TOPOLOGY OF THE CONVERGENCE IN PROBABILITY ON A LINEAR SPAN OF A SEQUENCE OF INDEPENDENT RANDOM VARIABLES

BY

K. PIETRUSKA-PALUŚ AND W. SMOLEŃSKI (WARSAW)

Abstract. Let \( X_1, X_2, \ldots \) be a sequence of independent symmetric Hilbert space valued non-degenerated random variables and let \( L_X \) denote the closed linear span of \( \{X_n\} \) in \( L_0(\Omega, \mathcal{F}, P; H) \). If \( L_X \) is a locally convex subspace of \( L_0 \), then \( L_X \) is Banach iff \( L_X \) does not contain an isomorphic copy of \( \mathbb{R}^\infty \) iff

\[
\sup_n P(X_n = 0) < 1.
\]

If, moreover, \( X_n \) are equidistributed and \( P(X_n = 0) = 0 \), then

\[
\left\{ Y \in L_X : P\left( \|Y\| > \frac{1}{201}\right) < \frac{1}{201} \right\}
\]

is a bounded neighbourhood of zero.

In this note we will investigate the topology of the convergence in probability for random variables of the form \( \sum a_n X_n \), \( n = 1, 2, \ldots \), where \( a_n \) are real numbers, \( \{X_n\} \) is a fixed sequence of independent symmetric non-degenerated Hilbert space valued random variables and the series converges in probability. We denote the linear space of random variables of this form by \( L_X \). It is easy to see that \( L_X \) endowed with the topology \( \tau_P \) of the convergence in probability is a complete separable linear-metric space.

**Theorem 1.** If \( (L_X, \tau_P) \) is locally convex, then the following conditions are equivalent:

(i) \( (L_X, \tau_P) \) is a Banach space;

(ii) \( L_X \) does not contain a subspace isomorphic to \( \mathbb{R}^\infty \);

(iii) \( \sup P(X_n = 0) < 1 \).

Before proving Theorem 1, we will introduce some notation and prove some lemmas. We use "\( : = \)" as "equal by definition".
For \( n = 1, 2, \ldots \) and \( t \in \mathbb{R} \) we have \( Q_n(t) := E \min(1, ||tX_n||^2) \). It is easy to see that \( Q_n(0) = 0 \),

\[
\lim_{t \to \infty} Q_n(t) = 1 - P(X_n = 0),
\]

\( Q_n(t) = Q_n(-t) \) and, for \( t_1 \geq t_2 \geq 0 \), \( Q_n(t_1) \geq Q_n(t_2) \).

For \( \varepsilon > 0 \)

\[
U_\varepsilon := \{ Y \in L_X : Y = \sum a_n X_n \text{ and } \sum Q_n(a_n) < \varepsilon \},
\]

\[
V_\varepsilon := \{ Y \in L_X : P(||Y|| > \varepsilon) < \varepsilon \}.
\]

**Lemma 1.** \( \varepsilon U_\varepsilon \subset V_{2\varepsilon} \subset U_{400\varepsilon} \) for \( 0 < \varepsilon < 1/400 \).

**Proof.** The inclusions follow directly from the following beautiful estimates [4]:

1. if \( 0 < \varepsilon < 1/200 \) and \( P(||\sum a_n X_n|| > \varepsilon) < \varepsilon \), then \( \sum Q_n(a_n) < 200\varepsilon \);
2. \( P(||\sum a_n X_n|| > \varepsilon) < 2\sum Q_n(a_n \varepsilon^{-1}) \) for every \( \varepsilon > 0 \).

**Remark.** Propositions (1) and (2) are stated in [4] under the assumption that \( X_1, X_2, \ldots \) are equidistributed real random variables. But those assumptions are not used in the proof, which can be rewritten (with obvious changes) in the Hilbert space case.

**Lemma 2.** If \( \text{conv } U_\varepsilon \subset U_\eta \) for some \( 0 < \varepsilon < 1 - \sup P(X_n = 0) \) and \( \eta > 0 \), then

\[
\forall \varepsilon > 0 \exists r = r(\varepsilon) > \forall n \in \mathbb{N} \forall t \in \mathbb{R} \quad Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \delta.
\]

**Proof.** Let us assume that the implication is false. Then for some \( \delta > 0 \) there exist sequences \( (n_k) \) and \( (t_k) \) of positive integers such that

\[
Q_{n_k}(t_k) < \varepsilon \quad \text{and} \quad Q_{n_k}(\frac{t_k}{k}) \geq \delta.
\]

Since \( \delta < 1 - \sup P(X_n = 0) \), we have

\[
\forall n \in \mathbb{N} \exists t_n > 0 \forall t > t_n \quad Q_n(t) > \varepsilon.
\]

