PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 10, Fasc. 1 (1989), p. 27–44

# FIRST HITTING TIMES AND POSITIONS OF CONCENTRIC SPHERES FOR TESTING THE DRIFT OF A DIFFUSION PROCESS

BY

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*Abstract.* Consider $X_t$ a diffusion process on $\boldsymbol{R}^m$, $m \geqslant 2$, with drift vector $\theta b(u)$ depending of an unknown real parameter $\theta$ with small known variance matrix $\varepsilon\sigma(u)$. The aim of this paper is testing $\theta = \theta_0$ vs $\theta > \theta_0$ with $\theta_0 \geqslant 0$ from the observation of the first hitting times and positions of concentric spheres centered at $x = X_0$ with radii $r \leqslant R$ for given $R$. We obtain the asymptotic behaviour of this process as $\varepsilon \to 0$ when the trajectory of the corresponding dynamical system leaves any sphere centered at $x$ within finite time. We then construct a test on $\theta$ and study its asymptotic properties by means of contiguity. When $\theta_0 > 0$, the test is locally asymptotically most powerful (LAMP). We also consider a test based on the first hitting times of spheres only.

Drift estimation for one-dimensional diffusion processes for which only the first hitting times of increasing levels are observed has been investigated in [4]. In this paper, we consider drift testing for an $m$-dimensional diffusion process $(X_t)_{t \geqslant 0}$ based on the observation of the first hitting times and positions of concentric spheres centered at $X_0$. The *diffusion* $(X_t)$ is defined as the solution of the stochastic differential equation

$$dX_t = \theta b(X_t)\,dt + \varepsilon\sigma(X_t)\,dW_t, \qquad X_0 = x,$$

where $(W_t)$ is a standard $m$-dimensional Brownian motion ($m \geqslant 2$), $\theta$ a real unknown parameter, $x \in \boldsymbol{R}^m$, $\varepsilon > 0$; the $m$-vector field $b(u)$ and the $(m \times m)$-matrix field $\sigma(u)$ are known.

Let $T_r = \inf\{t \geqslant 0;\ |X_t - x| = r\}$ be the first hitting time of the sphere $S(x, r)$ with center $x$ and radius $r$. From the observation $(X_{T_r}, T_r)_{r \leqslant R}$, for given $R > 0$, we study the testing problem $H_0$: $\theta = \theta_0$ vs $H_1$: $\theta > \theta_0$ with $\theta_0 \geqslant 0$ and asymptotic framework $\varepsilon \to 0$.

Let $(x_\theta(t))$ be the solution of the deterministic system corresponding to $\varepsilon = 0$. Under the (main) assumption that, for $\theta > 0$, $x_\theta(t)$ leaves any sphere centered at $x$ within finite time, we obtain a convergence in distribution theorem as $\varepsilon \to 0$ for the process $(X_{T_r}, T_r)$ to a Gaussian process (after suitable centering and normalization). For $\theta = 0$, the asymptotic behaviour of the observed process is not Gaussian. We then construct a test $\tilde{\Phi}_\varepsilon$ based on this observation and study its asymptotic properties as $\varepsilon \to 0$ by means of contiguity [7]. For $\theta_0 > 0$, the contiguous alternative is $\theta_0 + \varepsilon z$ and $\tilde{\Phi}_\varepsilon$ is locally asymptotically most powerful (LAMP). For $\theta_0 = 0$, $\tilde{\Phi}_\varepsilon$ is not LAMP but the contiguous alternative becomes $\varepsilon^2 z$. We also study a test $\bar{\Phi}_\varepsilon$ based on the observation of the hitting times $(T_r)_{r \leqslant R}$ only.

In Section 1 we consider the diffusion $X$ solution of

$$dX_t = b(\varepsilon, X_t)\, dt + \varepsilon \sigma(X_t)\, dW_t, \qquad X_0 = x.$$

The parameter $\theta$ is not introduced in this section. The main assumption is that the solution $x(t)$ of the deterministic equation corresponding to $\varepsilon = 0$ satisfies the inequality $(x(t) - x) b(0, x(t)) > 0$ for all $t > 0$. Then, the function $n(t) = |x(t) - x|$ being increasing, one can define its inverse function $t(r)$ for $0 \leqslant r < n(+\infty) = N$. In Theorem 1 and Corollary 1 we show that

$$\big(\varepsilon^{-1}(X_{T_r} - x t(r)),\ \varepsilon^{-1}(T_r - t(r))\big)_{0 \leqslant r < N}$$

converges in distribution as $\varepsilon \to 0$ to a continuous Gaussian proces. In Corollary 2 we obtain that, for smooth $\varphi$,

$$D_R(\varphi) = \int_0^{T_r} \varphi(X_s)\, ds - \int_{[0,R)} \varphi(X_{T_r})\, dT_r$$

satisfies $\varepsilon^{-1} D_R(\varphi) \equiv o_p(1)$. For $b \equiv 0$ the law of $(X_{T_r}, \varepsilon^2 T_r)_{r \geqslant 0}$ is independent of $\varepsilon$ (Proposition 1).

In Section 2 we study the statistical model of diffusion with drift $b(\varepsilon, u) = \theta b(u)$. The law of the diffusion is denoted by $P_\theta^\varepsilon$. We assume that the drift vector $b$ has the form $b = e\nabla V$, where $e = \sigma({}^t\sigma)$ and $\nabla V$ is the gradient vector of a function $V$: $\boldsymbol{R}^m \to \boldsymbol{R}$ such that $V(x) = 0$. In Theorem 2 we show that, for $\theta_0 > 0$ and $z > 0$, $\theta_\varepsilon = \theta_0 + \varepsilon z$, the distributions $(P_{\theta_0}^\varepsilon)$ and $(P_{\theta_\varepsilon}^\varepsilon)$ stopped at $T_R$ are contiguous and that $(X_{T_r}, T_r)_{r \leqslant R}$ is asymptotically sufficient for $\theta_0$. When $\theta_0 = 0$, the contiguous alternative is $\theta_\varepsilon = \varepsilon^2 z$. We then consider the test $\tilde{\Phi}_\varepsilon$ based on the statistic (estimator of $\theta$)

$$\tilde{\theta}_\varepsilon = V(X_{T_R}) / \int_{[0,R)} v(X_{T_r})\, dT_r \qquad \text{with } v = {}^t\nabla V e \nabla V.$$

The asymptotic properties of $\tilde{\Phi}_\varepsilon$ are stated in Corollaries 3 and 4. For $\theta_0 > 0$ this test is LAMP. We also construct a test $\bar{\Phi}_\varepsilon$ based on the first hitting times $(T_r)_{r \leqslant R}$ only, whose asymptotic properties are given in Proposition 2. Some examples are considered in the last section.

