
(c) $\tilde{\mu} = \mu_\epsilon$, $\tilde{\mu}' = \mu'_\epsilon$.

2. Wold decomposition for q.m. continuous $V$-bounded stochastic processes. Let $H$ be a (fixed) complex Hilbert space; one may choose, e.g., $H = L^2(\Omega, A, P)$, where $(\Omega, A, P)$ is a probability space. In this paper a stochastic process is always a mapping $x: R \to H$.

Let $x(t), t \in R$, be a stochastic process. For $t \in R$, by $\overline{s_p} \{ x; t \}$ we denote the closed linear subspace in $H$ spanned by the set $\{ x(s) | s \leq t \}$, and by $s_p \{ x \}$ we denote the closed linear subspace in $H$ spanned by the set $\{ x(s) | s \in R \}$. Furthermore we put

$$
\overline{s_p} \{ x; -\infty \} = \bigcap_{t \in R} \overline{s_p} \{ x; t \}.
$$

The stochastic process $x(t), t \in R$, is called purely non-deterministic if $\overline{s_p} \{ x; -\infty \} = \{ 0 \}$; it is called deterministic if $\overline{s_p} \{ x; -\infty \} = s_p \{ x \}$.

Let $x(t), t \in R$, be a stochastic process. The decomposition

$$
v_x(t) = P_{\overline{s_p} \{ x; -\infty \}} (x(t)), \quad u_x(t) = x(t) - v_x(t), \quad t \in R,
$$

is called the Wold decomposition for $x(t), t \in R$; it is the only decomposition for $x(t), t \in R$, in the form $x(t) = v_x(t) + u_x(t), t \in R$, with the properties (cf. [4], Theorem 1):

(W1) $v_x(t), t \in R$, is deterministic; $u_x(t), t \in R$, is purely non-deterministic;
(W2) $\overline{s_p} \{ v_x \} \perp \overline{s_p} \{ u_x \}$;
(W3) $\overline{s_p} \{ v_x; t \} \subset \overline{s_p} \{ x; t \}$, $\overline{s_p} \{ u_x; t \} \subset \overline{s_p} \{ x; t \}$ for all $t \in R$.

Recall that a stochastic process $x(t), t \in R$, is q.m. continuous if the mapping $x: R \to H$ is continuous; and it is in addition $V$-bounded if there exists a uniquely determined bounded vector measure $\mu: C_0(R) \to H$, the spectral measure of $x(t), t \in R$, such that

$$
x(t) = \int e^{it\lambda} d\mu(\lambda), \quad t \in R
$$

(cf. [6], [8]-[10] and the references given there).
If \( x(t), t \in \mathbb{R} \), is a q.m. continuous \( V \)-bounded stochastic process and if \( \mu \) is its spectral measure, then \( \overline{\mathcal{P}} \{ x \} = \overline{\mathcal{P}} \{ \mu \} \).

In this paper we are concerned with the construction of the Wold decomposition for a given q.m. continuous \( V \)-bounded stochastic process \( x(t), t \in \mathbb{R} \), in terms of the Lebesgue decomposition for its spectral measure \( \mu \) with respect to the Lebesgue measure \( m \) on \( \mathbb{R} \).

**Example 1.** Let \( x(t), t \in \mathbb{R} \), be a stationary stochastic process, i.e., \( (x(s)|x(t)) \) depends only on \( s-t, s, t \in \mathbb{R} \). If \( x(t), t \in \mathbb{R} \), is in addition q.m. continuous, then it is even \( V \)-bounded and its spectral measure \( \mu \) is orthogonally scattered. Put

\[
(2) \quad x_s(t) = \int e^{it\lambda} d\mu_s(\lambda), \quad x_c(t) = \int e^{it\lambda} d\mu_c(\lambda), \quad t \in \mathbb{R},
\]

where \( \mu = \mu_s + \mu_c \) is the Lebesgue decomposition for \( \mu \) with respect to \( m \).

Then:

(i) \( x_s(t), t \in \mathbb{R} \), is deterministic and \( x_c(t), t \in \mathbb{R} \), is either deterministic or purely non-deterministic;

(ii) \( x(t), t \in \mathbb{R} \), is deterministic if and only if \( x_c(t), t \in \mathbb{R} \), is deterministic;

(iii) if \( x_c(t), t \in \mathbb{R} \), is purely non-deterministic, then

\[
\overline{\mathcal{P}} \{ x_s; -\infty \} = \overline{\mathcal{P}} \{ v_x \} = \overline{\mathcal{P}} \{ \mu_s \}, \quad \overline{\mathcal{P}} \{ u_x \} = \overline{\mathcal{P}} \{ \mu_c \}
\]

(cf. [13], p. 115 and 116).

