ON MARCINKIEWICZ-ZYGMUND LAWS OF LARGE NUMBERS IN BANACH SPACES AND RELATED RATES OF CONVERGENCE

BY

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Abstract. The paper studies asymptotic almost sure and tail behavior of sums \( (X_1 + \ldots + X_n) / n^{1/p} \), \( 1 < p < 2 \), for independent, centered random vectors \( X_n, n = 1, 2, \ldots \), taking values in Banach space \( E \). The obtained results are in the spirit of Mazurkiewicz-Zygmund, Hsu-Robbins-Erdős-Spitzer, and Brunk theorems for real random variables and show the essential role played by the geometry of \( E \) in the infinite-dimensional case.

1. Introduction and preliminaries. Let \( (E, \| \cdot \|) \) be a real separable Banach space. In the present paper we study strongly measurable random vectors \( X \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with values in \( E \). If \( E \|X\| < \infty \), then \( EX \) stands for the Bochner integral, and throughout the paper \( (X_i)_{i=1,2,\ldots} \) will be independent random vectors in \( E \), with \( S_0 = 0 \), \( S_n = X_1 + \ldots + X_n \), \( n = 1, 2, \ldots \), and \( (r_i) \) will stand for a Rademacher sequence, i.e., a sequence of real independent random variables with \( \mathbb{P}(r_i = \pm 1) = 1/2 \).

We recall a couple of definitions (for more information cf., e.g., [14]).

Definition 1.1. Let \( 1 < p < 2 \). A Banach space \( E \) is said to be of Rademacher type \( p \) (R-type \( p \)) if there exists \( C \) such that for every \( x, \ldots, x_n \in E \)

\[
E \left\| \sum_{i=1}^n r_i x_i \right\| \leq C \left( \sum_{i=1}^n \| x_i \|^p \right)^{1/p}.
\]

Definition 1.2. Let \( 1 < p < 2 \). \( l_p \) is said to be finitely representable in \( E \) if for every \( \varepsilon > 0 \) and every \( n \in \mathbb{N} \) there exist \( x_1, \ldots, x_n \in E \) such that for all \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \)

\[
\left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p} \leq \left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq (1 + \varepsilon) \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}.
\]
Example 1.1. \( l_p \) is of \( R \)-type \( \min(p, 2) \) for any \( p \geq 1 \). \( l_p \) is finitely representable in \( l_q \) for any \( q \leq p \), but \( l_p \) is not finitely representable in \( l_q \) if \( q > p \). On the other hand, by Dvoretzky's theorem, \( l_2 \) is finitely representable in \( E \) for any infinite dimensional \( E \).

Definition 1.3. A sequence \((X_i)\) of random vectors in \( E \) is said to have uniformly bounded tail probabilities by tail probabilities of a real random variable \( X_0 \) if there exists \( C > 0 \) such that for every \( t > 0 \) and every \( i \in \mathbb{N} \)

\[
P(\|X_i\| > t) \leq CP(\|X_0\| > t).
\]

The main results of the paper deal with the almost sure convergence of sums \( S_n/n^{1/p} \) and with the rate of convergence to zero of tail probabilities \( P(\|S_n/n^{1/p}\| > \varepsilon) \) under restrictions on individual random vectors \( X_i \) and on geometric structure of \( E \). For real-valued independent identically distributed \((X_i)\) (\( E = \mathbb{R} \)) the problem of rates of convergence was studied in a series of papers by Erdös [3], Spitzer [12], Baum and Katz [1], and in the case of a general Banach space \( E \) certain interesting results have been obtained by Jain [4].

As far as the strong and weak laws of large numbers of Marcinkiewicz-Zygmund type (i.e., for \( S_n/n^{1/p} \) and i.i.d. \((X_i)\)) are concerned the following is known:

In the case \( p = 1 \), R. Fortet and M. Mourier proved in 1953 that, without any restrictions on \( E \), if \((X_i)\) are i.i.d., \( E\|X_1\| < \infty \) and \( E X_1 = 0 \), then \( S_n/n \to 0 \) a.s. On the other hand, Maurey and Pisier [10] have shown that \( (r_1 x_1 + \ldots + r_n x_n)/n^{1/p} \to 0 \) a.s. for any bounded sequence \((x_n) \subseteq E \) if and only if \( l_p \) is not finitely representable in \( E \) (1 \( \leq p < 2 \)). In 1977, Marcus and Woyczyński [8], [9] proved that \( S_n/n^{1/p} \to 0 \) in probability for any i.i.d. \((X_i)\) satisfying the condition

\[
\lim_{n \to \infty} n^p P(\|X_i\| > n) = 0
\]

if and only if \( l_p \) is not finitely representable in \( E \).

In this paper we show, in particular, that for independent \((X_i)\) with uniformly bounded tail probabilities the implication "if \( E\|X_i\|^p < \infty \) and \( E X_i = 0 \), then \( S_n/n^{1/p} \to 0 \) a.s." also depends in an essential way on \( l_p \) not being finitely representable in \( E \). We also prove that a Banach space analogue of Brunk's strong law of large numbers (cf. [2], [11]) depends on the \( R \)-type of \( E \). Brunk's type strong law is particularly useful in cases where one has information about existence of moments of \( X_i \)'s of orders greater than 2. Such information may not be utilized in the framework of Kolmogorov-Chung's strong law.