## 1. ASYMPTOTICS OF THE FIRST HITTING TIMES AND POSITIONS OF CONCENTRIC SPHERES

**1.1. Framework.** Let $(W_t)_{t \geqslant 0}$ be a standard $m$-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$, adapted to a right-continuous filtration $(\mathscr{F}_t)_{t \geqslant 0}$. We consider the diffusion $X^\varepsilon$ solution on $\Omega$ of the stochastic differential equation (s.d.e.),

$$(1)\qquad \begin{aligned} dX_t^\varepsilon &= b(\varepsilon, X_t^\varepsilon)\,dt + \varepsilon\sigma(X_t^\varepsilon)\,dW_t, \\ X_0^\varepsilon &= x, \qquad x \in \boldsymbol{R}^m, \end{aligned}$$

where the $m$-vector field $b(\varepsilon, u)$ and the $(m \times m)$-matrix field $\sigma(u)$ satisfy the following conditions:

(H1) $b\colon [0, +\infty) \times \boldsymbol{R}^m \to \boldsymbol{R}^m$ is $C^2$ as a function of $(\varepsilon, u)$,
$\sigma\colon \boldsymbol{R}^m \to \boldsymbol{R}^m \otimes \boldsymbol{R}^m$ is $C^2$.

(H2) For all $u$, $\sigma(u)$ is invertible.

(H3) There exists a positive constant $K$ such that, for all $u \in \boldsymbol{R}^m$ and $\varepsilon \geqslant 0$,

$$|b(\varepsilon, u)|^2 + |\sigma(u)|^2 \leqslant K(1 + |u|^2)$$

($|\cdot|$ denotes the usual Euclidian norm).

In matrix products, m-vectors are identified to the column-matrix of their components and $\cdot$ denotes the usual inner product. For $r \geqslant 0$ let us define

$$(2)\qquad T_r(X^\varepsilon) = T_r^\varepsilon = \inf\{t \geqslant 0;\ |X_t^\varepsilon - x| = r\}.$$

Under (H1) and (H3), $X^\varepsilon$ is a Markov process with continuous sample paths uniquely determined on $[0, \infty)$, $\boldsymbol{P}(T_r^\varepsilon < \infty) = 1$ for all $r \geqslant 0$ and $\boldsymbol{P}(T_0^\varepsilon = T_{0+}^\varepsilon = 0) = 1$, where $T_{0+}^\varepsilon = \lim_{r \downarrow 0} T_r^\varepsilon$, (see e.g. [6]).

Let $x(t)$ and $n(t)$ be defined by

$$(3)\qquad dx(t) = b\big(0, x(t)\big)\,dt, \qquad x(0) = x$$

and

$$(4)\qquad n(t) = |x(t) - x|.$$

The following conditions will be needed:

(H4) $\forall_{t>0}\quad \big(x(t) - x\big)\cdot b\big(0, x(t)\big) > 0.$

(H5) $n'(t) = \big(x(t) - x\big)\cdot b\big(0, x(t)\big)/n(t)$ has a positive limit when $t \to 0$.

Under (H4), the trajectory $x(t)$ will leave any sphere centered at $x$ within finite time, and $n(t)$ being increasing, one may define its inverse function

$$t(r) = n^{-1}(r), \quad 0 \leqslant r < n(+\infty) = N, \tag{5}$$

which is $C^1$ on $(0, N)$. Under the additional assumption (H5), $t$ will be $C^1$ on $[0, N)$ (see § 2.4, examples).

In what follows, we shall use the stochastic Taylor expansion of $X^\varepsilon$ which is available under (H1) up to order two (see [1]) and is recorded hereafter.

THEOREM A. *Under* (H1) *and* (H3) *there exist a continuous Gaussian process* $(g(t))_{t\geqslant 0}$ *and processes* $R_i^\varepsilon(t)$, $i = 1, 2$, *such that, for all* $t \geqslant 0$,

$$X_t^\varepsilon = x(t) + \varepsilon R_1^\varepsilon(t)$$

$$X_t^\varepsilon = x(t) + \varepsilon g(t) + \varepsilon^2 R_2^\varepsilon(t).$$

$$\lim_{\substack{\varepsilon \to 0 \\ k \to +\infty}} \boldsymbol{P}\big(\sup_{s \leqslant t} |R_i^\varepsilon(s)| \geqslant k\big) = 0, \quad i = 1, 2.$$

The Gaussian process $(g(t))$ is defined on $\Omega$ by

$$\begin{aligned} dg(t) &= \sigma\big(x(t)\big)\, dW_t + \Big(b^{(1)}\big(0, x(t)\big) g(t) + \frac{\partial b}{\partial \varepsilon}\big(0, x(t)\big)\Big) dt \\ g(0) &= 0, \end{aligned} \tag{6}$$

where, for $u = {}^t(u^1, \ldots, u^m)$ in $\boldsymbol{R}^m$, $b^{(1)}(0, u)$ is the following linear mapping:

$$b^{(1)}(0, u)\, y = \sum_{i=1}^{m} \frac{\partial b}{\partial u^i}(0, u)\, y^i, \quad y = {}^t(y^1, \ldots, y^m).$$

If $Q_t$ is the $m \times m$ invertible matrix such that

$$dQ_t = -Q_t b^{(1)}\big(0, x(t)\big)\, dt, \quad Q_0 = I, \tag{7}$$

then the solution of (6) is given by

$$g(t) = Q_t^{-1}\Big(\int_0^t Q_s \frac{\partial b}{\partial \varepsilon}\big(0, x(s)\big)\, ds + \int_0^t Q_s \sigma\big(x(s)\big)\, dW_s\Big). \tag{8}$$

We may now define the one- and $m$-dimensional Gaussian processes:

$$\begin{aligned} G(r) &= -\big(x(t(r)) - x\big) \cdot g\big(t(r)\big) / \big(x(t(r)) - x\big) \cdot b\big(0, x(t(r))\big), \quad r > 0 \\ G(0) &= 0, \end{aligned} \tag{9}$$

$$H(r) = g\big(t(r)\big) + G(r)\, b\big(0, x(t(r))\big). \tag{10}$$

Under (H1)-(H5) these processes are continuous on $[0, N)$.

**1.2. Limit theorem for the process $(X^{\varepsilon}_{T^{\varepsilon}_r}, T^{\varepsilon}_r)_{0 \leqslant r < N}$.**

THEOREM 1. *For all $h > 0$ and $R_0$, $R$ satisfying $0 < R_0 < R < N$, we have under* (H1)-(H4),

(i) $$\lim_{\varepsilon \to 0} \boldsymbol{P}\Big( \sup_{R_0 \leqslant r \leqslant R} |\varepsilon^{-1}(T^{\varepsilon}_r - t(r)) - G(r)| > h\Big) = 0,$$

(ii) $$\lim_{\varepsilon \to 0} \boldsymbol{P}\Big( \sup_{R_0 \leqslant r \leqslant R} |\varepsilon^{-1}(X^{\varepsilon}_{T^{\varepsilon}_r} - x(t(r))) - H(r)| > h\Big) = 0.$$

We first prove

LEMMA 1. *Under* (H1)-(H4), *for all $h > 0$ and $R \in [0, N[$,*

$$\lim_{\varepsilon \to 0} \boldsymbol{P}\Big( \sup_{0 \leqslant r \leqslant R} |T^{\varepsilon}_r - t(r)| > h\Big) = 0.$$

To simplify notation, let us omit all superscripts $\varepsilon$.

