ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS*

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Abstract. The paper contains direct proofs of two important theorems. First of them (Theorem 0.6) was formulated and proved by Dynkin and Mandelbaum [2], the second one (Theorem 1.2) — by Mandelbaum and Taqqu [3].

0. Introduction. The purpose of this note is to give a direct proof of some recent important results of Dynkin and Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process.

Let us set up some notation. Let \((X, \Sigma, \mu)\) be a probability space and \((X^k, \Sigma^k, \mu^k)\) be the \(k\)-fold produce probability space. Let \(h_k(x_1, \ldots, x_k)\) be a symmetric function of \(k\) variables. We call it canonical if

\[ \int h_k(x_1, \ldots, x_{k-1}, y) \, d\mu = 0 \quad \text{for all } x_1, \ldots, x_{k-1} \in X^{k-1}. \]

Let \(X_1, \ldots, X_n\) be an i.i.d. \(X\)-valued random variable on a probability space with distribution \(\mu\). As in [2], define

\[ \sigma^n_k(h_k) = \begin{cases} \sum_{1 \leq s_1 < \cdots < s_k \leq n} h_k(X_{s_1}, \ldots, X_{s_k}) & \text{for } k \leq n, \\ 0 & \text{for } k > n. \end{cases} \]

Let

\[ H = \{ (h_0, h_1, \ldots) : h_k \text{ canonical and } \sum_{k=1}^{\infty} \frac{1}{k!} \|h_k\|_2^2 < \infty \}, \]

where \(h_0\) is a constant and \(\|\cdot\|_2\) is the norm in \(L^2(X^k, \Sigma^k, \mu^k)\). On \(H\) define

\[ \|h\|^2 = \sum_{k=0}^{\infty} \|h_k\|_2^2/k! \cdot \]

\(H\) is the so-called exponential (Foch) space of \(L_2^2(X, \Sigma, \mu)\) \((\varphi \in L^2(X, \Sigma, \mu) \text{ with } E\varphi(X) = 0)\). It is a Hilbert space under coordinate addition, scalar

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multiplication and $\|\|$. For each $\phi \in L^2_0(X, F, \mu)$, $h^\rho \in H$ with $h^\rho = \phi (x_1), \ldots, \phi (x_n)$. It can be easily seen that $sp \{h^\rho: \phi \in L^2_0(X, F, \mu)\}$ is dense in $H$. Define, for each $h \in H$,

$$Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma_k^*(h_k).$$

Since $\sigma_k^*(h_k) = 0$ for $k > n$, this is a finite sum. Also, let

$$Y'_n(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma_k^{(n)}(h_k).$$

The main purpose is to show directly that

$$Y_n(h) \overset{p}{\to} \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!},$$

where $\overset{p}{\to}$ denotes convergence in distribution and $I_k(h_k)$ denotes Itô-Wiener multiple integral of $h_k$ with respect to Gaussian random measure $W$ with $EW(A)W(A') = \mu(A \cap A').$

In the next section we discuss the convergence of $Y'_n(h)$. We observe that for $\phi \in L^2_0(X, \Sigma, \mu)$

$$Y_n(h^\rho) = \sum_{k=0}^{n} n^{-k/2} \sum_{1 \leq s_1 < \ldots < s_k \leq n} \phi(X_{s_1}) \ldots \phi(X_{s_k})$$

$$= \sum_{k=0}^{n} \frac{\phi(X_{s_1}) \ldots \phi(X_{s_k})}{\sqrt{n}} = \prod_{1}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right).$$

Let us observe that, for any $\varepsilon > 0$,

$$\sum_j P(|\phi(X_j)| > \sqrt{\varepsilon}) = \sum_j P(|\phi(X_j)|^2 > \varepsilon) \leq C\|\phi\|_2^2 < \infty.$$

Hence by Borel-Cantelli lemma, a.s. (for $j \leq n$) $|\phi(X_j)| \leq \sqrt{\varepsilon} \leq \sqrt{\varepsilon} \sqrt{n}$ for $j \geq$ some $N(\omega)$ ($N(\omega) < \infty$). But

$$\prod_{1}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right) = \prod_{1}^{N(\omega)} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right) \prod_{N(\omega)}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right)$$

giving for a.s. $w$, so

$$\lim_{n} Y_n(h^\rho) = \lim_{n} \prod_{1}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right).$$

Thus WLOG, we can assume for $n$ large $|\phi(X_j)/\sqrt{n}| < 1$ a.s. for all $j \leq n$ and

$$Y_n(h^\rho) = \prod_{1}^{n} \left(1 + \frac{\phi(X_j)}{\sqrt{n}}\right).$$
Taking log on both sides and expanding log(1 + x) we have
\[
\log \prod_{1}^{n} \left(1 + \frac{\varphi(X_j)}{\sqrt{n}}\right) = \sum_{1}^{n} \frac{\varphi(X_j)}{\sqrt{n}} - \frac{1}{2} \sum_{1}^{n} \frac{\varphi(X_j)^2}{n} + \varepsilon_n(\varphi),
\]
where \(\varepsilon_n(\varphi) \xrightarrow{L} 0\) by the WLLN and since \(\max |\varphi(X_j)/\sqrt{n}| \xrightarrow{L} 0\) by Chebychev's Inequality, i.e. the \((Y_n(h_{(n)})) \xrightarrow{D} \exp[1_{1}(\varphi) - \frac{1}{2}||\varphi||^2_2]\). Using Cramér-Wold device and the above argument we get