The following theorem can be proved as Theorem 11 in [8], by applying Theorem 1 (cf. [1] and [6], Theorem 5).

**Theorem 2.** Let \( x: \mathbb{R} \to H \) be a q.m. continuous \( V \)-bounded stochastic process and let \( \mu: C_0(\mathbb{R}) \to \overline{\mathcal{P}} \{ x \} \) be its spectral measure. Then there exist a Hilbert space \( H' \) and a q.m. continuous stationary stochastic process \( x': \mathbb{R} \to H' \) such that \( \mu, H' \) and the spectral measure \( \mu': C_0(\mathbb{R}) \to \overline{\mathcal{P}} \{ x' \} \) of \( x'(t), t \in \mathbb{R} \), satisfy the conditions stated in Theorem 1, and by applying the notation introduced in (2) and Theorem 1:

\[
\begin{align*}
    j(x(t)) &= P_{J\{\mathcal{P}(\lambda)\}}(x'(t)), \quad t \in \mathbb{R}, \\
    j(x_s(t)) &= P_{J\{\mathcal{P}(\lambda)\}}(x'_s(t)), \quad t \in \mathbb{R}, \\
    j(x_c(t)) &= P_{J\{\mathcal{P}(\lambda)\}}(x'_c(t)), \quad t \in \mathbb{R}.
\end{align*}
\]

The following lemma is due to Abreu [1].

**Lemma 2.** Let \( x: \mathbb{R} \to H \) be a stochastic process. Suppose there exist a stochastic process \( x': \mathbb{R} \to H' \) and a bounded linear mapping \( A: \overline{\mathcal{P}} \{ x' \} \to \overline{\mathcal{P}} \{ x \} \) such that \( x(t) = A(x'(t)), t \in \mathbb{R} \). Then:

(i) \( A(\overline{\mathcal{P}} \{ x'; -\infty \}) \subseteq \overline{\mathcal{P}} \{ x; -\infty \} \);

(ii) if \( x'(t), t \in \mathbb{R} \), is deterministic, then \( x(t), t \in \mathbb{R} \), is deterministic.

Remark. The inclusion relation stated in Lemma 2 (i) may be strict.
LEMMA 3. Let \( x(t), t \in R, \) be a stochastic process and let \( M \subset \text{sp} \{x; -\infty\} \) be a closed linear subspace in \( \text{sp} \{x\}. \) Put
\[
y(t) = P_M(x(t)), \quad z(t) = x(t) - y(t), \quad t \in R.
\]
Then
(i) \( y(t), t \in R, \) is deterministic;
(ii) \( \text{sp} \{y; t\} \subset \text{sp} \{x; t\}, \quad \text{sp} \{z; t\} \subset \text{sp} \{x; t\}, \quad t \in R; \quad \text{sp} \{y\} \subset \text{sp} \{x\}, \)
\( \text{sp} \{z\} \subset \text{sp} \{x\}; \quad \text{sp} \{y; -\infty\} \subset \text{sp} \{x; -\infty\}, \quad \text{sp} \{z; -\infty\} \subset \text{sp} \{x; -\infty\}; \)
(iii) \( \text{sp} \{y\} \perp \text{sp} \{z\}; \)
(iv) \( \text{sp} \{x; -\infty\} = M \oplus \text{sp} \{z; -\infty\}; \)
(v) \( v_x(t) = y(t) + v_x(t), \ u_x(t) = u_x(t), \ t \in R. \)

Proof. Since \( A4 \subset \text{sp} \{x; -\infty\}, \) we have
\[
x(t) = P_M(x(t)), \quad t \in R.
\]
Thus, it follows from Lemma 2 that \( y(t), t \in R, \) is deterministic, proving (i).

Assertions (ii) and (iii) follow immediately from the definitions of \( y(t), t \in R, \) and \( z(t), t \in R. \)

In order to prove (iv), we first note that the inclusion \( M \oplus \text{sp} \{z; -\infty\} \subset \text{sp} \{x; -\infty\} \) is clear. On the other hand, suppose \( w \in \text{sp} \{x; -\infty\}. \) Put \( w = w_1 + w_2, \) where \( w_1 \in M \) and \( w_2 \in \text{sp} \{x; -\infty\}, w_2 \perp M. \) In order to show that \( w_2 \in \text{sp} \{z; -\infty\}, \) note that for any \( \varepsilon > 0 \) and for any \( w' \in \text{sp} \{x; t\}, t \in R, \) of the form
\[
w' = \sum_{k=1}^{n} a_k x(t_k), \quad a_k \in C, \ t_k \leq t, \ k = 1, \ldots, n,
\]
satisfying \( \|w - w'\| < \varepsilon, \) we have
\[
\| (I - P_M)(w - w') \| < \varepsilon
\]
and
\[
(I - P_M)(w - w') = w_2 - \sum_{k=1}^{n} a_k z(t_k).
\]
Thus, the fact that \( w \in \text{sp} \{x; -\infty\} \) implies \( w_2 \in \text{sp} \{z; -\infty\}, \) proving (iv).

