REMARK ON A MULTIPLICATIVE DECOMPOSITION OF PROBABILITY MEASURES

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Abstract. The aim of this note is to define a rather wide class of probability measures admitting a multiplicative decomposition.

Let $P$ be the set of all Borel probability measures on the real line. Given $\mu, \nu \in P$, by $\mu \nu$ we shall denote the probability distribution of the product $XY$ of two independent random variables $X$ and $Y$ with probability distributions $\mu$ and $\nu$, respectively. It is evident that the binary operation $\mu \nu$ is commutative, associative and distributive with respect to convex combinations of probability measures. In what follows $\mu^n$ will denote the $n$-th power under this operation. Further, by $\delta_c$ we denote the probability measure concentrated at the point $c$. It is easy to check that

$$ (\mu \nu)(E) = \int_{x \neq 0} \mu(x^{-1} E) \nu(dx) + \nu(0) \delta_0(E). $$

Put $I = (0, 1]$. By $P_I$ we denote the subset of $P$ consisting of all measures concentrated on $I$. We say that $\mu \in Q$ if $\mu \in P$ and, for every $x \in I$, there exists a positive number $c$ such that $\mu(x^{-1} E) \leq c \mu(E)$ for all Borel subsets $E$ of the real line.

Denote by $q(\mu, x)$ the infimum of all those numbers $c$. It is clear that $q(\mu, x) \geq 1$ whenever $x \in I$ and $q(\mu, 1) = 1$. Moreover, denoting by $\{E_n\}$ the sequence of all open intervals with rational endpoints, we have

$$ \{x: q(\mu, x) \leq c\} = \bigcap_{n=1}^{\infty} \{x: \mu(x^{-1} E_n) \leq c \mu(E_n)\}, $$

which shows that the function $q(\mu, \cdot)$ is Borel measurable on $I$. 
A standard calculation leads to the following inequalities for \( \mu \in Q \):

\[
q(\mu, xy) \leq q(\mu, x) q(\mu, y), \quad x, y \in I,
\]

(1) \[
\int_0^1 q(\mu, x)(\lambda_1 \lambda_2 \ldots \lambda_n) (dx) \leq \prod_{j=1}^n \int_0^1 q(\mu, x) \lambda_j (dx)
\]

for any \( \lambda_1, \lambda_2, \ldots, \lambda_n \in P_I \);

(2) \[
(\lambda \mu)(E) \leq \mu(E) \int_0^1 q(\mu, x) \lambda (dx) \quad \text{for} \quad \lambda \in P_I.
\]

We note that the set \( Q \) is closed under convolution and convex combinations. Moreover,

\[
q(\mu * v, x) \leq q(\mu, x) q(v, x)
\]

and

\[
q(c \mu + (1-c) v, x) \leq \max(q(\mu, x), q(v, x)).
\]

As a simple example of measures belonging to \( Q \) we quote the Gaussian measure \( q \) with the mean \( m \) and the variance \( \sigma^2 \). Then we have

\[
q(\mu, x) = x^{-1} \exp \left( \frac{m^2 (1 - x)}{2 \sigma^2 (1 + x)} \right), \quad x \in I.
\]

Setting for any \( b > 0 \)

\[
\mu_b(E) = b \int_{E \cap I} x^{b-1} dx,
\]

we have also \( \mu_b \in Q \) and \( q(\mu_b, x) = x^{-b} (x \in I) \). Furthermore, it is easy to check that all unimodal distributions with the mode at \( 0 \) belong to \( Q \).

**Theorem.** Let \( \mu \in Q \). For every \( \lambda \in P_I \) satisfying the condition

(3) \[
\int_0^1 q(\mu, x) \lambda (dx) < 2 \lambda (\{1\})
\]

there exists a measure \( v \in P \), absolutely continuous with respect to \( \mu \), such that \( \lambda v = \mu \).

**Proof.** The measure \( \lambda \) can be written in the form

\[
\lambda = p \delta_1 + (1-p) \eta,
\]

where \( p = \lambda (\{1\}) \), \( \eta \in P_I \) and \( \eta (\{1\}) = 0 \). In the case \( p = 1 \) we have \( \lambda = \delta_1 \).
and our assertion is obvious with \( v = \mu \). Suppose that \( p < 1 \). Since \( q(\mu, x) \geq 1 \), we have by (3) the inequality \( p > 1/2 \). Consequently,

\[
0 < r = \frac{1-p}{p} < 1
\]

and

\[
s = r \left( \int_0^1 q(\mu, x) \eta(dx) \right) < 1.
\]

Further, inequalities (1) and (2) yield

\[
(\eta^n \mu)(E) \leq \left( \int_0^1 q(\mu, x) \eta(dx) \right)^n \mu(E), \quad n = 1, 2, \ldots
\]

Setting

\[
\beta = (1-r)^{-1} (\mu - r\eta \mu)
\]

and taking into account (4) and (5) we infer that

\[
\beta(E) = (1-r)^{-1} (\mu(E) - r \left( \int_0^1 \mu(x^{-1} E) \eta(dx) \right))
\]

\[
\geq (1-r)^{-1} \mu(E) \left( 1 - s \int_0^1 q(\mu, x) \eta(dx) \right) \geq 0.
\]

Since \( \beta \) is normed on the real line, we conclude that \( \beta \in P \). Put

\[
v = (1-r^2) \sum_{k=0}^{\infty} r^{2k} \eta^{2k} \beta,
\]

where \( \eta^0 = \delta_1 \). Obviously, \( v \in P \) and, by (7),

\[
v = (1+r) \sum_{n=0}^{\infty} (-1)^n r^n \eta^n \mu.
\]

Consequently, by (5) and (6),

\[
v(E) \leq (1+r) \sum_{n=0}^{\infty} r^n (\eta^n \mu)(E) \leq \frac{1+r}{1-s} \mu(E),
\]
which shows that \( v \) is absolutely continuous with respect to \( \mu \). Further, by (4) and (8),

\[
\eta v = \frac{1+r}{r} \mu - \frac{1}{r} v = \frac{1}{1-p} \frac{1}{1-p} v.
\]

Thus

\[
\lambda v = p(\delta_{1} v) + (1-p)(\eta v) = pv + (1-p)(\eta v) = \mu,
\]

which completes the proof.

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