Proof. Let $R \in [0,N[$ and $h$, $h_1$, $T > 0$ such that $t(R+h_1) < T$. Then

$$A(h) = \{\sup_{t \leqslant T} |X_t - x(t)| < h\}$$

is included in

$$\{ \sup_{0 \leqslant r \leqslant R} |T_r - t(r)| \leqslant \omega(h)\},$$

where

$$\omega(h) = \sup \{|\mathrm{t}(\mathrm{r}') - \mathrm{t}(\mathrm{r}'')|;\ |r' - r''| \leqslant 2h,\ 0 \leqslant r', r'' \leqslant R + h_1\}.$$

From the continuity of $(t(r))$, fix $\eta > 0$ and $h > 0$ such that $\omega(h) < \eta$. Lemma 1 then follows from Theorem A.

Proof of Theorem 1.

(i) From Theorem A, for $r \geqslant 0$, we have

(11) $$X_{T_r} - x = x(T_r) - x + \varepsilon g(T_r) + \varepsilon^2 R_2(T_r)$$

and (see (4) and (5))

(12) $$\varepsilon^{-1}(r - n(T_r)) = r^{-1}(x(t(r)) - x) \cdot g(t(r)) + Y_r + \varphi_r$$

with

(13) $$Y_r = 2(r + n(T_r))^{-1}(x(T_r) - x) \cdot g(T_r) - r^{-1}(x(t(r)) - x) \cdot g(t(r)))$$

and

(14) $$\varphi_r = (r + n(T_r))^{-1}(\varepsilon |g(T_r)|^2 + \varepsilon R_2(T_r) \cdot (2(x(T_r) - x) + 2\varepsilon g(T_r) + \varepsilon^2 R_2(T_r)).$$

An application of Taylor's formula yields

(15) $$T_r - t(r) = (n(T_r) - r)t'(r) + \tfrac{1}{2}(n(T_r) - r)^2 t''(r^*) \quad \text{with } r^* \in (r, n(T_r)).$$

Thus using (4), (5) and (9), we obtain

(16) $$\varepsilon^{-1}(T_r - t(r)) = G(r) + \varrho_1(r)$$

with

$$\varrho_1(r) = -t'(r)(\varphi_r + Y_r) + \varepsilon^{-1}\big(n(T_r) - r\big)^2 t''(r^*)/2. \tag{17}$$

Fix $R_0$, $R$ such that $0 < R_0 < R < N$. We now check that $\varrho_1(r)$ is uniformly $o_p(1)$ on $[R_0, R]$ as $\varepsilon \to 0$.

Because of Lemma 1, $\big(r + n(T_r)\big)^{-1}$ converges uniformly on $[R_0, R]$ to $(2r)^{-1}$ in probability. Let $T > 0$ be such that $t(R) < T$ in order to ensure

$$\lim_{\varepsilon \to 0} P(T_R < T) = 1.$$

On $(T_R < T)$, $\sup_{R_0 \leqslant r \leqslant R} |\varphi_r|$ is bounded from above by a random variable which is $o_p(1)$ in view of (14) and Theorem A. Thus $\sup_{R_0 \leqslant r \leqslant R} |\varphi_r| = o_p(1)$.

To see that $Y_r$ is also uniformly $o_p(1)$ on $[R_0, R]$, it remains to show that $\sup_{r \leqslant R} \big|Z(T_r) - Z\big(t(r)\big)\big|$, with $Z(t) = \big(x(t) - x\big) \cdot g(t)$, is $o_p(1)$. This is a straightforward consequence of Lemma 1 and of the continuity of the process $\big(Z(t)\big)$. So, in view of (12),

$$\sup_{R_0 \leqslant r \leqslant r} \varepsilon^{-1}\big(n(T_r) - r\big)^2 = o_p(1).$$

Now to see that $\sup_{R_0 \leqslant r \leqslant R} |t''(r^*)|$ is bounded in probability, choose $k > 0$ such that $0 < t(R_0 - k)$ and again $t(R) < T$. On $C = \{t(R_0 - k) < T_{R_0}, T_R < T\}$, $r^*$ remains in $[R_0 - k, n(T)]$ and $\lim_{\varepsilon \to 0} P(C) = 1$. Thus

$$\sup_{R_0 \leqslant r \leqslant R} |\varrho_1(r)| = o_p(1),$$

which (see (16) and (17)) achieves the proof of (i).

(ii) Formula (11) and a Taylor expansion for $x(T_r) - x\big(t(r)\big)$ yield that

$$\varepsilon^{-1}\big(X_{T_r} - x\big(t(r)\big)\big) = H(r) + \varrho_2(r),$$

where

$$\varrho_2(r) = b\big(0, x\big(t(r)\big)\big)\varrho_1(r) + \frac{\varepsilon}{2} x''(t_r^*)\big((T_r - t(r))/\varepsilon\big)^2 + \\ + g(T_r) - g\big(t(r)\big) + \varepsilon R_2(T_r),$$

and $t_r^* \in \big(t(r), T_r\big)$.

On $(T_R < T)$, $t_r^* \in [0, T]$. So we proceed as in (i) to get

$$\sup_{R_0 \leqslant r \leqslant R} |\varrho_2(r)| = o_p(1)$$

and (ii).

Remark 1. Even when a higher order expansion of $X^\varepsilon$ in powers of $\varepsilon$ is available (e.g. if $b$, $\sigma$ are $C^k$, $k > 2$), it is not possible to improve the expansion of $(T_r^\varepsilon, X_{T_r^\varepsilon}^\varepsilon)$ to within $o(\varepsilon^2)$ becuase $g(t)$ is not differentiable ([2], p. 59).

Remark 2. A useful consequence of Lemma 1 and Theorem A is that, for any continuous $m$-vector field $\psi$,

$$\int_0^{T_R^\varepsilon} \psi(X_s^\varepsilon)\cdot dW_s \xrightarrow[\varepsilon\to 0]{P} \int_0^{t(R)} \psi(x(s))\cdot dW_s,$$

which can be checked by the classical Lenglart inequalities.

The following two corollaries of Theorem 1 are the basement of the statistical study of Section 2.

Corollary 1. *Under* (H1)-(H4) *and the additional assumption* (H5), *the result of Theorem* 1 *remains true with* $R_0 = 0$.

Proof. Under (H5), the processes $(G(r))$ and $(H(r))$ are right-continuous and nul at 0. Since this is also true in probability for $T_r$, Corollary 1 follows.

Corollary 2. *Let* $\varphi\colon \mathbf{R}^m \to \mathbf{R}$ *be* $C^2$. *For* $R \in [0, N[$, *let*

$$D_R^\varepsilon(\varphi) = \int_0^{T_R^\varepsilon} \varphi(X_s^\varepsilon)\,ds - \int_{[0,R)} \varphi(X_{T_r^\varepsilon}^\varepsilon)\,dT_r^\varepsilon,$$

*where the previous integral is a stochastic integral with respect to the increasing left-continuous process* $(T_r^\varepsilon)$. *Under* (H1)-(H5), $\varepsilon^{-1} D_R^\varepsilon(\varphi) \to 0$ *in probability as* $\varepsilon \to 0$.