As far as the rates of convergence are concerned a number of simple remarks are in order here. Directly from definitions and from Chebyshev's inequality one can obtain the following "trivial" rate:
PROPOSITION 1.1. Let $1 \leq p \leq 2$ and let $E$ be of $R$-type $p$. If $(X_i)$ are i.i.d. with $E \|X_1\|^p < \infty$ and $EX_1 = 0$, then

$$P(\|S_n/n\| \geq \varepsilon) = O(n^{1-p}) \quad \text{for every } \varepsilon > 0.$$  

Also some exponential rates can be immediately obtained without any restrictions on the geometric structure of $E$.

PROPOSITION 1.2. If $(X_i)$ are i.i.d. with $EX_1 = 0$ and with the property that for every $\varepsilon > 0$ there exist $C_\varepsilon$ and $\beta_\varepsilon$ such that for every $\beta \leq \beta_\varepsilon$ 

$$E \exp \left[ \beta \|X_1\| \right] \leq C_\varepsilon \exp \left[ \beta \varepsilon \right],$$

then for every $\varepsilon > 0$ there exists $\alpha < 1$ such that

$$P(\|S_n/n\| > \varepsilon) = O(\alpha^n).$$

Proof. By Chebyshev's inequality and for $\delta < \varepsilon$ we get

$$P(\|S_n/n\| > \varepsilon) \leq \exp \left[ -\beta_\varepsilon n \varepsilon \right] E \exp \left[ \beta_0 \|S_n\| \right] \leq \exp \left[ -\beta_\varepsilon n \varepsilon \right] (E \exp [\beta_0 \|X_1\|])^n \leq C_\delta \exp [(\delta - \varepsilon) \beta_\varepsilon]^n.$$

It is also interesting to notice that a sufficiently rapid rate of convergence to zero of tail probabilities $P(\|S_n/a_n\| > \varepsilon)$ implies similar rates of convergence in the strong law, i.e., for the suprema.

PROPOSITION 1.3. Let $E$ be a Banach space and let $(X_i)$ be independent symmetric random vectors in $E$. Let $(a_i)$, $(b_i)$, $(c_i) \in \mathbb{R}$ be such that

$$0 < a_i \uparrow \infty, \quad b_i, c_i \downarrow 0 \quad \text{and} \quad \sum_{i=1}^{j} 2^i b_{2^i} = O(2^i c_{2^i})$$

and let

$$\sum_{n=1}^{\infty} c_n P(\|S_n/a_n\| > \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.$$  

Then

$$\sum_{n=1}^{\infty} b_n P(\sup_{k \geq n} \|S_k/a_k\| > \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.$$  

Proof. Grouping the terms in exponential blocks ($n$: $2^j < n \leq 2^{j+1}$) we get

$$A \equiv \sum_{n=1}^{\infty} b_n P(\sup_{k \geq n} \|S_k/a_k\| > \varepsilon) \leq \sum_{i=1}^{\infty} b_{2^i} \cdot 2^i P(\sup_{k \geq 2^i} \|S_k/a_k\| > \varepsilon) \leq \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} b_{2^i} \cdot 2^i P(\max_{2^j < k \leq 2^{j+1}} \|S_k/a_k\| > \varepsilon)$$
and, by Lévy's inequality,
\[
A \leq 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{2^i} \cdot 2^i \mathbb{P}(\|S_{2^i+1}/a_{2^i+j+1}\| > \varepsilon)
\]
\[
= 2 \sum_{j=1}^{\infty} \left( \sum_{i=1}^{j} b_{2^i} \cdot 2^i \right) \mathbb{P}(\|S_{2^i+1}/a_{2^i+j+1}\| > \varepsilon)
\]
\[
\leq 2 C \sum_{j=1}^{\infty} c_{2^j} \cdot 2^j \mathbb{P}(\|S_{2^j+1}/a_{2^j+j+1}\| > \varepsilon).
\]

Now, by the symmetry assumptions, grouping the terms again as follows:
\[
S_n = S_{2^j+1} - X_{2^j+1} - X_{2^{j+1}-1} - \ldots - X_{n+1}, \quad 2^{j-1} \leq n < 2^j,
\]
we obtain
\[
A \leq 8 C \sum_{n=1}^{\infty} c_n \mathbb{P}(\|S_n/a_n\| > 2\varepsilon).
\]

Two special cases of Proposition 1.3 will be of interest later on.

**Corollary 1.1.** Let $E$ be a Banach space and let $(X_i)$ be independent symmetric random vectors in $E$. Then
(i) for every $q > 1$ there exists $C > 0$ such that
\[
\sum_{n=1}^{\infty} n^{-q} \mathbb{P}(\sup_{k \geq n} \|S_k/a_k\| > \varepsilon) \leq C \sum_{n=1}^{\infty} n^{-q} \mathbb{P}(\|S_n/a_n\| > \varepsilon);
\]
(ii) there exists $C > 0$ such that
\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P}(\sup_{k \geq n} \|S_k/a_k\| > \varepsilon) \leq C \sum_{n=1}^{\infty} n^{-1} (\log n) \mathbb{P}(\|S_n/a_n\| > \varepsilon).
\]

2. Rates of convergence based on the Marcinkiewicz-Zygmund inequality.

In Proposition 1.1 we could have only used moments of order $p$, $1 \leq p \leq 2$, and in Proposition 1.2 exponential moments were needed. The following analogue of the Marcinkiewicz-Zygmund inequality (cf. also results by P. Assouad and B. Maurey and G. Pisier quoted in [14]) permits us to use the information on moments of arbitrary order.

