ON LÉVY (SPECTRAL) MEASURES 
of Integral Form on Banach Spaces

By

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Dedicated to my Teacher and Master
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the occasion of his sixtieth birthday

Abstract. In order to be a Lévy measure some necessary and
sufficient conditions are given for measures of integral form. In
particular, a complete proof for elements from classes $L_\alpha$, $\alpha > 0$, is
presented. Also some other examples are quoted.

It is well-known that a measure $M$ on a Hilbert space $H$ is a Lévy (spectral)
measure iff $M$ integrates the function $\min(1, \|x\|^2)$ over $H$. On the contrary, on
general Banach spaces this condition is neither necessary nor sufficient.
Moreover, there is no function $g$ such that the integrability of $g(\|x\|)$ with
respect to $M$ would be a necessary and sufficient one for $M$ to be a Lévy
measure on an arbitrary Banach space ([1], Chapter III, Theorem 6.3). On the
other hand, studying random integrals of the deterministic real-valued function
with respect to Lévy processes or stable measures we have to show that some
mixtures of (Lévy) measures are Lévy measures as well [5–8]. Some of the
previous proofs have appealed to random integral arguments to conclude that
a measure is Lévy. Here our justifications go throughout the series of
independent Banach space valued random variables and some type of
“comparison principle” (cf. Proposition 1). Proposition 2 shows that if the
$\lambda$-mixture of $T_t G$, $t > 0$, is Lévy, then so is $G$. The opposite implication is
considered in Proposition 3. Finally, in Section 4 some examples are discussed.

1. NOTATION AND SOME BASIC FACTS

Let $E$ be a real separable Banach space with the norm $\| \cdot \|$, the topological
dual $E'$ and the bilinear form $\langle \cdot , \cdot \rangle$ between $E'$ and $E$. Let $\text{ID}(E)$ denote the set
of all infinitely divisible measures on $E$. A $\sigma$-finite Borel measure $M$ on $E$, such
that $M(\{0\}) = 0$ and the function

$$\varphi_M(y) = \exp \int_{E(\{0\})} \left[ e^{i\langle y, x \rangle} - 1 - i \langle y, x \rangle_B(x) \right] M(dx), \quad y \in E',$$
is a characteristic function (of a probability measure \(\bar{\varepsilon}(M)\)), is called Lévy (spectral) measure (cf. [1], p. 117–118, where \(\bar{\varepsilon}(M)\) is denoted by \(c_1\) Pois\(M\)). The importance of Lévy measures follows from the fact that \(\mu \in \text{ID} \iff \mu = \delta_x \ast \gamma \ast \bar{\varepsilon}(M)\), where \(x \in E\), \(\gamma\) is the symmetric Gaussian measure and \(M\) is a Lévy (spectral) measure. The triple \(x, \gamma\) and \(M\) is uniquely determined by \(\mu\) (cf. [1], Theorem 6.2, p. 136).

In the sequel \(\mathcal{M}(E)\) denotes the set of all Lévy measures on Banach space \(E\), \(B_r := \{x \in E: \|x\| \leq r\}, \ r > 0\), is a ball and \(\mathcal{B}_0\) is the family of all Borel subsets of \(E_0 := E \setminus \{0\}\). If \(H\) is a Hilbert space, then

\[
M \in \mathcal{M}(H) \quad \text{iff} \quad \int_H \min \left(1, \|x\|^2\right) M(\text{d}x) < \infty.
\]

But on the general Banach space formula (1.0) is not longer true. However, some sufficient conditions are available. Namely, we have the following (cf. [1], Chapter III, Theorems 4.7 and 6.3):

\[
\begin{align*}
(1.1) & \quad M(E) < \infty \text{ implies } M \in \mathcal{M}(E) \text{ and } e(M) := e^{-M(E)} \sum_{k=0}^{\infty} \frac{M^k}{k!} \in \text{ID}(E); \\
(1.2) & \quad \int_E \min(1, \|x\|) M(\text{d}x) < \infty \text{ implies } M \in \mathcal{M}(E); \\
(1.3) & \quad \text{if } 0 \leq N \leq M \text{ and } M \in \mathcal{M}(E), \text{ then } N, M-N \text{ are in } \mathcal{M}(E); \\
(1.4) & \quad \text{if } M, N \in \mathcal{M}(E), \text{ then } M+N \in \mathcal{M}(E).
\end{align*}
\]