Let us fix $R \in [0, N[$, and omit the superscripts $\varepsilon$ for the following proofs. First we prove

Lemma 2. *Assume* (H1)-(H5).

(i) *Let* $(f(r, \omega))_{0 \le r \le R}$ *be a random continuous function adapted to* $(\mathcal{F}_{t(r)})_{0 \le r \le R}$. *Then*

$$\int_{[0,R)} f(r)\,dT_r^\varepsilon \xrightarrow[\varepsilon\to 0]{} \int_0^R f(r)\,dt(r) \text{ in probability.}$$

(ii) *If* $f$ *is* $C^1$, *then*

$$\varepsilon^{-1} \int_{[0,R)} f(r)\big(dT_r^\varepsilon - dt(r)\big) \xrightarrow[\varepsilon\to 0]{} \int_0^R f(r)\,dG(r) \text{ in probability,}$$

*where the above limit is a stochastic integral with respect to the continuous semi-martingale* $(G(r))$.

Proof. (i) Consider

$$f_n(r)=\sum_{k=0}^{[2^nR]} f(2^{-n}k)\,\mathbb{1}_{(2^{-n}k\leqslant r<2^{-n}(k+1)\wedge R)}.$$

By Lemma 1

$$\int_{[0,R)} f_n(r)\,dT_r \xrightarrow[\varepsilon\to 0]{} \int_0^R f_n(r)\,dt(r) \text{ in probability.}$$

Now, on $(T_R<T)$ with $t(R)<T$,

$$\Big|\int_{[0,R)} f_n(r)\big(dT_r-dt(r)\big)\Big| \leqslant 2T\omega(f,2^{-n})+\Big|\int_{[0,R)} f_n(r)\big(dT_r-dt(r)\big)\Big|,$$

where $\omega(f,\delta)=\sup\{|f(r)-f(r')|;\ |r-r'|\leqslant\delta,\ 0\leqslant r,\ r'\leqslant R\}$.

Result (i) follows from the continuity of $f$ and Lemma 1.

(ii) Since $f$ is $C^1$, by Theorem 1 and Corollary 1,

$$\varepsilon^{-1}\int_{[0,R)} f(r)\big(dT_r-dt(r)\big)=\varepsilon^{-1}\Big(f(R)\big(T_R-t(R)\big)-\int_0^R f'(r)\big(T_r-t(r)\big)\,dr\Big)$$

converges in probability to

$$f(R)\,G(R)-\int_0^R f'(r)\,G(r)\,dr=\int_0^R f(r)\,dG(r)$$

because the integration by parts formula is also valid for the semi-martingale $G(r)$.

Proof of Corollary 2. The random variable $D_R(\varphi)$ may be written as $D_R(\varphi)=A+B+C$ with

$$A=\int_{[0,R)} \varphi(X_{t(r)})\big(dt(r)-dT_r\big),$$

$$B=\int_{[0,R)} \big(\varphi(X_{t(r)})-\varphi(X_{T_r})\big)\,dT_r,\qquad C=\int_{t(R)}^{T_R} \varphi(X_s)\,ds.$$

An application of Taylor's formula to $\varphi(X_{t(r)})-\varphi\big(x(t(r))\big)$ and $\varphi(X_{T_r})-\varphi\big(x(t(r))\big)$ yields, by (10),

$$\varepsilon^{-1}A=-\int_0^R \varphi\big(x(t(r))\big)\,dG(r)+o_p(1) \tag{18}$$

and

$$\varepsilon^{-1}B=-\int_0^R \nabla\varphi\big(x(t(r))\big)\cdot G(r)\,b\big(0,\,x(t(r))\big)\,dt(r)+o_p(1), \tag{19}$$

when $\nabla\varphi$ is the gradient vector of $\varphi$. Also

(20) $$\varepsilon^{-1} C = \varphi\big(x(t(R))\big) G(R) + o_p(1).$$

In the three above equalities we have used Theorem A, Theorem 1 and Corollary 1, and Lemma 2 to see that the remainder terms are $o_p(1)$. Now,

$$d\varphi\big(x(t(r))\big) = \nabla\varphi\big(x(t(r))\big)\cdot b\big(0,\, x(t(r))\big)\, dt(r)$$

and

$$\varphi\big(x(t(R))\big) G(R) = \int_0^R \varphi\big(x(t(r))\big)\, dG(r) + \int_0^R G(r)\, d\varphi\big(x(t(r))\big).$$

together with (18)-(20), achieve the proof of Corollary 2.

Thus under (H1)-(H5) we have obtained an asymptotically Gaussian behaviour with rate $\varepsilon$ for the process $(X^{\varepsilon}_{T^{\varepsilon}_r}, T^{\varepsilon}_r)$ and the main consequence of this result is, in view of Corollary 2, that the whole information carried (on the drift vector $b$) by the observation $(X^{\varepsilon}_s,\, s \leqslant T^{\varepsilon}_R)$ will be contained in $(X^{\varepsilon}_{T^{\varepsilon}_r}, T^{\varepsilon}_r)_{0\leqslant r\leqslant R}$ as is seen in Section 2. For the purpose of testing $b \equiv 0$ from the observation $(X^{\varepsilon}_{T^{\varepsilon}_r}, T^{\varepsilon}_r)$, we also need to specify its behaviour under this hypothesis.

PROPOSITION 1. *Let* $b \equiv 0$ *in* (1) *and assume* (H1)-(H3).

*The distribution of the process* $(X^{\varepsilon}_{T^{\varepsilon}_r}, \varepsilon^2 T^{\varepsilon}_r)_{r\geqslant 0}$ *is independent of* $\varepsilon$. (*This law is on the space of left-continuous with right-hand limits function on* $[0, +\infty)$, *taking values in* $\boldsymbol{R}^m \times [0, +\infty)$ *endowed with the Skorokhod Borel* $\sigma$-*algebra*).

Proof. The process $B^{\varepsilon}_t = \varepsilon W_{\varepsilon^{-2}t}$ is a standard Brownian motion and $Y^{\varepsilon}_t = X^{\varepsilon}_{\varepsilon^{-2}t}$ satisfies

$$Y^{\varepsilon}_t = x + \int_0^t \sigma(Y^{\varepsilon}_s)\, dB^{\varepsilon}_s.$$

Thus the law of $Y^{\varepsilon}$ does not depend on $\varepsilon$. Since

$$\varepsilon^2 T_r(X^{\varepsilon}) = \inf\{t \geqslant 0;\, |Y^{\varepsilon}_t - x| = r\} = T_r(Y^{\varepsilon})$$

and $X^{\varepsilon}_{T_r(X^{\varepsilon})} = Y^{\varepsilon}_{T_r(Y^{\varepsilon})}$, we obtain the result of Proposition 1.