(0.3) LEMMA. For any finite subset \(\{\varphi_1, \ldots, \varphi_k\} \subseteq L^2(X, \Sigma, \mu)\)
\[
(Y_n(h_{(n)}^1), \ldots, Y_n(h_{(n)}^k)) \xrightarrow{D} \exp(1_{1}(\varphi_1) - \frac{1}{2}||\varphi_1||^2_2, \ldots, \exp(1_{1}(\varphi_k) - \frac{1}{2}||\varphi_k||^2_2).
\]

As a consequence, we get for \(\{\varphi_i, i \in I\}\) a finite subset of \(L^2(X, \Sigma, \mu)\) and \(\{c_i, i \in I\} \subseteq \mathbb{R}\),

(0.3)
\[
Y_n(\sum_{i \in I} c_i h_{(n)}^i) \xrightarrow{D} \sum_{k = 0}^{\infty} \frac{I_k[\sum_{i \in I} c_i h_{(n)}^i]}{k!}.\]

We now observe that for \(h, h' \in H\),

(0.4)
\[
E[Y_n(h) - Y_n(h')]^2 = \sum_{k} \binom{n}{k} n^{-k} ||h_k - h_k'||^2 \leq E||h - h'||^2,
\]
since \(E\sigma_k(h_k - h_k') \sigma_k(h_i - h_i) = \binom{n}{k} ||h_k - h_k'||^2 \delta_{ki}\) (by [2], p. 744). Also,

(0.5)
\[
E \left( \sum_{k = 0}^{\infty} \frac{I_k(h_k)}{k!} - \sum_{k = 0}^{\infty} \frac{I_k(h_k')}{k!} \right)^2 = ||h - h'||^2.
\]

Thus we get

(0.6) THEOREM. For any \(h \in H\),
\[
Y_n(h) \xrightarrow{D} W(h) = \sum_{k = 0}^{\infty} \frac{I_k(h_k)}{k!}.
\]

Proof. Let \(h \in H\) and \(\varepsilon > 0\). Choose

\[
h' = \sum_{i \in I} c_i h_{(n)}^i
\]
such that \(||h - h'||^2 < \varepsilon/2\). Now consider, for \(t \in \mathbb{R}\),
\[
|E(e^{itY_n(h)} - e^{itW(h)})| \leq E|e^{itY_n(h)} - e^{itY_n(h')}| + E|e^{itY_n(h')} - e^{itW(h')}| + E|e^{itW(h')} - e^{itW(h)}|.
\]

Using Schwartz's Inequality and the fact \(|e^{ix} - 1| \leq |x|\), we get that the first and third terms of the above inequality are dominated by \(t^2 E||h - h'||^2\) using (0.4) and (0.5). Hence, by (0.3),
As $\varepsilon$ is arbitrary, we get the result.

Finally, we make some observations to be used later:

(0.7) \[ Y_n(h^\varphi) = \sum_{k=0}^{[nt]} n^{-k/2} \sum_{1 \leq s_1 < \ldots < s_k \leq [nt]} \varphi(X_{s_1}) \ldots \varphi(X_{s_k}) = \prod_{i=1}^{[nt]} \left( 1 + \frac{\varphi(X_i)}{\sqrt{n}} \right). \]

Also, \( \min(t, s) \mu(A \cap A') \) is a covariance on \([0, \infty) \times \Sigma\) giving that there exists a centered Gaussian process \( W(t, A) \) with \( EW(t, A) W(s, A') = \min(t, s) \mu(A \cap A') \). Let, for \( T < \infty \),

\[ H_T = \{ (h_0, h_1, \ldots) \in H: \sum_{k=0}^{T} \frac{\|h_k\|^2}{k!} < \infty \}. \]

1. Invariance Principle. Let \( D[0, T], T \leq \infty, \) be the space of right continuous functions on \([0, T) \times [0, \infty)\) with left limits at each \( t \leq T \). The space \( D[0, T] \) is endowed with Skorohod topology [1]. The topology in \( D[0, \infty) \) is the one described in Whitt [4]. We note that

\[ X_{[nt]} = \sum_{i=1}^{[nt]} \left( \frac{\varphi^2(X_i) - EW^2}{n} \right) \]

has stationary independent increments. So, for \( \varepsilon > 0 \),

\[ P\left( \sup_{0 \leq t \leq t} |X_{[nt]}| > \varepsilon \right) \leq C P\left( |X_{[nt]}| > \varepsilon \right) \to 0 \]

by the weak law of large numbers. Using this, the arguments preceding Lemma 0.3, invariance principle and Cramér-Wold device we get the following analogue of Lemma 0.3:

**Lemma 1.1.** \( (Y_n(h^\varphi), \ldots, Y_n(h^{\varphi_k})) \) \( \xrightarrow{D^k} \) \( (\exp(I'(\varphi_1) - \frac{1}{2} t \|\varphi_1\|^2), \ldots, \exp(I'(\varphi_k) - \frac{1}{2} t \|\varphi_k\|^2)) \), where \( I'(\varphi_j) = \int \left( \frac{1}{2} \int_0^t \varphi_j(x) W_k(du, dx) \right) \).

Here \( \xrightarrow{D^k} \) denotes convergence in \( D^k [0, T] \) with respect to product topology.

We note that \( W(t, A) \) is a Brownian motion for each \( A \in \Sigma \). Thus we can choose \( I'(\varphi) \) continuous for each \( \varphi \) and a martingale in \( t \) as \( I'(\varphi) = \int \varphi(x) W(t, dx) \). We get, for \( \{c_1, \ldots, c_k\} \subseteq R \) (\( k \) finite),

\[ Y'(\sum_{j=1}^{k} c_j h^\varphi) \to \sum_{j=1}^{k} c_j \exp(I'(\varphi_j) - \frac{1}{2} t \|\varphi_j\|^2). \]

Let \( \varphi \in L^2_0(X, \Sigma, \mu), \|\varphi\| = 1, \) and write

\[ (\varphi^k)^j = \varphi(x_1) \ldots \varphi(x_k) 1_{(0, t)}(u_1) \ldots 1_{(0, t)}(u_k). \]

Define \( I_k(\varphi^k)^j = k! H_k(t, I(\varphi)), \) where \( H_k \) is Hermite polynomial, i.e.

\[ \sum_{k=0}^{\infty} \gamma^k H_k(t, x) = \exp(\gamma x - \frac{1}{2} \gamma^2 t). \]
For $\varphi \in L^2_0(X, \Sigma, \mu), \|\varphi\| = 1$, we define for $(h^\varphi)' = (1, \varphi', (\varphi^2)', \ldots)$,

$$W(h^\varphi)' = \sum_{k=0}^{\infty} \frac{I_k(\varphi^k)'}{k!},$$

and extend it linearly to $(\Sigma C_j(h^\varphi)')$. It is a martingale.

Let $h \in H_T$, a sequence in $sp \{(h^\varphi)', \varphi \in CONS \in L^2_0(X, \Sigma, \mu)\} \subseteq H_T$; then

$$P\left(\sup_{t \leq T} |W^T(h(n) - h(m))| \geq \varepsilon\right) \leq E|W^T(h(m) - h(n))|^2$$

using Doob's inequality and argument as in (0.5). Define, for $h \in H^t$, $W^t(h) = -\lim W^T(h_k)$, where the limit is uniform on compact for $h_k \to h$. Then $W^t(h)$ is right continuous martingale and has the same distribution as $\Sigma_k I_k'(h_k)/k!$. Now we derive the main theorem of [3].

**Theorem 1.2.** $Y_n(h) \overset{D}{\to} W^t(h)$ in $D[0, T]$ for $h \in H_T$ for each $T < \infty$.

**Proof.** Let $h \in H$ and $\varepsilon > 0$, choose $h_k \in sp \{h^\varphi: \varphi \in L^2_0(X, \Sigma, \mu)\} \ni h_k \to h$. Now define $X_{n,k} = Y_n(h_k), Z_n = Y_n(h), X_k = W^t(h_k)$ and $X = W^t(h)$. Then $X_{n,k} \overset{D}{\to} X_k$ as $n \to \infty$ in $D[0, T]$ for each $T < \infty$ by Lemma 1.1. Also $X_k \overset{D}{\to} X$ as $n \to \infty$ in $D[0, T]$ for each $T < \infty$. In addition,

$$P\left(\sup_{0 \leq t \leq T} |X_{n,k} - Z_n| \geq \varepsilon\right) \leq E|Y_n^T(h_n) - h_n)|^2 \leq T ||h - h_n||$$

giving

$$\lim_{k \to \infty} \lim_{n \to \infty} P\left(\rho(X_{n,k}, Z_n) \geq \varepsilon\right) \to 0$$

with $\rho$ being the Skorohod metric on $D[0, T]$. This implies (by [1], Thm 4.2, p. 25) that $Z_n \overset{D}{\to} W^t(h)$ in $D[0, T]$ ($T < \infty$) giving the result.

**Remark.** In the above arguments we may use an interpolated version of $Y_n(h)$ from the beginning and use appropriate version of Donsker's Invariance Principle to conclude above convergence occurs in $D[0, T]$ in sup norm giving $W^t(h)$ continuous.

**REFERENCES**


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