**Proposition 2.1.** Let $1 < p \leq 2$ and $q \geq 1$. The following properties of $E$ are equivalent:
(i) $E$ is of $R$-type $p$.
(ii) There exists $C$ such that for every $n \in \mathbb{N}$ and for any sequence $(X_i)$ of independent random vectors in $E$ with $EX_i = 0$
\[
E \left\| \sum_{i=1}^{n} X_i \right\|^q \leq C E \left( \sum_{i=1}^{n} \|X_i\|^p \right)^{q/p}.
\]
Proof. (i) ⇒ (ii). Let \((\tilde{X}_i) = (X_i - X_i')\) be a symmetrization of \((X_i)\) and let \((r_i)\) be independent of \((X_i)\) and \((X_i')\). Then
\[
E\left\| \sum_{i=1}^{n} X_i \right\|^q \leq E\left\| \sum_{i=1}^{n} \tilde{X}_i \right\|^q = E\left\| \sum_{i=1}^{n} r_i \tilde{X}_i \right\|^q \\
\leq C E\left( \sum_{i=1}^{n} \| \tilde{X}_i \|^p \right)^{q/p} \leq C \cdot 2^q E\left( \sum_{i=1}^{n} \| X_i \|^p \right)^{q/p},
\]
where the first inequality follows from the condition \(E X_i = 0\), and because \((X_i')\) are independent of \((X_i)\), the equality holds by symmetry of \((\tilde{X}_i)\), the second inequality by \(R\)-type of \(E\) and Fubini's theorem, and the third one by the triangle inequality.

The implication (ii) ⇒ (i) follows from the proof of Theorem 3.1 given in the sequel.

Corollary 2.1. Let \(E\) be of \(R\)-type \(p\) and \(q \geq p\). If \((X_n)\) are i.i.d. random vectors in \(E\) with \(E\|X_1\|^q < \infty\) and \(EX_1 = 0\), then \(E\|S_n\|^q = O(n^{q/p})\).

Proof. If \(p = q\), the estimate follows directly from the definition of \(R\)-type \(p\). If \(q > p\), then by Hölder's inequality with exponents \(q/p\) and \(q/(q-p)\) and by Proposition 2.1 we have
\[
E\left\| \sum_{i=1}^{n} X_i \right\|^q \leq C E\left( \sum_{i=1}^{n} \| X_i \|^p \right)^{q/p} \\
\leq C E\left( \sum_{i=1}^{n} \| X_i \|^q \right)^{(q-p)/p} = C n^{q/p} E\|X_1\|^q.
\]
Hence, by Chebychev's inequality we obtain immediately

Corollary 2.2. Let \(E\) be of \(R\)-type \(p\) and \(q \geq p\). If \((X_n)\) are i.i.d. with \(E\|X_1\|^q < \infty\) and \(EX_1 = 0\), then
\[
P\left( \|S_n/n\| > \epsilon \right) = O(n^{q(1/p - 1)}) \quad \text{for every } \epsilon > 0.
\]

Remark 2.1. Jurek and Urbanik [5], studying stable measures on \(E\), define \(E\) as being of type \((s, r)\), \(s \geq 0, r > 0\), whenever there exists \(C\) such that for all \((X_i)\) independent and symmetric in \(E\)
\[
E\left\| \sum_{i=1}^{n} X_i \right\|^r \leq C n^s \sum_{i=1}^{n} E\|X_i\|^r.
\]
Proposition 2.1 implies (as in the proof of Corollary 2.1) that if \(E\) is of \(R\)-type \(p\), then
\[
E\left\| \sum_{i=1}^{n} X_i \right\|^q \leq C n^{q/p - 1} \sum_{i=1}^{n} E\|X_i\|^q \quad \text{for every } q \geq p,
\]
i.e. \(E\) is also of Jurek-Urbanik's type \((q/p - 1, q)\) or, equivalently, \(E\) is of type \((s, p(s+1))\) for every \(s \geq 0\). One can also show (as in Theorem 3.1
below) that if for some \( s > 0 \) the space \( E \) is of type \((s, p(s+1))\), then \( E \) is of \( R\)-type \( p \).

3. Brunk's type strong law and related rates of convergence. The following result extends the Kolmogorov-Chung type strong law in \( E \) obtained by the author and J. Hoffmann-Jörgensen and G. Pisier (cf. [14], p. 390, where \( E \) is of \( R\)-type \( p \), \( 1 \leq p \leq 2 \), and \( q = 1 \)). In the case \( E = \mathbb{R} \), \( p = 2 \), \( q \geq 1 \), the theorem is due to Brunk [2] and Prohorov [11].

**Theorem 3.1.** (a) Let \( 1 \leq p \leq 2 \), let \( E \) be of \( R\)-type \( p \), and \( q \geq 1 \). If \((X_n)\) are independent zero-mean random vectors in \( E \) such that

\[
\sum_{n=1}^{\infty} \frac{E\left\|X_n\right\|^{pq}}{n^{pq + 1 - q}} < \infty,
\]

then \( S_n/n \to 0 \) a.s. in norm.

