A FUNCTIONAL CALCULUS BASED ON FEYNMAN–KAC FORMULA

BY
ANDRZEJ HULANICKI (WROCLAW)

Abstract. It is proved that if
\[ Hf = \int_{-\infty}^{\infty} \lambda E(\lambda) f \]
is a spectral resolution of a Schrödinger operator \( H = -A + V \) on \( \mathbb{R}^d \) with \( V \in K_{\mathrm{loc}}, V(x) \geq 0 \) and \( V(x) \geq C|x|^\alpha \) for some \( \alpha > 0 \) and \( |x| \geq c \), then there exists an \( N \) such that if \( K \in C_0^N \), then the operator
\[ \int_{-\infty}^{\infty} K(\lambda) dE(\lambda) \]
is bounded on \( L^p(\mathbb{R}^d), 1 \leq p < \infty \).

Let \( H \) be a self-adjoint (unbounded) operator on \( L^2(\mathcal{M}) \), where \( \mathcal{M} \) is a measure space. We write its spectral resolution
\[ Hf = \int_{-\infty}^{+\infty} \lambda dE(\lambda) f. \]

As we know, if \( K \in L^\infty(\mathbb{R}) \), then
\[ E_K = \int_{-\infty}^{+\infty} K(\lambda) dE(\lambda) \]
is a bounded operator on \( L^2(\mathcal{M}) \) and
\[ L^\infty(\mathcal{M}) \ni K \rightarrow E_K \in \mathcal{B} \left( L^2(\mathcal{M}) \right) \]
is a \(*\)-homomorphism.

This is the simplest and the best known functional calculus.

**QUESTION.** Are there any reasonable conditions on \( K \) under which \( E_K \) is bounded on some \( L^p(\mathcal{M}), p \neq 2 \)?

Of course in this generality the answer is "no".

In his book *Topics in Harmonic Analysis*... Stein [3] proved the following theorem, perhaps still the best one, specifying conditions on \( H \) under which the question has an answer.
A. Hulanicki

Stein assumes that the operator $H$ is the infinitesimal generator of a semi-group of operators \( \{T_t\}_{t \geq 0} \) such that

\[
\|T_t\|_{L^p, L^p} \leq 1 \quad \text{for all } 1 \leq p \leq \infty.
\]

**Theorem (E. M. Stein).** Condition (2) and

\[
K(\lambda) = \lambda \int_0^\infty e^{-\lambda \xi} m(\xi) d\xi \quad \text{for some } m \in L^\infty(R^+),
\]

imply that \( \|E_k\|_{L^p, L^p} \leq C_p \) for all \( 1 < p < \infty \).

As we see, condition (3) implies that \( K \) is holomorphic in the right half-plane. However for some specific operators \( H \) the class of functions \( K \) on \( R^+ \) for which \( E_K \) is bounded on some \( L^p, p \neq 2 \), contains functions with compact support. This is the case of some Schrödinger operators.

These are operators of the form

\[
H = -\frac{1}{2}A + V(x),
\]

where \( A \) is the laplacian on \( R^d \) and \( V \) is the potential, i.e. the operator of multiplication by the function \( V \).

The following condition on \( V \) has been introduced by M. Aizenman and B. Simon in 1982 (cf. e.g. [1]):

\[
(K^{\text{loc}}) \quad \lim_{a \to 0} \sup_{|x-x_0| < a} \int_{|y| < a} V(y) \varphi(x-y) dy = 0,
\]

where

\[
\varphi(x) = \begin{cases} 
|x|^{-d+2} & \text{if } d > 2, \\
\log|x| & \text{if } d = 2, \\
1 & \text{if } d = 1.
\end{cases}
\]

**Theorem.** Assume that \( V \) satisfies \((K^{\text{loc}})\), \( V(x) \geq 0 \), and, for some \( \alpha > 0 \), \( V(x) \geq |x|^\alpha \) for \( |x| > C' \). Let

\[
N \geq \frac{d}{2(x \wedge 2)} + 3.
\]

Then, if \( K \in C^N[0, \infty) \) and

\[
\sup \{ e^{\lambda N} |K(\lambda)|: \lambda > 0 \} < \infty, \quad j = 0, \ldots, N,
\]

then \( \|E_K\|_{L^1, L^1} < \infty \), which, by interpolation, implies

\[
\|E_K\|_{L^p, L^p} < \infty \quad \text{for all } 1 \leq p \leq \infty.
\]

**Remark.** The class of functions defined by (4) is an algebra in which \( C^N_c[0, \infty) \) is dense.
Proof. The proof is based on an old idea of Y. Katznelson (cf. e.g. [2]) which has been used many times by various authors.

Let $e(\xi) = e^{it} - 1$. If $F \in C^1(\pi, \pi)$ and $F(0) = 0$, then

$$F(\xi) = \sum \hat{F}(n)(e^{in\xi} - 1) + \sum \hat{F}(n) = \sum \hat{F}(n)e(n\xi).$$

Since, for a fixed $n$,

$$e(n\xi) = \sum_{k=1}^{\infty} \frac{(in)^k e^{\lambda_k}}{k!},$$

we have $\|e(nA)\|_{L^1, L^1} < \infty$.