## 2. CONTIGUITY PROPERTIES AND APPLICATIONS TO DRIFT TESTING

**2.1. Assumptions and notations.** We now assume that the drift $b(\varepsilon, u) = \theta b(u)$ does not depend on $\varepsilon$ and depends on an unknown linear parameter $\theta \in [0, +\infty)$. Let $\big(C, \mathscr{C}, (\mathscr{C}_t)_{t\geqslant 0}, (X_t)_{t\geqslant 0}, P^{\varepsilon}_{\theta}\big)$ be the canonical diffusion solution of the s.d.e. (1) with drift $\theta b(u)$, where $C = C(R^+, R^m)$, $(X_t)$ are the canonical coordinates of $C$,

$$\mathscr{C}_t = \bigcap_{s>t} \mathscr{C}_t^0, \quad \mathscr{C}_t^0 = \sigma(X_s, s \leqslant t), \quad \mathscr{C} = \bigvee_{t \geqslant 0} \mathscr{C}_t.$$

Let $T_r(X) = T_r$, $(x_\theta(t))$ be the solution of $x_\theta'(t) = \theta b(x_\theta(t))$, $x_\theta(0) = x$ and $n_\theta(t) = |x_\theta(t) - x|$.

We assume

(K1) For all $\theta > 0$, the functions $\theta b(u)$, $\sigma(u)$ and $x_\theta(t)$ satisfy (H1)-(H5).

(K2) $b = e\nabla V$, where $e = \sigma({}^t\sigma)$, $V$: $R^m \to R$ is $C^3$ and $V(x) = 0$.

Clearly,

$$x_\theta(t) = x(\theta t), \qquad n_\theta(t) = n(\theta t), \tag{21}$$

where $x(t) = x_1(t)$ and $n(t) = n_1(t)$ correspond to $\theta = 1$.

Thus, for $\theta > 0$,

$$n_\theta(+\infty) = n(+\infty) = N, \qquad t_\theta(r) = \theta^{-1} t(r) \tag{22}$$

with $t_\theta = n_\theta^{-1}$ $t = n^{-1}$, $r < N$ and $x_\theta(t_\theta(r)) = x(t(r))$.

Let $(G_\theta(r))$ and $(H_\theta(r))$ be the processes defined in (9) and (10), associated to the drift $\theta b(u)$ $(\theta > 0)$. They are continuous centered (because $\partial b / \partial \varepsilon \equiv 0$) Gaussian processes and the covariance function of $(G_\theta(r))$ has, in view of (6) -(9), the following form:

$$\mathrm{Cov}(G_\theta(r), G_\theta(r')) = \theta^{-3} \gamma(r, r').$$

From Section 1, under $P_\theta^\varepsilon$, $\theta > 0$,

$$\sup_{r \leqslant R} |T_r - \theta^{-1} t(r)| \xrightarrow[\varepsilon \to 0]{} 0, \qquad 0 \leqslant R < N,$$

and

$$(\varepsilon^{-1}(T_r - \theta^{-1} t(r)), \varepsilon^{-1}(X_{T_r} - x(t(r)))) \xrightarrow[\varepsilon \to 0]{\mathscr{D}} (G_\theta(r), H_\theta(r))$$

in the Skorokhod space $D([0, N[)$.

Let us define:

$$\begin{aligned} \alpha(r) &= V(x(t(r))), \qquad r < N, \\ v(u) &= {}^t\nabla V(u) e(u) \nabla V(u), \qquad u \in R^m. \end{aligned} \tag{23}$$

(Note that $\alpha'(r) = v(x(t(r))) t'(r)$).

**2.2. Testing $\theta$ from the observation $(X_{T_r}, T_r)_{0 \leqslant r \leqslant R}$.** For given $R \in [0, N[$, the first hitting times and positions of the spheres $S(x, r)$ with $r \leqslant R$ are observed. We are concerned with the testing problem:

$$H_0\colon \theta = \theta_0 \text{ vs } H_1\colon \theta > \theta_0 \text{ with } \theta_0 \geqslant 0.$$

Under (K1)-(K2), for all $\theta \geqslant 0$, the distributions $P^\varepsilon_\theta/\mathscr{C}_{T_R}$ and $P^\varepsilon_0/\mathscr{C}_{T_R}$ are equivalent and the likelihood of $(X_t)_{t \leqslant T_R}$ is given by

(24) $$(dP^\varepsilon_\theta/dP^\varepsilon_0)/\mathscr{C}_{T_R} = L_{T_R}(\theta)$$

with

(25) $$L_{T_R}(\theta) = \exp\left[\varepsilon^{-2}\left(\theta \int_0^{T_R} \nabla V(X_s)\cdot dX_s - \frac{\theta^2}{2}\int_0^{T_R} v(X_s)\,ds\right)\right].$$

We set:

(26) $$l(\theta, \theta_0) = \log\left(L_{T_R}(\theta)/L_{T_R}(\theta_0)\right),$$

(27) $$\tilde{\Delta}_R(\theta_0) = \varepsilon^{-1}\left(V(X_{T_R}) - \theta_0 \int_{[0,R)} v(X_{T_r})\,dT_r\right).$$

THEOREM 2. *Assume* (K1)-(K2).

(i) *For* $\theta_0 > 0$, $z > 0$, $\theta_\varepsilon = \theta_0 + \varepsilon z$, *under* $P^\varepsilon_{\theta_0}$, *as* $\varepsilon \to 0$, *we have*

(28) $$l(\theta_\varepsilon, \theta_0) = z\tilde{\Delta}_R(\theta_0) - \theta_0^{-1}\alpha(R)z^2/2 + o_p(1)$$

*with*

(29) $$\tilde{\Delta}_R(\theta_0) \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \theta_0^{-1}\alpha(R)\right).$$

*So, the distributions* $(P^\varepsilon_{\theta_0})$ *and* $(P^\varepsilon_{\theta_\varepsilon})$, *considered on* $\mathscr{C}_{T_R}$, *are contiguous as* $\varepsilon \to 0$.

(ii) *For* $\theta_0 = 0$, $z > 0$, $\theta_\varepsilon = \varepsilon^2 z$, *under* $P^\varepsilon_0$, *the distribution of* $l(\theta_\varepsilon, 0)$ *is independent of* $\varepsilon$. *The distributions* $(P^\varepsilon_0)$ *and* $(P^\varepsilon_{\theta_\varepsilon})$, *considered on* $\mathscr{C}_{T_R}$, *are contiguous as* $\varepsilon \to 0$.