On the other hand, the following properties are necessary:

\[
\begin{align*}
(1.5) & \quad M \in \mathcal{M}(E) \text{ implies } M(B_r) < \infty \text{ for each } r > 0, \text{i.e. } M \text{ is finite outside every neighbourhood of zero; } \\
(1.6) & \quad M \in \mathcal{M}(E) \text{ implies } \int_E \min(1, \langle y, x \rangle^2) M(\text{d}x) < \infty \text{ for each } y \in E', \text{ because } \\
& \quad \text{where } \Pi_y M \in \mathcal{M}(R), \text{ where } \Pi_y : E \to R \text{ is given by } \Pi_y x := \langle y, x \rangle.
\end{align*}
\]

For further references let explicitly state that (cf. (1.3) and (1.4))

\[
(1.7) \quad M \in \mathcal{M}(E) \iff M^0 := M + M^- \in \mathcal{M}(E), \text{ where } M^-(A) := M(-A) \text{ for } \\
& \quad A \in \mathcal{B}_0.
\]

Thus we can consider symmetric measures \(M\) only. The measure \(M^0\) in (1.7) is called the symmetrization of a measure \(M\). We complete this introductory section with a lemma which will be repeatedly applied later on.

**Lemma 1.** If \(M\) is a symmetric Lévy measure supported by the unit ball \(B_1\), \(\xi_n\)'s are \(E\)-valued independent rv's with distributions

\[ e(M|_{\|x\| \leq n^{-1}}) \quad \text{for } n = 1, 2, \ldots, \]

and \(\xi\) has the distribution \(\bar{\varepsilon}(M)\), then \(\sum_n \xi_n\) converges to \(\xi\) in \(L_p\)-norm for each \(p > 0\).
Proof. By Lemma 4.4 and Theorem 2.10, Chapter III, in [1], we infer that \( \sum \xi_n \) converges a.s. to \( \xi \). From [1], Corollary 3.3, we infer that \( \xi \) has all exponential moments, in particular all \( p \)-moments. From the Lévy inequality and Theorem 2.11 in [1] we conclude the proof of Lemma 1.

2. FUNDAMENTAL INEQUALITIES

Let \( \lambda \) be a measure on \( R^+ = (0, \infty) \) and \( m \) a Borel measure on \( E \). Then the measure \( m^{(\lambda)} \), defined on Borel sets \( A \) by

\[
 m^{(\lambda)}(A) = \int \int_{R^+} 1_A(tx) \lambda(dt) m(dx) = \int_{R^+} m(t^{-1} A) \lambda(dt),
\]

is the \( \lambda \)-mixture of measures \( (T_t m)(\cdot) = m(t^{-1} \cdot), t \in R^+ \). Measures of the form (2.1) appeared in the study of stable measures (cf. [6], p. 272–273, or [1], p. 165), random integrals (cf. [7], p. 250, or [4], Theorems 1.3 and 3.2) and in fractional calculus in probability theory (cf. [8]). In all of these circumstances one has to determine whether \( m^{(\lambda)} \) is a Lévy measure for a particular given measure \( \lambda \). Here (in Section 3) we will discuss this question as well as the opposite one in general.

**Proposition 1.** For \( 1 \leq j \leq k \) let \( \lambda_j \) be finite measures on \( R^+ \) with mean values \( v_j \) and let \( m_j \) be finite Borel measures on \( E \) with zero mean values. Then

\[
 \int_{E} \|x\| T_{a_1} m_1 \ast \ldots \ast T_{a_k} m_k(dx) \leq \int_{E} \|x\| m_1^{(\lambda_1)} \ast \ldots \ast m_k^{(\lambda_k)}(dx)
\]

\[
 \leq c_k \int_{E} \|x\| m_1 \ast \ldots \ast m_k(dx),
\]

where \( a_i = v_i \prod_{j \neq i} \lambda_j(R^+) \) for \( 1 \leq i \leq k \) and

\[
 c_k = 2 \int_{R^+} \ldots \int_{R^+} \max(t_1, \ldots, t_k) \lambda_1(dt_1) \ldots \lambda_k(dt_k).
\]

**Proof.** Let \( A_k \) be the middle term in the above inequality. By Lemma 2.12, p. 108, from [1], we get

\[
 A_k = \int_{R^+} \ldots \int_{R^+} \int_{E} \|t_1 x_1 + \ldots + t_k x_k\| m_1(dx_1) \ldots m_k(dx_k) \lambda_1(dt_1) \ldots \lambda_k(dt_k)
\]

\[
 \leq c_k \int_{E} \|x\| m_1 \ast \ldots \ast m_k(dx),
\]

which gives the right-hand side inequality. Since the norm of an integral is not greater than the integral of the norm of a function, we have

\[
 A_k \geq \int_{E} \ldots \int_{R^+} \left( t_1 x_1 + \ldots + t_k x_k \right) \lambda_1(dt_1) \ldots \lambda_k(dt_k) \| m_1(dx_1) \ldots m_k(dx_k)
\]