(b) Conversely, if \( q \geq 1 \), \( 1 \leq p \leq 2 \), and, for each \((x_i) \subset E \) such that

\[
\sum_{i=1}^{n} \left\|x_i\right\|^{pq}/i^{pq + 1 - q} < \infty,
\]

\( \sum_{i=1}^{n} r_i x_i/n \to 0 \) as \( n \to \infty \)

a.s. in norm, then \( E \) is of \( R\)-type \( p \).

**Proof.** (a) For \( q = 1 \) the theorem boils down to the Kolmogorov-Chung type strong law as mentioned above.

Assume \( q > 1 \). Then \( \left\|S_n\right\|^{pq} \) is a real submartingale and, by the well-known Hajek-Rényi-Chow type inequality, we get

\[
\varepsilon^{pq} P\left(\sup_{j \geq n} \left\|S_j/j\right\| > \varepsilon\right) = \varepsilon^{pq} \lim_{m \to \infty} P\left(\sup_{m \leq j \leq m+n} \left\|S_j/j\right\|^{pq} > \varepsilon^{pq}\right)
\]

\[
\leq n^{-pq} E\left\|S_n\right\|^{pq} + \sum_{j=n+1}^{\infty} j^{-pq} E\left(\left\|S_j\right\|^{pq} - \left\|S_{j-1}\right\|^{pq}\right)
\]

for every \( \varepsilon > 0 \).

By Proposition 2.1 and by Hölder's inequality,

\[
E\left\|S_j\right\|^{pq} \leq C E\left(\sum_{i=1}^{j} \left\|X_i\right\|^{p}\right)^{q} \leq C^{q-1} \sum_{i=1}^{j} E\left\|X_i\right\|^{pq},
\]

so that by (3.1) and Kronecker's lemma we obtain

\[
j^{-pq} E\left\|S_j\right\|^{pq} \to 0 \quad \text{as} \quad j \to \infty.
\]

Also the series on the right-hand side of (3.2) converges because of Proposition 2.1. Hence, summing by parts,

\[
\sum_{j=1}^{n} (j-1)^{-pq} + j^{-pq}) E\left\|S_j\right\|^{pq} \leq \sum_{j=1}^{n} (j-1)^{-pq} + j^{-pq}) j^{-pq-1} \sum_{i=1}^{j} E\left\|X_i\right\|^{pq}
\]

\[
\leq C \sum_{j=1}^{n} E\left\|X_j\right\|^{pq}/j^{pq + 1 - q} + \sum_{i=1}^{n} E\left\|X_i\right\|^{pq}/n^{pq + 1 - q}.
\]
Therefore, for every $\varepsilon > 0$,

$$P(\sup_{j \geq n} \|S_j/j\| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$  

(b) Kahane's theorem (cf. [14], p. 275) states that, for any Banach space $E$ and any $p$ ($0 < p < \infty$), all the $L_p(E)$-norms are equivalent on the span of $(r_i x_i), (x_i) \subset E$. Hence, in view of the closed graph theorem, there exists $C$ such that for all $(x_i) \subset E$

$$E \left\| \sum_{i=1}^{n} r_i x_i n^{-1} \right\| \leq C \left( \sum_{i=1}^{n} \|x_i\|^{p} \right)^{1/pq},$$

so that

$$E \left\| \sum_{i=1}^{n} n^{-1 + (1-q)/pq} r_i x_i \right\| \leq C \left( \sum_{i=1}^{n} \|x_i\|^{p} \right)^{1/pq} \quad \text{for all } (x_i) \subset E.$$  

Hence

$$E \left\| \sum_{i=1}^{n} r_i x_i \right\| = E \left\| \sum_{i=n+1}^{2n} r_i x_i \right\| \leq n^{-(1-q)/pq} E \left\| \sum_{i=1}^{n} \frac{i^{1+(1-q)/pq}}{2n} r_i x_i + \sum_{i=n+1}^{2n} \frac{i^{1+(1-q)/pq}}{2n} r_i x_i \right\| \leq n^{-(1-q)/pq} C \cdot 2^{1/pq} \left( \sum_{i=1}^{n} \|x_i\|^{p} \right)^{1/pq}.$$

Now, since for any $\alpha, \beta$ ($0 < \alpha, \beta < \infty$) and $a_i \geq 0$ the inequality

$$\left( \sum a_i^\alpha \right)^{1/\alpha} \leq n^{1/\alpha - 1/\beta} \left( \sum a_i^\beta \right)^{1/\beta}$$

holds, we have

$$E \left\| \sum_{i=1}^{n} r_i x_i \right\| \leq C \cdot 2^{1/pq} n^{-(1-q)/pq} n^{1/pq - 1/p} \left( \sum_{i=1}^{n} \|x_i\|^{p} \right)^{1/p} \leq C \cdot 2^{1/pq} \left( \sum_{i=1}^{n} \|x_i\|^{p} \right)^{1/p}.$$

The following "rate of convergence" result for the weak law is associated with the strong law above.

**Theorem 3.2.** Let $1 \leq p \leq 2$ and $q \geq 1$. The following properties of a Banach space $E$ are equivalent:

(i) $E$ is of $R$-type $p$.