Proof. (i) Let

(30) $$\Delta_R(\theta_0) = \varepsilon^{-1}\int_0^{T_R} \nabla V(X_s)\cdot\left(dX_s - \theta_0\, e(X_s)\nabla V(X_s)\,ds\right).$$

From (25)-(26) we get

(31) $$l(\theta_\varepsilon, \theta_0) = z\Delta_R(\theta_0) - (z^2/2)\int_0^{T_R} v(X_s)\,ds.$$

An application of Ito's formula yields

(32) $$\Delta_R(\theta_0) = \tilde{\Delta}_R(\theta_0) + \theta_0\varepsilon^{-1}\left(\int_{[0,R)} v(X_{T_r})\,dT_r - \int_0^{T_R} v(X_s)\,ds\right) + \varepsilon\int_0^{T_R} h(X_s)\,ds,$$

where

$$h(u) = \frac{1}{2}\sum_{1 \leqslant i,j \leqslant m} \frac{\partial^2 V}{\partial u^i\,\partial u^j}(u)\,e_{ij}(u).$$

By Theorem 1 and its corollaries we get

(33) $$\Delta_R(\theta_0)-\tilde{\Delta}_R(\theta_0)=o_p(1) \text{ under } P^\varepsilon_{\theta_0} \quad \text{as } \varepsilon \to 0.$$

By remark 2 (at the end of Theorem 1),

$$\Delta_R(\theta_0) \xrightarrow[\varepsilon\to 0]{P^\varepsilon_{\theta_0}} \int_0^{\theta_0^{-1}t(R)} \nabla V(x(\theta_0 s))\cdot\sigma(x(\theta_0 s))\,dW_s,$$

which is a centered Gaussian variable with variance (see (23))

$$\int_0^{\theta_0^{-1}t(R)} |{}^t\sigma(x(\theta_0 s))\nabla V(x(\theta_0 s))|^2\,ds = \theta_0^{-1}\alpha(R).$$

Moreover, $\int_0^{T_R} v(X_s)\,ds$ and $\int_{[0,R)} v(X_{T_r})\,dT_r$ converge in $P^\varepsilon_{\theta_0}$-probability to

(34) $$\int_0^{\theta_0^{-1}t(R)} v(x(\theta_0 s))\,ds = \theta_0^{-1}\int_0^R v(x(t(r)))\,dt(r) = \theta_0^{-1}\alpha(R).$$

In view of (30)-(32) we obtain the first part of (i). The contiguity follows from the fact that

$$l(\theta_\varepsilon, \theta_0) \xrightarrow[\varepsilon\to 0]{\mathscr{D}} \mathscr{N}(-\sigma^2 z^2/2, \sigma^2 z^2)$$

with $\sigma^2=\theta_0^{-1}\alpha(R)$ under $P^\varepsilon_{\theta_0}$ (see [7], chap. 1).

(ii) When $\theta_\varepsilon=\varepsilon^2 z$, we have

(35) $$l(\theta_\varepsilon, 0) = z\Big(V(X_{T_R})-\varepsilon^2\int_0^{T_R} h(X_s)\,ds\Big)-(z^2\varepsilon^2/2)\int_0^{T_R} v(X_s)\,ds$$
$$= z\Big(V(Y^\varepsilon_{\tau^\varepsilon_R})-\int_0^{\tau^\varepsilon_R} h(Y^\varepsilon_s)\,ds\Big)-(z^2/2)\int_0^{\tau^\varepsilon_R} v(Y^\varepsilon_s)\,ds,$$

where $Y^\varepsilon_t = X_{\varepsilon^{-2}t}$, $\tau^\varepsilon_R = T_R(Y^\varepsilon)=\varepsilon^2 T_R(X)$.

Under $P^\varepsilon_0$, the law of $Y^\varepsilon$ is independent of $\varepsilon$ (see the proof of Proposition 1), which yields (ii).

Theorem 2 leads us to consider the following estimator $\tilde{\theta}_\varepsilon$ of $\theta$ and the test of level $a$, $0 \leqslant a \leqslant 1$, based on $\tilde{\theta}_\varepsilon$:

(36) $$\tilde{\theta}_\varepsilon = V(X_{T_R})\Big/\int_{[0,R)} v(X_{T_r})\,dT_r,$$

(37) $$\tilde{\Phi}_\varepsilon = \mathbb{1}_{(\tilde{\theta}_\varepsilon > \tilde{c}_\varepsilon(a,\theta_0))}+\tilde{\gamma}_\varepsilon(a,\theta_0)\,\mathbb{1}_{(\tilde{\theta}_\varepsilon = \tilde{c}_\varepsilon(a,\theta_0))},$$

where $\tilde{c}_\varepsilon(a,\theta_0)$ and $\tilde{\gamma}_\varepsilon(a,\theta_0)$ are determined by the equality $E^\varepsilon_{\theta_0}\tilde{\Phi}_\varepsilon = a$. (We denote by $\mathscr{N}(x)$ the distribution function of the normal law $\mathscr{N}(0, 1)$).

COROLLARY 3. *Let* $\theta_0 > 0$, $z > 0$, $\theta_\varepsilon = \theta_0 + \varepsilon z$ and assume (K1)-(K2). *For testing* $\theta_0$ *vs* $\theta > \theta_0$, $\tilde{\Phi}_\varepsilon$ *is locally asymptotically most powerful* (LAMP) *of level* $a$, *i.e., for any other* $\mathscr{C}_{T_R}$*-measurable test function* $\Phi_\varepsilon$ *of level* $a$,

$$\lim_{\varepsilon \to 0} \inf_{\theta_0 < \theta < \theta_0 + \varepsilon z} E_\theta^\varepsilon \tilde{\Phi}_\varepsilon - E_\theta^\varepsilon \Phi_\varepsilon \geqslant 0$$

(see e.g. [5], Def. 1.4.1, p. 17).

*Moreover,*

(38) $$\tilde{c}_\varepsilon(a, \theta_0) = \theta_0 + \varepsilon\big(\theta_0 \alpha(R)^{-1}\big)^{1/2} \mathcal{N}^{-1}(a) + o(\varepsilon)$$

*and*

(39) $$\lim_{\varepsilon \to 0} E_{\theta_\varepsilon}^\varepsilon \tilde{\Phi}_\varepsilon = \mathcal{N}\big(z(\theta_0^{-1} \alpha(R))^{1/2} + \mathcal{N}^{-1}(a)\big).$$

Proof. From (34) and (36) we infer that

(40) $$\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0) = \theta_0 \alpha(R)^{-1} \tilde{\Delta}_R(\theta_0) + o_p(1) \quad \text{under } P_{\theta_0}^\varepsilon.$$

This equality together with Theorem 2 (i) yield that $\tilde{\Phi}_\varepsilon$ is LAMP according to Theorem 1.4.1, p. 18, of [5]. It also implies that, under $P_{\theta_0}^\varepsilon$,

$$\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0) \xrightarrow[\varepsilon \to 0]{\mathscr{D}} \mathcal{N}\big(0, \theta_0 \alpha(R)^{-1}\big).$$

Thus $\big(\tilde{c}_\varepsilon(a, \theta_0) - \theta_0\big)\varepsilon^{-1}\big(\alpha(R)/\theta_0\big)^{-1/2}$ must converge to $\mathcal{N}^{-1}(a)$ because $\tilde{\Phi}_\varepsilon$ has the level $a$, whereas $P_{\theta_0}^\varepsilon\big(\tilde{\theta}_\varepsilon = \tilde{c}_\varepsilon(a, \theta_0)\big) \to 0$ as $\varepsilon \to 0$, yielding (38).

Using (28) and (40), we get that $\big(l(\theta_\varepsilon, \theta_0), \varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0)\big)$ converges under $P_{\theta_0}^\varepsilon$ to the degenerate two-dimensional Gaussian law

$$\mathcal{N}_2\left(\begin{pmatrix} -z^2 \theta_0^{-1} \alpha(R)/2 \\ 0 \end{pmatrix}, \begin{pmatrix} z^2 \theta_0^{-1} \alpha(R) & z \\ z & \theta_0/\alpha(R) \end{pmatrix}\right).$$

By the contiguity, it follows that

$$\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \theta_0) \xrightarrow[\varepsilon \to 0]{\mathscr{D}} \mathcal{N}\big(z, \theta_0/\alpha(R)\big)$$

under the contiguous alternative $P_{\theta_\varepsilon}^\varepsilon$ (see [7], Chap. 1, Theorem 7.2) which by (38) leads to (39).