Thus the boundedness of \( (n_k) \) would entail the boundedness of \( (t_k) \). But for \( (n_k) \) and \( (t_k) \) bounded we would have

\[
\lim_{k \to \infty} Q_{n_k}(\frac{t_k}{k}) = 0.
\]

Hence we can assume that \( (n_k) \) is strictly increasing.
Consider the following sequence of elements of $\text{conv } U_e$:

\[
Y_1 = t_1 X_{n_1}, \\
Y_2 = \frac{1}{2} t_2 X_{n_2} + \frac{1}{2} t_3 X_{n_3}, \\
\ldots \ldots \ldots \ldots \ldots \\
Y_m = \sum_{k=m}^{2m-1} \frac{1}{m} t_k X_{n_k}.
\]

It is clear that

\[
\sum_{k=m}^{2m-1} Q_n(t_k/m) \geq \sum_{k=m}^{2m-1} Q_n(t_k/k) \geq m\delta.
\]

This contradicts the assumption of the lemma that $Y_m$ belongs to $U_e$.

**Lemma 3.** Let $\varepsilon, \lambda > 0$ and let $Z = \sum b_n X_n$, $n = 1, 2, \ldots$, be an element of $U_\varepsilon$. If $Q_n(b_n) < \lambda$ for every $n$, then $\lambda Z/(\lambda + \varepsilon)$ is an element of $\text{conv } U_\lambda$.

**Proof.** Since $Q_n(b_n) < \lambda$, there exist positive integers $M$ and $1 = n_0 < n_1 < n_2 < \ldots < n_M$ such that

\[
\sum_{n=1}^{n_1-1} Q_n(b_n) = \lambda_1 < \lambda \quad \text{and} \quad Q_{n_1}(b_{n_1}) \geq \lambda - \lambda_1, \\
\sum_{n=n_1}^{n_2-1} Q_n(b_n) = \lambda_2 < \lambda \quad \text{and} \quad Q_{n_2}(b_{n_2}) \geq \lambda - \lambda_2, \\
\ldots \ldots \ldots \ldots \ldots \\
\sum_{n=n_M-1}^{n_M-1} Q_n(b_n) = \lambda_M < \lambda \quad \text{and} \quad Q_{n_M}(b_{n_M}) \geq \lambda - \lambda_M, \\
\sum_{n=n_M}^{\infty} Q_n(b_n) < \lambda.
\]

Consequently, random variables

\[
Z_k = \sum_{n=n_k-1}^{n_k-1} b_n X_n \quad (k = 1, 2, \ldots, M) \quad \text{and} \quad Z_{M+1} = \sum_{n=n_M}^{\infty} b_n X_n
\]

are elements of $U_\lambda$ such that $Z_1 + Z_2 + \ldots + Z_M + Z_{M+1} = Z$.

Obviously $M+1 \leq \varepsilon/\lambda + 1$. Thus $\lambda Z/(\lambda + \varepsilon) \in U_\lambda$. 

then \((L_X, \tau_F)\) is isomorphic to \(R^\infty\).

Proof. We have to prove that:

(a) for every sequence of real numbers \((a_n)\) the series \(\sum a_n X_n,\)

(b) the sequence

\[
\left( \sum_{n=1}^\infty a_{nk} X_n \right)_{k=1}^\infty
\]

de elements of \(L_X\) converges to zero in probability iff

\[
\lim_{k \to \infty} a_{nk} = 0 \quad \text{for every } n.
\]

Both (a) and (b) follow immediately from the Borel-Cantelli Lemma.

Proof of the Theorem 1. (i) \(\Rightarrow\) (ii) is obvious.

\(~(iii) \Rightarrow ~(ii).\) Let \((n_k)\) be an increasing sequence of positive integers such that \(P(X_{n_k} = 0) > 1 - 1/2^k\). By Lemma 4, the closed linear span of \((X_{n_k})\)

is isomorphic to \(R^\infty\).

(iii) \(\Rightarrow\) (i). It is enough to prove the existence of a bounded neighborhood of zero. Thus, by Lemma 1, it is enough to show that

\[
\exists_{\varepsilon > 0} \forall_{\eta > 0} \exists_{s > 0} \quad s U_\varepsilon \subset U_\eta.
\]

Let us take \(\delta > 0\). Local convexity of \((L_X, \tau_F)\) and Lemma 1 imply the existence of an \(\varepsilon > 0\) such that \(\operatorname{conv} U_\varepsilon \subset U_\delta\). We can assume that \(\varepsilon < 1 - \sup P(X_n = 0)\).