(ii) for every $\varepsilon > 0$ there exists $C_\varepsilon$ such that for any independent zero-mean $(x_i)$ in $E$

$$\sum_{n=1}^{\infty} n^{-1} P(\|S_n/n\| > \varepsilon) \leq C_\varepsilon \sum_{n=1}^{\infty} \frac{E \left\| X_n \right\|^{pq}}{n^{pq + 1 - q}}.$$

Proof. (i) ⇒ (ii). By the Chebyshev, Marcinkiewicz-Zygmund (Proposition 2.1) and Hölder inequalities we get

\[
\sum_{n=1}^{\infty} n^{-1} P(\|S_n\| > \varepsilon n) \leq \sum_{n=1}^{\infty} n^{-1} n^{-pq} \varepsilon^{-pq} E\|S_n\|^{pq}
\]

\[
\leq \varepsilon^{-pq} C \sum_{n=1}^{\infty} n^{-1+(q-1)-pq} \sum_{k=1}^{n} E\|X_k\|^{pq}
\]

\[
\leq C\varepsilon^{-pq} \sum_{k=1}^{\infty} E\|X_k\|^{pq} \sum_{n=k}^{\infty} n^{-pq+q-2}
\]

\[
\leq C\varepsilon^{-pq} \sum_{k=1}^{\infty} E\|X_k\|^{pq/k^{pq+1-q}}.
\]

(ii) ⇒ (i) follows directly from the proof of (b) in Theorem 3.1.


**Theorem 4.1.** Let \(1 < p < 2\). Then the following properties of a Banach space \(E\) are equivalent:

(i) \(l_p\) is not finitely representable in \(E\).

(ii) For any sequence \((X_i)\) of zero-mean independent random vectors in \(E\) with tail probabilities uniformly bounded by tail probabilities of an \(X_0 \in L^p\), the series

\[
\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}}
\]

converges a.s. in norm.

(iii) For any sequence \((X_i)\) as in (ii), \(S_n/n^{1/p} \to 0\) a.s.

The proof of Theorem 4.1 will be based on the following

**Lemma 4.1.** Let \(1 \leq p < 2\), let \(l_p\) be not representable in \(E\), and let \((X_n)\) satisfy assumptions of Theorem 4.1 (ii). Then the series

\[
\sum_{n=1}^{\infty} (X_n - EY_n)/n^{1/p},
\]

where \(Y_n = X_n I (\|X_n\| \leq n^{1/p})\), converges a.s.

**Proof.** Since

\[
\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(\|X_n\| > n^{1/p}) \leq C \sum_{n=1}^{\infty} P(|X_0| > n^{1/p}) \leq C_1 E|X_0|^p < \infty,
\]

in view of the Borel-Cantelli lemma it suffices to show that the series \(\sum(Y_n - EY_n)/n^{1/p}\) converges a.s.
Let $r > p$. Then
\[ \sum_{n=1}^{\infty} E \| Y_n - E Y_n \|^r n^{-r/p} \leq 2^{r+1} \sum_{n=1}^{\infty} E \| Y_n \|^r n^{-r/p} \]
\[ = 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \int_{\| X_n \| < n^{1/p}} \| X_n \|^r dP = 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \int_0^{n^{1/p}} t^r dP (\| X_n \| < t) \]
\[ = 2^{r+1} \sum_{n=1}^{\infty} n^{-r/p} \left( n^{r/p} \int_0^{n^{1/p}} P (\| X_n \| < t) dt \right) \]
\[ \leq C_1 \sum_{n=1}^{\infty} \left( 1 - n^{-r/p} \int_0^{n^{1/p}} t^r (1 - P (\| X_n \| > t)) dt \right) \]
\[ = C_1 \sum_{n=1}^{\infty} n^{-r/p} \int_0^{n^{1/p}} t^{-1} P (\| X_n \| > t) dt = C_1 \sum_{n=1}^{\infty} \int_0^{n^{1/p}} P (\| X_n s^{-1/p} \| > n^{1/p}) ds \]
\[ \leq C_2 E |X_0|^p \int_0^{n^{1/p}} s^{-r/p} ds = C_2 \frac{r}{r-p} E |X_0|^p < \infty. \]

By Maurey-Pisier's theorem (see [10] and [14], p. 371) and by assumption, there exists $r > p$ such that $E$ is of $R$-type $r$. Therefore, the estimate above and Theorem V.7.5 in [14] give the desired a.s. convergence of $\sum (Y_n - EY_n) n^{-1/p}$.

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii). In view of Lemma 4.1 it is sufficient to prove the absolute convergence of the series $\sum EY_n n^{-1/p}$. Since $EX_n = 0$ and $p > 1$, we have
\[ \sum_{n=1}^{\infty} \| EY_n \|^n n^{-1/p} \leq \sum_{n=1}^{\infty} n^{-1/p} \int_0^{n^{1/p}} tdP (\| X_n \| < t) \]
\[ = - \sum_{n=1}^{\infty} n^{-1/p} \int_0^{n^{1/p}} tdP (\| X_n \| > t) \]
\[ = \sum_{n=1}^{\infty} \left( P (\| X_n \| > n^{1/p}) + \int_0^{n^{1/p}} P (\| X_n s^{-1/p} \| > n^{1/p}) ds \right) \leq C E |X_0|^p, \]
which gives (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows by a straightforward application of Kronecker’s lemma.