Suppose

$$\|e(nA)\|_{L^1, L^1} \leq C|n|^M.$$  \hspace{1cm} (4)

Then, of course, for $F \in C^{M+2}(\pi, \pi)$ and $F(0) = 0$,

$$F(A) = \sum \hat{F}(n)e(nA) \in \mathfrak{B}(L^1, L^1).$$

So, if $A = E_\varphi$, and the range of $\varphi$ is contained in $(0, \pi)$, then, by (1),

$$E_{\varphi} = \int_{-\infty}^{+\infty} F(\varphi(\lambda)) dE(\lambda) \in \mathfrak{B}(L^1, L^1).$$

Now assume $H$ is a Schrödinger operator which satisfies the assumption of the theorem. Then $H$ is essentially self-adjoint, and non-negative. Let

$$Hf = \int_{0}^{\infty} \lambda dE(\lambda)$$

be its spectral resolution. We write

$$T_t f = \int_{0}^{\infty} e^{-\lambda t} dE(\lambda) f.$$  \hspace{1cm} (5)

The Feynman-Kac formula says

$$T_t f(x) = E \exp \left[ - \int_{0}^{t} v(b_s) ds \right] f(b_t),$$

where $b$ is the Brownian motion in $R^d$. Hence, since $V(x) \geq 0$,

$$|T_t f(x)| \leq E |f(b_t)| = |f| p_t, \quad \text{where} \quad p_t(x) = (2\pi t)^{-d/2} \exp \left[ - \frac{|x|^2}{2t} \right].$$

Hence $\|T_t\|_{L^1, L^1} \leq 1$.

We put $T = T_1$ and estimate $\|e(nT)f\|_{L^1}$ in terms of $\|f\|_{L^1}$.

First we note that $e(nT) = AT$, where, by the spectral theorem,

$$\|A\|_{L^2, L^2} \leq \sup \{ |\lambda^{-1}(e^{-in\lambda} - 1)| : \lambda > 0 \}.$$
We write
\[ \|e(nT)f\|_{L^1} = \int |e(nT)f| \, dx = \int_{|x| \leq m} |f| + \int_{|x| > m} |f| = I_1 + I_2, \]
where \( |x| = \max |x_i|, \ x = (x_1, \ldots, x_d) \). Then, by the Schwarz inequality,
\begin{equation}
I_1 \leq m^{d/2} \|e(nT)f\|_{L^2} \leq m^{d/2} \|A\|_{L^2,L^2} \|Tf\|_{L^2} \leq m^{d/2} |n| C_T \|f\|_{L^1},
\end{equation}
since, by M. Aizenman, B. Simon (cf. [1]), \( V \in K_{d}^{\text{loc}} \), \( V(x) \geq 0 \) implies \( \|Tf\|_{L^2} \leq C_T \|f\|_{L^1} \). On the other hand,
\[ I_2 \leq \int_{|x| > m} \sum_{k=1}^{\infty} |n|^k E \exp \left[ - \int_0^1 V(b_k) \, ds \right] \|f(b_k)\| \, dx. \]

Now we use the following well-known, and easy to prove fact (cf. [1]):
\[ P_x \left\{ \inf_{0 \leq s \leq 1} |b_s| < \frac{1}{2} |x| \right\} \leq P_0 \left\{ \sup_{0 \leq s \leq 1} |b_s| \geq \frac{1}{2} |x| \right\} \]
\[ \leq 2dP_0 \left\{ \sup_{0 \leq s \leq 1} b_s^1 \geq \frac{1}{2} |x| \right\} = 4dP_0 \left\{ b_1^1 \geq \frac{1}{2} |x| \right\} \leq Ce^{-\epsilon |x|^2} \]
for some \( C \) and \( \epsilon > 0 \) which depend only on \( d \), and \( b_1^1 \) denotes the one-dimensional Brownian motion. Hence, for \( |x| > C' \),
\[ E \exp \left[ - \int_0^1 V(b_s) \, ds \right] \|f(b_k)\| \leq E \exp \left[ - \int_0^1 V(b_s) \, ds \right] \|f(b_k)\| \]
\[ \leq P_x \left\{ \inf_{0 \leq s \leq 1} |b_s| < \frac{1}{2} |x| \right\} E \|f(b_k)\| + \exp \left[ - \frac{1}{2} |x|^2 \right] E \|f(b_k)\| \]
\[ \leq (Ce^{-\epsilon |x|^2} + e^{-|x|^{\alpha/2}}) \|f\| \|x\|^\alpha \cdot p_1(x). \]

Consequently,
\[ \int_{|x| > m} E \exp \left[ - \int_0^k V(b_s) \, ds \right] \|f(b_k)\| \leq c'e^{-\epsilon |x^{\alpha/2}|^2} \|f\|_{L^1} \]
for some \( c' \) and \( c' > 0 \). Thus \( I_2 \leq c'e^{|n|} e^{-\epsilon |x|^{\alpha/2}} \|f\|_{L^1} \).

Putting \( m = c|n|^{1/(\alpha + 2)} \) for sufficiently large \( c \), by (5), we obtain
\[ \|e(nT)f\|_{L^1,L^1} \leq C|n|^{d/2(\alpha + 2) + 1}. \]

Thus for every \( F \in C(N(-\pi, \pi)) \) such that \( F(0) = 0 \) the function
\begin{equation}
K(\lambda) = F(e^{-\lambda})
\end{equation}
has the property \( \|E_k\|_{L^1,L^1} < \infty \). It is easy to verify that functions of the form
(6) are precisely the ones which satisfy (4). This completes the proof of the theorem.

REFERENCES


Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw, Poland

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