Remark 3. The locally asymptotically normal representation (28) of the loglikelihood ratio shows that the observation $(X_{T_r}, T_r)_{r \leqslant R}$ is asymptotically sufficient for $\theta_0$ when $\theta_0 > 0$. By Corollary 2, this is also true for general drift $b(\theta, u)$ depending on an unknown parameter $\theta \in \boldsymbol{R}^k$ if $b(\theta, u)$ satisfies (H1)-(H5) and smoothness assumptions with respect to $(\theta, u)$.

COROLLARY 4. *Let* $\theta_0 = 0$, $z > 0$, $\theta_\varepsilon = \varepsilon^2 z$.

*The distribution of* $\varepsilon^{-2}\tilde{\theta}_\varepsilon$ *under* $P_0^\varepsilon$ *and* $P_{\theta_\varepsilon}^\varepsilon$ *is independent of* $\varepsilon$.

Proof. It is a consequence of Proposition 1 and Theorem 2 (ii).

Thus, $\tilde{\Phi}_\varepsilon$ can be used for testing $\theta = 0$ vs $\theta > 0$, with $\tilde{c}_\varepsilon(a, 0) = \varepsilon^2 c(a)$, $\tilde{\gamma}_\varepsilon(a, 0) = \gamma(a)$ and the power $E^\varepsilon_{\theta_\varepsilon} \tilde{\Phi}_\varepsilon$ at the contiguous alternative $\theta_\varepsilon = \varepsilon^2 z$ is independent of $\varepsilon$. The optimality properties of $\tilde{\Phi}_\varepsilon$ are lost, but the separating rate $\varepsilon^2$ of $H_0$ and $H_1$ is improved.

**2.3. Testing $\theta$ from the observation $(T_r)_{0 \leq r \leq R}$.** From (22), the limit $x(t(r))$ of $X_{T_r}$ is independent of the unknown $\theta$. Replacing $X_{T_r}$ by its limit in (36), we define

$$\bar{\theta}_\varepsilon = \alpha(R) / \int_{[0,R)} v(x(t(r)))\, dT_r \tag{41}$$

and the test of $\theta_0$ vs $\theta > \theta_0$, based on $\bar{\theta}_\varepsilon$,

$$\bar{\Phi}_\varepsilon = \mathbb{1}_{(\bar{\theta}_\varepsilon > \bar{c}_\varepsilon(a,\theta_0))} + \bar{\gamma}_\varepsilon(a, \theta_0)\, \mathbb{1}_{(\bar{\theta}_\varepsilon = \bar{c}_\varepsilon(a,\theta_0))} \text{ with } E^\varepsilon_{\theta_0} \bar{\Phi}_\varepsilon = a.$$

Proposition 2. *Assume* (K1)-(K2).

(i) *Let* $\theta_0 > 0$, $z > 0$ *and* $\theta_\varepsilon = \theta_0 + \varepsilon z$. *There exists a* $J(R) > 0$, $\varrho(R) \in [-1, 1]$, *not depending on* $\theta_0$ *such that*

$$\bar{c}_\varepsilon(a, \theta_0) = \theta_0 + \varepsilon(\theta_0 J(R))^{1/2} + o(\varepsilon) \tag{42}$$

and

$$\lim_{\varepsilon \to 0} E^\varepsilon_{\theta_\varepsilon} \bar{\Phi}_\varepsilon = \mathcal{N}\left[z\varrho(R)(\theta_0^{-1}\alpha(R))^{1/2} + \mathcal{N}^{-1}(a)\right]. \tag{43}$$

*The test* $\bar{\Phi}_\varepsilon$ *is LAMP iff* $\varrho(R) = 1$.

(ii) *Let* $\theta_0 = 0$, $z > 0$, $\theta_\varepsilon = \varepsilon^2 z$. *The distribution of* $\varepsilon^{-2}\bar{\theta}_\varepsilon$ *under* $P^\varepsilon_0$ *and* $P^\varepsilon_{\theta_\varepsilon}$ *is independent of* $\varepsilon$.

Proof. Let

$$\bar{\Delta}_R(\theta_0) = \varepsilon^{-1}\theta_0 \int_{[0,R)} v(x(t(r)))(\theta_0^{-1}\, dt(r) - dT_r). \tag{44}$$

By Theorem 1, its Corollaries and Lemma 2, we have (see (30)), under $P^\varepsilon_{\theta_0}$,

$$(\Delta_R(\theta_0), \bar{\Delta}_R(\theta_0)) \xrightarrow[\varepsilon \to 0]{\mathcal{D}} \Big( \int_0^{\theta_0^{-1} t(R)} \nabla V(x(\theta_0 s)) \cdot \sigma(x(\theta_0 s))\, dW_s - \tag{45}$$

$$- \theta_0 \int_0^R v(x(t(r)))\, dG_{\theta_0}(r) \Big),$$

where $(G_{\theta_0}(r))$ is the limiting process of $(T_r - \theta_0^{-1} t(r))$. From (41) and the fact that

$$\int_{[0,R)} v(x(t(r)))\, dT_r \xrightarrow[\varepsilon \to 0]{} \theta_0^{-1}\alpha(R)$$

under $P^{\varepsilon}_{\theta_0}$ (see (23) and Lemma 2), we get

(46) $$\varepsilon^{-1}(\bar{\theta}_{\varepsilon}-\theta_0)=\theta_0\,\alpha(R)^{-1}\,\bar{\Delta}_R(\theta_0)+o_p(1).$$

We have already obtained (see (28) and (33))

(47) $$l(\theta_{\varepsilon},\theta_0)=z\Delta_R(\theta_0)-\theta_0^{-1}\alpha(R)z^2/2+o_p(1).$$

Formulae (45)-(47) yield that $\left(l(\theta_{\varepsilon},\theta_0)+\theta_0^{-1}\alpha(R)z^2/2,\ \varepsilon^{-1}(\bar{\theta}_{\varepsilon}-\theta_0)\right)$ converges in distribution to a centered Gaussian vector with covariance matrix

$$\begin{pmatrix} z^2\theta_0^{-1}\alpha(R) & zC(R)\\ zC(R) & \theta_0 J(R)\end{pmatrix},$$

where

(48) $$J(R)=\alpha(R)^{-2}\operatorname{Var}\left(\int_0^R v\big(x(t(r))\big)\,d\big(\theta_0^{3/2}G_{\theta_0}(r)\big)\right)$$

and

(49) $$C(R)=\alpha(R)^{-1}\operatorname{Cov}\Big(\sqrt{\theta_0}\int_0^{\theta_0^{-1}t(R)}\nabla V\big(x(\theta_0 s)\big)\cdot\sigma\big(x(\theta_0 s)\big)\,dW_s-$$
$$-\int_0^R v\big(x(t(r))\big)\,d\big(\theta_0^{3/2}G_{\theta_0}(r)\big)\Big).$$