(iii) $\Rightarrow$ (i). This implication is essentially due to Maurey and Pisier [10] (cf. also [14], p. 389). We quote the proof for the sake of completeness.

In view of Kronecker’s lemma it suffices to construct, in any Banach space $E$ such that $l_p$ is finitely representable in $E$, a sequence $(x_n) \subset E$, 

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\( \|x_n\| \leq 1, n = 1, 2, \ldots, \) such that for a sequence \((N_k) \subset N, N_k \to \infty, \) for all choices of \( \varepsilon_n = \pm 1 \) and for all \( k \in N \)

\[
N_k^{-1/p} \left\| \sum_{i=1}^{N_k} \varepsilon_i x_i \right\| > \frac{1}{2}.
\]

Put \( N_1 = 1 \) and choose any \( x_1 \in E, \|x_1\| = 1, \) Suppose \( N_1, \ldots, N_k \) and \( x_1, \ldots, x_N \) have been chosen so that \( \|x_i\| \leq 1, i = 1, \ldots, N_k, \) and for all \( \varepsilon_i = \pm 1 \) inequality (4.1) is satisfied. Choose \( N_{k+1} \in N \) large enough for

\[
N_{k+1}^{-1/p} \left[ \frac{2}{3} (N_{k+1} - N_k)^{1/p} - N_k \right] > \frac{1}{2}.
\]

Since \( l_p \) is finitely representable in \( E, \) we can find \( x_{N_k+1}, \ldots, x_{N_k+1} \) such that for all \( (\alpha_k) \subset R \)

\[
\frac{2}{3} \left( \sum_{i=N_k+1}^{N_{k+1}} |\alpha_i|^p \right)^{1/p} \leq \left| \sum_{i=N_k+1}^{N_{k+1}} \alpha_i x_i \right| \leq \left( \sum_{i=N_k+1}^{N_{k+1}} |\alpha_i|^p \right)^{1/p}.
\]

Therefore

\[
N_{k+1}^{-1/p} \left\| \sum_{i=1}^{N_{k+1}} \varepsilon_i x_i \right\| \geq N_{k+1}^{-1/p} \left\| \sum_{i=N_k+1}^{N_{k+1}} \varepsilon_i x_i \right\| - \left\| \sum_{i=1}^{N_k} \varepsilon_i x_i \right\| > N_{k+1}^{-1/p} \left[ \frac{2}{3} (N_{k+1} - N_k)^{1/p} - N_k \right] > \frac{1}{2} \quad \text{for all } \varepsilon_n = \pm 1.
\]

For spaces \( E \) such that \( l_1 \) is not finitely representable in \( E, \) i.e., for \( B \)-convex spaces (see [14], Chapter VII), Lemma 4.1 permits to prove the following

**Theorem 4.2.** The following properties of a Banach space \( E \) are equivalent:

(i) \( l_1 \) is not finitely representable in \( E. \)

(ii) For any sequence \((X_i)\) of independent zero-mean random vectors in \( E \) with tail probabilities uniformly bounded by tail probabilities of an \( X_0 \in L \log^+ L, \)

the series

\[
\sum_{n=1}^{\infty} \frac{X_n}{n}
\]

converges a.s.

(iii) For any sequence \((X_i)\) as in (ii), \( S_n/n \to 0 \) a.s. as \( n \to \infty. \)

**Proof.** (i) \( \Rightarrow \) (ii). In view of Lemma 4.1 it suffices to prove that \( \sum \|EY_n\| n^{-1} \) converges whenever \( X_0 \in L \log^+ L. \) Since \( EX_n = 0, \) by integration by parts
we obtain
\[
\sum_{n=1}^{\infty} \|EY_n\| n^{-1} < \sum_{n=1}^{\infty} n^{-1} \sum_{n=1}^{\infty} n^{-1} \int dt P(\|X_n\| < t) = \sum_{n=1}^{\infty} P(\|X_n\| > n) n^{-1} \sum_{n=1}^{\infty} P(\|X_n\| > n) dt
\]
\[
\leq C_1 \mathbb{E}|X_0| + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^{-1} P(|X_0| > k)
\]
\[
= C_1 \mathbb{E}|X_0| + \sum_{k=1}^{\infty} (\log k) P(|X_0| > k)
\]
\[
\leq C_1 \mathbb{E}|X_0| + \sum_{k=1}^{\infty} \log^+ |X_0| < \infty.
\]

(ii) \implies (iii) follows directly from Kronecker's lemma, and (iii) \implies (i) can be proved exactly as (iii) \implies (i) in Theorem 4.1.

**Theorem 4.3.** (a) Let \( E \) be a Banach space, 1 < \( p \) < 2, and let \( \alpha \geq 1/p \). Then \( l_p \) is not finitely representable in \( E \) if and only if for each independent zero-mean \((X_i)_i \) in \( E \) with tail probabilities uniformly bounded by tail probabilities of an \( X_0 \in L^p \) we have
\[
\sum_{n=1}^{\infty} n^{2p-1} P(\max_{1 \leq i \leq n} \|S_i\| > n^{\alpha} \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.
\]

(b) Let \( E \) be a Banach space and let 1 \( \leq p \) < 2. Then \( l_p \) is not finitely representable in \( E \) if and only if for each independent zero-mean \((X_i)_i \) in \( E \) with tail probabilities uniformly bounded by tail probabilities of an \( X_0 \in L^p \log L \) we have
\[
\sum_{n=1}^{\infty} n^{-1} (\log n) P(\|S_n\| > n^{1/p} \varepsilon) < \infty \quad \text{for every } \varepsilon > 0.
\]