Finally, assertion (v) follows immediately from (iv).

The lemma is proved.

The forthcoming theorem follows now from Theorem 2, Example 1, Lemmas 2 and 3. (Statement (ii) in Theorem 3 was already presented in [10], Theorem 3.)

THEOREM 3. Let \( x(t), t \in R, \) be a q.m. continuous \( V \)-bounded stochastic process and let \( \mu = \mu_++\mu_- \) be the Lebesgue decomposition with respect to \( m \) for its spectral measure \( \mu: C_0(R) \rightarrow \text{sp} \{x\}. \) Then, by applying the notation introduced in (2):
(i) \( \text{sp} \{\mu_+\} \subset \text{sp} \{x; -\infty\}. \)
(ii) If \( \mu_c = 0 \), then \( x(t) \), \( t \in \mathbb{R} \), is deterministic.

(iii) Put

\[
w(t) = x_c(t) - P_{\text{sp}(\mu_c)}(x_c(t)), \quad t \in \mathbb{R}.
\]

Then

\[
v_x(t) = x_s(t) + P_{\text{sp}(\mu_c)}(x_c(t)) + v_w(t), \quad u_x(t) = u_w(t), \quad t \in \mathbb{R}.
\]

(iv) \( u_x(t) \) and \( v_x(t) \) for \( t \in \mathbb{R} \) are q.m. continuous \( V \)-bounded stochastic processes with spectral measures

\[
\mu_v = \mu_s + P_{\text{sp}(\mu_c)} \circ \mu_c + P_{\text{sp}(\mathbb{R}^\infty)} \circ (\mu_c - P_{\text{sp}(\mu_c)} \circ \mu_c)
\]

and

\[
\mu_u = (I - P_{\text{sp}(\mathbb{R}^\infty)}) \circ (\mu_c - P_{\text{sp}(\mu_c)} \circ \mu_c),
\]

respectively.

Remark. (i) The stochastic process \( w(t) \), \( t \in \mathbb{R} \), defined in Theorem 3, is a q.m. continuous \( V \)-bounded stochastic process with an \( m \)-continuous spectral measure.

(ii) If a q.m. continuous stationary stochastic process has an \( m \)-continuous spectral measure, then it is either deterministic or purely non-deterministic. The following example shows that, in general, this statement is not valid for q.m. continuous \( V \)-bounded stochastic processes.

Example 2. For convenience we consider here only the discrete time case. The example can be transformed, in a straightforward way, into the continuous time case by applying a suitable smoothing function.

Suppose \( e_k \in H, \|e_k\| = 1, k = 1, 2, \) and \( e_1 \perp e_2 \). Define \( x(k), k \in \mathbb{Z} \), by

\[
x(0) = e_1, \quad x(k) = 0 \text{ for } k > 0, \quad x(k) = k^{-1}e_2 \text{ for } k < 0.
\]

Then \( x(k), k \in \mathbb{Z} \), is a \( V \)-bounded sequence with an \( m \)-continuous spectral measure (cf. [14], p. 183 and 184). Furthermore,

\[
v_x(k) = k^{-1}e_2 \text{ for } k < 0, \quad v_x(k) = 0 \text{ for } k \geq 0,
\]

\[
u_x(0) = e_1, \quad u_x(k) = 0 \text{ for } k \neq 0,
\]

i.e., \( v_x \neq 0 \) and \( u_x \neq 0 \) even if the spectral measure of \( x(k), k \in \mathbb{Z} \), is \( m \)-continuous.

The next theorem follows immediately from Theorem 12 in [10] and from Theorem 3. It can be considered as a vector-valued version of the well-known result by F. and M. Riesz concerning the \( m \)-continuity of scalar-valued bounded measures with Fourier-Stieltjes transforms vanishing on a half-line.

**Theorem 4.** Let \( x(t), t \in \mathbb{R} \), be a purely non-deterministic q.m. continuous \( V \)-bounded stochastic process. Then:

(i) the spectral measure \( \mu \) of \( x(t), t \in \mathbb{R} \), is \( m \)-continuous;
(ii) if there exists a Borel set $E \subset R$ such that $m(E) > 0$ and $\mu(E') = 0$ for all Borel sets $E' \subset E$, then $\mu = 0$ and, a fortiori, $x(t) = 0$, $t \in R$.