Let us fix an \(\eta > 0\) and let us take a \(\lambda > 0\) such that \(\operatorname{conv} U_\lambda \subset U_{\eta/2}\). By Lemma 2 there exists an \(r = r(\eta/2\varepsilon)\) such that

\[
\forall_{n \in N} \forall_{t \in \mathbb{R}} \quad Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \frac{\eta \lambda}{2\varepsilon}.
\]

We claim that

\[
(\ast) \quad s U_\varepsilon \subset U_\eta \quad \text{for } s = \min \left( \frac{1}{F}, \frac{\lambda}{\lambda + \varepsilon} \right).
\]

Let \(Y = \sum_{n=1}^\infty a_n X_n\) be an element of \(U_\varepsilon\). Let \(N_\lambda = \{n \in N : Q_n(a_n) \geq \lambda\}\). Since \(Q_n(a_n) < \varepsilon\), we have \(Q_n(ra_n) < \eta \lambda/2\varepsilon\). Obviously \(\text{card } N_\lambda \leq \varepsilon/\lambda\). Hence

\[
\sum_{n \in N_\lambda} Q_n(ra_n) < \frac{\eta \lambda}{2}\varepsilon.
\]
On the other hand, by Lemma 3, we have

\[ \frac{\lambda}{\lambda + \varepsilon} \sum_{n \in \mathbb{N}} a_n X_n \in \text{conv } U_\varepsilon \subset U_{\eta/2}. \]

Thus

\[ \sum_{n=1}^{\infty} Q_n(sa_n) < \eta, \]

q.e.d.

As a corollary we get

**Theorem 2.** If \( X_1, X_2, \ldots \) are equidistributed and \((L_X, \tau_\rho)\) is locally convex, then

(a) \( E\|X_1\|^p < \infty \), for every \( 0 < p < 1 \)

(b) if, moreover, \( P(X_1 = 0) = 0 \), then

\[ \left\{ Y \in L_X : P\left( \|Y\| > \frac{1}{201} \right) < \frac{1}{201} \right\} \]

is a bounded neighbourhood of zero in \((L_X, \tau_\rho)\).

**Proof.** (a) From Theorem 1 we know that \((L_X, \tau_\rho)\) is a Banach space. Thus, by a theorem of Nikishin ([5], Theorem 1)(1) there exists an \( A \in \mathcal{F} \), \( P(A) \geq \frac{1}{2} \), such that \( E\|X_n\|^p X_A \leq c_p \). Since \( X_n \) are equidistributed and independent, it follows that \( E\|X_1\|^p < \infty \) for every \( 0 < p < 1 \).

(b) In view of Lemma 1 it is enough to prove that

\[ \forall \eta > 0 \exists \varepsilon > 0 \quad sU_\varepsilon \subset U_\eta, \quad \text{where } \varepsilon = \frac{200}{201}. \]

Let us fix \( \eta > 0 \) and let us take \( \lambda > 0 \) such that \( \text{conv } U_\lambda \subset U_{\eta/2} \). Since \( Q_1 = Q_2 = \ldots \) and \( \lim_{t \to \infty} Q_1(t) = 1 \), there exists an \( r > 0 \) such that

\[ Q_n(t) < \varepsilon \Rightarrow Q_n(rt) < \frac{\eta \lambda}{2 \varepsilon}. \]

Now we can rewrite the part of the previous proof starting from (*).

**Remarks.** The case of \( H = \mathbb{R} \) and \( X_1, X_2, \ldots \) equidistributed symmetric random variables is better known.

1. It is proved in [1] that, for equidistributed real symmetric random variables, “locally convex” and “Banach” is the same for \((L_X, \tau_\rho)\) (see also [2] for a survey of results).

(1) It is stated for \( H = \mathbb{R} \) and \( \Omega = [0, 1] \) but, again, the proof can be just re-written to get what we want.
2. The case of $X_1$, $X_2$, ... real symmetric equidistributed, with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$, shows that $\{ Y \in L_X : P(\| Y \| > \frac{1}{2}) < \frac{1}{2} \}$ is not, in general, a bounded neighbourhood for a locally convex $\tau_p$. However, in this real case $\frac{1}{2} - \varepsilon$ works for every $\varepsilon > 0$. The last statement follows from the following estimate (obtained from Inequality II, p. 6, in [3] and from [6]): for every $0 < \lambda < \frac{1}{2}$, if $P(\| \sum a_n X_n \| > \varepsilon) < \lambda$, then

$$\sqrt{\sum a_n^2} < 4 \frac{\varepsilon}{1 - 2\lambda}.$$ 

3. For every $1 \leq p < 2$ there exists a sequence $X_1$, $X_2$, ... of equidistributed symmetric independent real r.v.'s such that $E|X|^p < \infty$, but $(L_X, \tau_p)$ is not locally convex.(2)

Indeed, let $(l_i)$ be an increasing sequence of positive integers such that

\[(*) \quad \sum_{i=1}^{\infty} l_i \left( \frac{l_{i-1}}{l_i} \right)^{2/p} i^{2/p} < \infty, \quad l_0 = 1 \]

(e.g. $l_i = 2^{l_{i-1}+1}$, $c > p/(2-p)$).