**Proof.** (a) We prove first the sufficiency of the condition of \( l_p \) not being finitely representable in \( E \). By Theorem 4.1, \( S_u/n^{1/p} \to 0 \) a.s. and, as is easy to see, also
\[
M_u/n^{1/p} \to 0 \text{ a.s.}, \quad \text{where } M_u = \max_{1 \leq i \leq [u]} \|S_i\|, \quad u \in \mathbb{R}, \quad [u] = \text{entier } u.
\]

Hence, if we introduce Chow's delayed sums
\[
S_{u,v} = \sum_{1 \leq j \leq v} X_{[u]+j}, \quad u, v \in \mathbb{R},
\]
we get

\[ M_{n,n} n^{-1/p} \leq (M_n + M_{2n}) n^{-1/p} \to 0 \text{ a.s. as } n \to \infty. \]

Now, in the case \( \alpha = 1/p \), since \( M_{2n,2n} \) (\( n = 1, 2, \ldots \)) are independent, from the Borel-Cantelli lemma we infer that

\[
\sum_{n=1}^{\infty} P(M_{2n,2n} > 2^{n/p} \varepsilon) = \sum_{n=1}^{\infty} P(M_{2n} > 2^{n/p} \varepsilon) \geq \int P(M_{2n} > 2^{(\alpha+1)/p} \varepsilon) \, dt
\]

\[
> (\log 2)^{-1} \int u^{-1} P(M_u > 2^{1/p} e u^{1/p}) \, du \quad \text{for every } \varepsilon > 0,
\]

so that \( \sum n^{-1} P(M_n > n^{1/p} \varepsilon) < \infty \) for every \( \varepsilon > 0 \).

In the case \( \alpha > 1/p \), for \( m \geq 1 \) we have

\[
(m+1)^{\alpha p/(\alpha p - 1)} \geq m^{\alpha p/(\alpha p - 1)} + \frac{\alpha p}{\alpha p - 1} m^{1/(\alpha p - 1)} + m^{1/(\alpha p - 1)},
\]

so that the random variables \( M_{m^{\alpha p/(\alpha p - 1)}, m^{1/(\alpha p - 1)}} \), \( m = 1, 2, \ldots \), are independent. Moreover, by Theorem 4.1,

\[
m^{-\frac{2}{\alpha p - 1}} M_{m^{\alpha p/(\alpha p - 1)}, m^{1/(\alpha p - 1)}} \leq m^{-\frac{2}{\alpha p - 1}} M_{m^{\alpha p/(\alpha p - 1)}, m^{\alpha p/(\alpha p - 1)}} \to 0 \text{ a.s. as } m \to \infty.
\]

Therefore, again by the Borel-Cantelli lemma we obtain

\[
\sum_{m=1}^{\infty} P(M_{m^{\alpha p/(\alpha p - 1)}, m^{1/(\alpha p - 1)}} \leq m^{2/(\alpha p - 1)} \varepsilon) = \sum_{m=1}^{\infty} P(M_{m^{1/(\alpha p - 1)}} \geq m^{2/(\alpha p - 1)} \varepsilon) \geq \int P(M_{1/(\alpha p - 1)} \geq (1 + 1)^{2/(\alpha p - 1)} \varepsilon) \, dt
\]

\[
\geq (\alpha p - 1) \int u^{\alpha p - 1} P(M_u \geq 2^{2/(\alpha p - 1)} u^\varepsilon) \, du,
\]

which gives the desired rate of convergence. The necessity of the condition of \( l_p \) not being representable in \( E \) follows directly from the example developed in the proof of (iii) \( \Rightarrow \) (i) in Theorem 4.1.

(b) Sufficiency. We may assume that \( X_i \)'s are symmetric. The case of zero expectations can be handled by adapting in the standard way the method presented below.
Put $Y_{kn} = X_k I (\|X_k\| < n^{1/p})$. Then
\[
\sum_{n=1}^{\infty} n^{-1} (\log n) P (\|S_n\| > n^{1/p} \varepsilon) \leq \sum_{n=1}^{\infty} n^{-1} (\log n) \left( \bigcup_{k=1}^{n} (\|X_k\| > n^{1/p} \varepsilon) \right) + \sum_{n=1}^{\infty} n^{-1} (\log n) P (\| \sum_{k=1}^{n} Y_{kn} \| > n^{1/p} \varepsilon).
\]

The series on the right-hand side can be estimated from above by
\[
C \sum_{n=1}^{\infty} (\log n) P (\|X_0\| > n^{1/p} \varepsilon) \leq C_1 E |X_0|^{p} \log^{+} |X_0| < \infty,
\]
and the convergence of the second series can be proved as follows.