We close this paper by considering a special case where the results stated in Theorem 3 can be improved.

Recall that a stochastic process $x(t)$, $t \in R$, is uniformly bounded linearly stationary (UBLS) if one of the following three equivalent conditions holds (cf. [16]):

(i) There exists a constant $M \geq 0$ such that

$$\left\| \sum_{j=1}^{n} a_j x(t_j + s) \right\| \leq M \left\| \sum_{j=1}^{n} a_j x(t_j) \right\|$$

for all $a_j \in C$, $s$, $t_j \in R$, $j = 1, \ldots, n$, $n \in N$.

(ii) There exists a uniquely determined group of operators $T_r^s$: $\mathfrak{sp} \{x\} \rightarrow \mathfrak{sp} \{x\}$, the shift operator group of $x(t), t \in R$, such that

$$T_r^s(x(t)) = x(t+s), \quad \left\| T_r^s \right\| \leq M \quad \text{for all } s, t \in R.$$

(iii) $x(t), t \in R$, has a stationary similarity $(y, B)$, i.e., there exist a stationary stochastic process $y(t)$, $t \in R$, and a bounded linear operator $B: \mathfrak{sp} \{y\} \rightarrow \mathfrak{sp} \{x\}$ with a bounded inverse $B^{-1}: \mathfrak{sp} \{x\} \rightarrow \mathfrak{sp} \{y\}$ such that

$$x(t) = B(y(t)), \quad t \in R.$$  

Remark. Since any UBLS stochastic process has a stationary similarity, any q.m. continuous UBLS stochastic process is even $V$-bounded (cf. [9], Theorem 4).

Statements (i)-(iii) in the following theorem were proved in [9] (Lemma 6, Theorems 7 and 8), statements (iv)-(vii) are implied by Lemma 1, Example 1 and Theorem 3.

**Theorem 5.** Let $x(t)$, $t \in R$, be a UBLS stochastic process and let $(y, B)$ be a stationary similarity of $x(t)$, $t \in R$. Then:

(i) $\mathfrak{sp} \{x; -\infty\} = B(\mathfrak{sp} \{y; -\infty\}), \quad \mathfrak{sp} \{y; -\infty\} = B^{-1}(\mathfrak{sp} \{x; -\infty\});$

(ii) $x(t)$, $t \in R$, is deterministic (respectively, purely non-deterministic) if and only if $y(t)$, $t \in R$, is deterministic (respectively, purely non-deterministic);

(iii) the stochastic processes

$$x'(t) = B(v_r(t)), \quad x''(t) = B(u_r(t)), \quad t \in R,$$

are UBLS stochastic processes having the same shift operator group as $x(t)$, $t \in R$; $x'(t), t \in R$, is deterministic and $x''(t), t \in R$, is purely non-deterministic.

Suppose, in addition, that $x(t), t \in R$, is q.m. continuous and that $\mu = \mu_{d} + \mu_{c}$ is the Lebesgue decomposition with respect to $m$ for the spectral measure $\mu$ of $x(t), t \in R$. If $x(t), t \in R$, is not deterministic, then by applying the notation introduced in (2):
Wold decomposition

(i) $x'(t) = x_s(t)$ and $x''(t) = x_c(t)$ for $t \in R$;
(ii) $\tilde{s}_p \{ x; -\infty \} = \tilde{s}_p \{ \mu_s \}$;
(iii) for all $t \in R$

$$v_x(t) = x_s(t) + P_{vP(\mu_g)}(x_c(t)), \quad u_x(t) = x_c(t) - P_{vP(\mu_g)}(x_c(t));$$

(vi) the spectral measures of $v_x(t)$ and $u_x(t)$ for $t \in R$ are

$$\mu_s = \mu_s + P_{vP(\mu_g)} \circ \mu_c \quad \text{and} \quad \mu_u = \mu_c - P_{vP(\mu_g)} \circ \mu_c,$$

respectively.

Remark. (i) A q.m. continuous UBLS stochastic process with an $m$-continuous spectral measure is either purely non-deterministic or deterministic.

(ii) In [9], Theorem 14, it is presented a necessary and sufficient condition for a so-called harmonizable UBLS stochastic process to be deterministic (respectively, purely non-deterministic).

Acknowledgements. The final form of this paper was prepared during the author’s stay at the Technical University of Wroclaw. The author wants to express his gratitude for the hospitality offered to him during his stay in Wroclaw.

REFERENCES


Department of Mathematics
University of Helsinki
SF-00100 Helsinki 10, Finland

*Received on 1. 6. 1979*