We put $a_i = (l_{i-1}/l_i)^2 i^2$, $i = 1, 2, \ldots$, then

\[(**) \quad \sum_{i=1}^{\infty} l_i a_i^{1/p} < \infty. \]

Let $g_1$, $g_2$, ... be a sequence of independent symmetric random variables with distribution

$$P(g_i = l_i) = P(g_i = -l_i) = a_i = \frac{1}{2} - \frac{1}{2} P(g_i = 0)$$

and let $(g_{ij})_{i=1}^{\infty}$, $(g_{i2})_{i=1}^{\infty}$, ... be independent copies of the sequence $(g_i)_{i=1}^{\infty}$. We put

$$X_j = \sum_{i=1}^{\infty} g_{ij}.$$ 

It follows from $(**)$ that $E|X_j|^p < \infty$.

Let

$$A_i = \frac{i}{l_i}, \quad k_i = \frac{1}{a_i} i = \left( \frac{l_{i-1}}{l_i} \right)^{-2} i^{-3}.$$ 

(2) We owe this remark to S. Kwapień.
For $0 < \delta < 1$ we have
\[
P\left(\left|\sum_{j=1}^{k_i} X_j\right| > \delta\right) \leq P\left(\left|\sum_{j=1}^{k_i} g_{s_j}\right| > \frac{\delta}{2}\right) +
\quad + P\left(\left|\sum_{j=1}^{k_i} g_{s_j}\right| > \frac{\delta}{2}\right) = I + II,
\]
\[
I \leq \sum_{j=1}^{k_i} \sum_{s=1}^{\infty} P(\{g_{s,j} \neq 0\}) = k_i \sum_{s=1}^{\infty} a_s \leq 2Mk_i a_i = \frac{2M}{i} \to 0
\]

\((\ast)\) implies that \(\sum_{s=1}^{\infty} a_s \leq Ma_i\) for some constant \(M\),
\[
II \leq \left(\frac{2}{\delta}\right)^2 E\left|\sum_{j=1}^{k_i} g_{s,j}\right|^2 = \frac{4}{\delta^2} A_i^2 k_i \left(\sum_{s=1}^{\infty} l_s^2 s^2\right)
\quad = \frac{4}{\delta^2} A_i^2 k_i \left(\sum_{s=1}^{\infty} l_s^2 s^2\right) \leq \frac{4}{\delta^2} M_1 A_i^2 k_i l_{i-1}^2 = \frac{4M_1}{\delta^2 i} \to 0
\]

\((\ast)\) implies that \(\sum_{s=1}^{\infty} l_s^2 s^2 \leq M_1 l_{i-1}^2\) for some constant \(M_1\).

Thus for every $0 < \delta < 1$ there exists an $i$ such that
\[
P\left(\left|\sum_{j=1}^{k_i} X_j\right| > \delta\right) < \delta.
\]

On the other hand, for every $i$ we have
\[
P\left(\frac{1}{i} \left|\sum_{j=1}^{k_i} X_j\right| + A_i \sum_{j=k_i+1}^{2k_i} X_j + \ldots + A_i \sum_{j=(i-1)k_i+1}^{ik_i} X_j\right) \geq 1/5
\]
\[
= P\left(\frac{A_i}{i} \left|\sum_{j=1}^{k_i} X_j\right| \geq 1/5\right) \geq \frac{1}{2} P\left(\frac{A_i}{i} \left|\sum_{j=1}^{k_i} g_{s,j}\right| \geq 1/5\right)
\]
\[
\geq \frac{1}{4} P\left(\max_{1 \leq j \leq ik_i} \frac{A_i}{i} g_{s,j} \geq 1/5\right) = \frac{1}{4} (1-(1-2a_i)^{ik_i})
\]
\[
\geq \frac{1}{4} (1-e^{-2a_i k_i}) = \frac{1}{4} (1-e^{-2}) \geq \frac{1}{5},
\]
which shows that \((L_X, \tau_p)\) is not locally convex.

4. In this case we can give a simple sufficient condition to have \((L_X, \tau_p)\) locally convex, namely, for $t > t_0$, \(tP(|X_1| > t)\) is decreasing.

It can be obtained by the calculating derivative of \(Q(x)/x\). This condition
is sufficient for $Q(x)/x$ to be decreasing in some small neighbourhood of zero, so that $Q$ can be replaced by an equivalent convex function $Q_1$.

Acknowledgement. We would like to thank S. Kwapięń for his helpful advice.

REFERENCES


Department of Mathematics
Warsaw University
PKiN, 9p.
00-901 Warsaw, Poland

Institute of Mathematics
Warsaw Technical University
Pl. Jedności Robot. 1
00-661 Warsaw, Poland

Received on 9. 12. 1986