Since $l_p$ is not finitely representable in $E$, by Maurey-Pisier's theorem mentioned before there exists $\delta > 0$ such that $E$ is of $R$-type $(p + \delta)$. Hence, making use of Chebyshev's inequality and integrating by parts we get
\[
\sum_{n=1}^{\infty} n^{-1} (\log n) P (\| \sum_{k=1}^{n} Y_{kn} \| > n^{1/p} \varepsilon)
\]
\[
\leq C_1 \sum_{n=1}^{\infty} n^{-1-(p+\delta)/p} (\log n) \sum_{k=1}^{n} E \| Y_{kn} \|^{p+\delta}
\]
\[
\leq C_2 \sum_{n=1}^{\infty} n^{-1-(p+\delta)/p} (\log n) \sum_{k=1}^{n} t^{p+\delta} dP (\|X_k\| \leq t)
\]
\[
\leq C_2 \sum_{n=1}^{\infty} n^{-(p+\delta)/p} (\log n) \int_{0}^{\infty} t^{p+\delta-1} P (|X_0| > t) dt
\]
\[
= C_2 \int_{0}^{1} t^{\delta/p} \sum_{n=1}^{\infty} (\log n) P (|X_0 s^{-1/p}| > n^{1/p}) ds
\]
\[
\leq C_3 E |X_0|^{p} \log^{+} |X_0| \int_{0}^{1} s^{-1+\delta/p} ds < \infty.
\]

This completes the proof of the sufficiency.

The necessity can be obtained exactly as in (a).

**Corollary 4.1.** If $l_p$ is not finitely representable in $E$, $1 < p < 2$, and $(X_i)$ are i.i.d. zero-mean random vectors in $E$ with $E \|X_1\|^{p} < \infty$, then

\[
P (\|S_n/n\| > \varepsilon) = o (n^{1-p}) \quad \text{for every } \varepsilon > 0.
\]

**Corollary 4.2.** Let $E$ be of $R$-type $p$, $1 < p \leq 2$, and let $(X_i)$ be independent zero-mean vectors in $E$ such that

\[
(4.2) \quad P (\|X_k\| > n) = o (n^{-p})
\]
uniformly in k. Then for every $\delta > 0, \epsilon > 0$

$$P (\| S_n/n \| > \epsilon) = o (n^{1-\frac{\delta}{p}}).$$

Proof. Since $E$ is of $R$-type $p$, $l^p - \delta$ is not finitely representable in $E$ for every $\delta > 0$. From (4.2) it follows also that $X_i$'s have tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^{p-\delta}$. Therefore, by Theorem 4.3,

$$\sum_{n=1}^{\infty} n^{p-\delta-2} P (\| S_n/n \| > \epsilon) < \infty,$$

so that

$$n^{p-\delta-2} P (\| S_n/n \| > \epsilon) = o (n^{-1}),$$

which gives the corollary.

From Corollary 1.1 and Theorem 4.3 we get immediately

**Corollary 4.3.** If $1 \leq p < 2$ and $l_p$ is not finitely representable in $E$, then for any sequence $(X_i)$ of independent zero-mean random vectors in $E$ with tail probabilities uniformly bounded by tail probabilities of an $X_0 \in L^p$ if $1 < p < 2$ and of an $X_0 \in L^{\log^+ L}$ if $p = 1$ we have

$$\sum_{n=1}^{\infty} n^{p-2} P (\sup_{k \geq n} \| S_n/k \| > \epsilon) < \infty \quad \text{for every } \epsilon > 0.$$

5. Concluding remarks.

5.1. Brunk's type strong law of large numbers in Banach spaces can be also obtained by using the methods developed by Kuelbs and Zinn [6] (J. Zinn — oral communication). These methods use however a rather powerful tool of exponential inequalities in Banach spaces.

5.2. In the i.i.d. case an alternative proof of results concerning rates of convergence is possible by applying a theorem of Jain [4] who proved that by and large, real-line "rates of convergence" results remain valid in general Banach spaces as long as $S_n/n^2$ are bounded in probability. In presence of our geometric restrictions on $E$ the latter is, of course, implied by the Marcinkiewicz-Zygmund type strong law. Other extensions along the lines of Jain's paper are also possible (e.g., Orlicz space type moment assumptions). We stuck to a simpler set up to emphasize the relation between geometric and probabilistic phenomena in $E$.

5.3. It also follows from Jain's paper that, for any Banach space $E$ and any i.i.d. zero-mean $(X_i)$, if $X_i \in L_1 (E)$, then $\sum n^{-2} P (\| S_n/n^2 \| > \epsilon) < \infty$ for every $\epsilon > 0$, and if, for an $\alpha \geq 1/p$, $\sum n^{p-2} P (\| S_n/n^p \| > \epsilon) < \infty$ for every $\epsilon > 0$, then $EX_1 = 0$ and $E \| X_1 \|^p < \infty$.

5.4. If $E$ is a Hilbert space, we can prove a result somewhat stronger than Corollary 4.2. Namely, if $(X_i)$ are i.i.d. in $E$ with $EX_1 = 0$ and
\[
P(\|X_1\| \geq n) = o(n^{-p}) \text{ for a } p > 1, \text{ then } P(\|S_n/n\| > \varepsilon) = o(n^{1-p}) \text{ for every } \varepsilon > 0.
\]

5.5. The validity of the Marcinkiewicz-Zygmund strong law of large numbers for i.i.d. \((X_n)\) in \(E\) is equivalent to \(E\) being of \(R\)-type \(p\) (A. de Acosta — oral communication).

5.6. Taylor and Wei [13] studied weighted sums of independent random vectors in Banach spaces under moment conditions similar to ours, but obtained only weak laws for them (i.e., with convergence in probability).

REFERENCES