Introducing the standard Brownian motion $B_t^{\theta_0}=\sqrt{\theta_0}\,W_{\theta_0^{-1}t}$, we deduce from (7)-(9), with $b(\varepsilon,u)$ replaced by $b(u)$,

(50) $$\theta_0^{3/2}G_{\theta_0}(r)=-\zeta(r)\cdot Q_{t(r)}^{-1}\int_0^{t(r)}Q_u\,\sigma\big(x(u)\big)\,dB_u^{\theta_0},$$

where $Q_t$ satisfies $dQ_t=-Q_t\,b^{(1)}\big(x(t)\big)\,dt$, $Q_0=I$, and the first random variable appearing in $C(R)$ is equal to

(51) $$\int_0^{t(R)}\nabla V\big(x(u)\big)\cdot\sigma\big(x(u)\big)\,dB_u^{\theta_0}.$$

So $J(R)$ and $C(R)$ do not depend on $\theta_0$. Let

$$\varrho(R)=C(R)/\big(J(R)\,\alpha(R)\big)^{1/2}$$

be the limiting correlation coefficient. Since

$$\varepsilon^{-1}(\bar{\theta}_{\varepsilon}-\theta_0)\xrightarrow[\varepsilon\to 0]{\mathscr{D}}\mathcal{N}[0,\theta_0 J(R)]$$

under $P^{\varepsilon}_{\theta_0}$ and $E^{\varepsilon}_{\theta_0}\Phi_{\varepsilon}=a$, we get (42). The previous joint convergence in distribution yields

$$\varepsilon^{-1}(\bar{\theta}_\varepsilon-\theta_0)\xrightarrow[\varepsilon\to 0]{\mathcal{D}}\mathcal{N}\big(z\varrho(R)(J(R)\alpha(R))^{1/2},\ \theta_0 J(R)\big)$$

under the contiguous alternative $P^\varepsilon_{\theta_\varepsilon}$ ([7], Theorem 7.2, Chap. 1) from which (43) is obtained, together with the fact that $\bar{\Phi}_\varepsilon$ is LAMP iff $\varrho(R)=1$. (ii) is a consequence of Proposition 1 and Theorem 2 (ii).

Remark 4. The limiting variance and correlation coefficient $J(R)$ and $\varrho(R)$ can be calculated using definitions (6)-(9) and formulae (48)-(51), but no simple expressions are available unless $\varrho(R)=1$. The limiting distribution of $\varepsilon^{-1}(\bar{\theta}_\varepsilon-\theta_0)$ under $P^\varepsilon_{\theta_\varepsilon}$, $\theta_\varepsilon=\theta_0+\varepsilon z$, can also be obtained by Theorem 1 with $b(\varepsilon, u)=\theta_\varepsilon b(u)$.

**2.4. Examples.**

**2.4.1.** *Brownian motion with drift.*

The model $X_t=\theta\vec{u}t+\varepsilon W_t$ stopped at $T_R$ has been studied in [3]. In this case, the last observation $(X_{T_R}, T_R)$ is (exactly) sufficient and has an explicitly known distribution. The test based on $\bar{\theta}_\varepsilon=R/T_R$ is LAMP.

**2.4.2.** *Linear drift.*

For the model, $dX^i_t=\theta X^i_t dt+\varepsilon dW^i_t$, $X^i_0=x^i$, $i=1,\dots,m$, ${}^t x={}^t(x^1,\dots,x^m)\neq 0$, $\theta>0$, we have:

$$x(t)=x\exp(t),$$

$$t(r)=\log(1+|x|^{-1}r),\quad 0\leqslant r<+\infty=N,$$

$$x(t(r))=x(1+|x|^{-1}r),$$

$$g_\theta(t)=\int_0^t\exp(\theta(t-s))\,dW_s.$$

Thus, $G_\theta(r)=Z_\theta(\theta^{-1}t(r))$ with

$$Z_\theta(t)=-\theta^{-1}x\int_0^t\exp(-\theta s)\,dW_s/|x|^2.$$

The statistics $\tilde{\theta}_\varepsilon$ and $\bar{\theta}_\varepsilon$ are given by:

$$\tilde{\theta}_\varepsilon=\frac{1}{2}(|X_{T_R}|^2-|x|^2)/\int_{[0,R)}|X_{T_r}|^2\,dT_r,$$

$$\bar{\theta}_\varepsilon=\frac{1}{2}((|x|+R)^2-|x|^2)/\int_{[0,R)}(|x|+r)^2\,dT_r.$$

In this case, as $(G_\theta(r))$ is a Gaussian martingale, easy computations yield

$\varepsilon^{-1}(\tilde{\theta}_\varepsilon - \bar{\theta}_\varepsilon) = o_p(1)$ under $P^\varepsilon_{\theta_0}$, $\theta_0 > 0$ as $\varepsilon \to 0$. Both tests $\tilde{\Phi}_\varepsilon$ and $\bar{\Phi}_\varepsilon$ are LAMP.

The model $dX^i_t = -\theta X^i_t dt + \varepsilon dW_t$, $X^i_0 = x^i$, $i = 1, \ldots, m$ ${}^t x = {}^t(x^1, \ldots, x^m) \neq 0$, $\theta > 0$ leads to:

$$x(t) = x\exp(-t),$$

$$t(r) = -\theta^{-1}\log(1-|x|^{-1}r) \quad \text{for } 0 \leqslant r < |x| = N.$$

For $R < |x|$, the tests $\tilde{\Phi}_\varepsilon$ and $\bar{\Phi}_\varepsilon$ based on the observation $(X_{T_r}, T_r)_{r \leqslant R}$ or $(T_r)_{r \leqslant R}$ are also LAMP.

**2.4.3.** *Bilinear diffusion.*

Consider:

$$dX^i_t = \theta X^i_t dt + \varepsilon X^i_t dW^i_t, \; X^i_0 = x^i, \quad i = 1, \ldots, m, \; \theta > 0.$$

When $x^i > 0$, $i = 1, \ldots, m$, $X^i_t > 0$ for all $t \geqslant 0$ a.s. for $i = 1, \ldots, m$. Thus

$$\sigma(X_t) = \mathrm{diag}(X^i_t, i = 1, \ldots, m)$$

is a.s. invertible and we can define

$$V(X_t) = \sum_{i=1}^{m} \log(X^i_t/x^i).$$

We obtain:

$$G_\theta(r) = Z_\theta(\theta^{-1}t(r)) \text{ with } Z_\theta(t) = -\theta^{-1}|x|^{-2}\sum_{i=1}^{m}(x^i)^2 W^i_t,$$

$$\tilde{\theta}_\varepsilon = \sum_{i=1}^{m} \log(X^i_{T_R}/x^i)/mT_R,$$

$$\bar{\theta}_\varepsilon = \log(1+|x|^{-1}R)/T_R \quad \text{and} \quad \varrho(R) = |x|^2/\Big\{\sum_{i=1}^{m}(x^i)^4\Big\}^{1/2}.$$

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*Received on 17. 2. 1987*