\[
 = \int_{E} \ldots \int_{E} a_1 x_1 + \ldots + a_k x_k \| m_1(dx_1) \ldots m_k(dx_k),
\]

which is the left-hand side inequality in Proposition 1.
We conclude this subsection with some simple properties of \( \lambda \)-mixtures \( m^{(\lambda)} \), which will be needed later on. Namely we have:

\[
(2.2) \quad (m^{(\lambda)})^0 = (m^0)^{(\lambda)}, \quad \text{where } m^0 \text{ is the symmetrization of } m; \\
(2.3) \quad \text{if } m_1 \leq m_2 \text{ or } \lambda_1 \leq \lambda_2, \text{ then } m^{(\lambda_1)}_1 \leq m^{(\lambda_2)}_2; \\
(2.4) \quad (\sum_j m_j)^{(\lambda)} = \sum_j m_j^{(\lambda)} \quad \text{and} \quad m^{(\lambda_1 \lambda_2)}_{\lambda_1} = \sum_j m^{(\lambda_2)}_j; \\
(2.5) \quad m^{(\beta a)}_a = T_\beta m \quad \text{and} \quad m^{(\beta \alpha \beta a)}_j = \sum \alpha_j T_\alpha m; \\
(2.6) \quad am^{(\lambda)} = (am)^{(\lambda)} = m^{(\alpha \lambda)} \quad \text{for } a \in R^+. 
\]

In formulas (2.2)–(2.6), \( m \) and \( m_j \)'s are measures on \( E \), and \( \lambda \) and \( \lambda_j \)'s are measures on \( R^+ \).

3. MIXTURES OF MEASURES

As before, \( E \) denotes a \( \cdot \)-Banach space, \( S_0 \) is a \( \sigma \)-algebra of Borel subset of \( E_0 := E \setminus \{0\} \) and \( M(E) \) is the family of all Lévy measures on \( E \).

**Proposition 2.** Let \( g \) be a measure on \( R^+ \) and \( G \) be a measure on \( S_0 \), both non-zero, such that \( G(g) \in M(E) \), i.e., \( G(g) \) is a Lévy measure. Then

(a) \[
\int_{R^+} G(B_{t^{-1}}) g(dt) = \int_{E_0} g(t : t > \|x\|^{-1}) G(dx) < \infty, 
\]

(b) \( g \) and \( G \) are Lévy measures on \( R^+ \) and \( E \), respectively.

**Proof.** From (1.5) and (2.1) we get

\[
G^{(g)}(B_t) = \int_{R^+} G(B_t^{-1}) g(dt) = \int_{E_0} g(t : t > \|x\|^{-1}) G(dx) < \infty, 
\]

which gives (a). Moreover, \( G \) and \( g \) are finite outside some neighbourhoods of zero in \( E \) and \( R^+ \). In fact, they are finite outside every neighbourhood of zero. Note that

\[
\int_{a^{-1}}^{\infty} G(B_{t^{-1}}) g(dt) < \infty \quad \text{for each } a > 0. 
\]

Thus there is a \( t_0 \in (a^{-1}, \infty) \) such that \( G(B_{t_0^{-1}}) < \infty \), i.e., \( G(B_t) < \infty \) for \( a > 0 \). Similarly, we show this for \( g \). Furthermore, \( \Pi_y G^{(g)} \in M(R) \) for all \( y \in E' \) (cf. (1.6)), and (1.0) gives

\[
\int_{R^+} \min (1, s^2) \Pi_y G^{(g)}(ds) = \int_{E} \min (1, t^2 \langle y, x \rangle^2) g(dt) G(dx) < \infty. 
\]

Hence there are \( x_0 \in E \) and \( y_0 \in E' \) such that \( w = \langle y_0, x_0 \rangle \neq 0 \) and \( g \) integrates \( \min (1, t^2 w^2) \) over \( R^+ \), i.e., \( g \in M(R^+) \).

To complete the proof of part (b) we can assume that \( G \) is symmetric and concentrated on the unit ball \( B_1 \) (cf. (1.7) and (1.1)). Also, without loss of the
generality, we can assume that $g$ is concentrated on a bounded set $A$ in $\mathbb{R}^+$ and that $g(A) = 1$, because $G^{(g)} \geq G^{(g(A))} \in \mathcal{M}(E)$ and $aG^{(g(A))} = G^{aG(A)}$ (cf. (2.3), (2.6) and (1.3)). Consequently, $G^{(g)}$ is a symmetric Lévy measure concentrated on the ball $B_r$, where $r := \sup A < \infty$ and $g_1 := g|_A$. Taking

$$I_n := B_{r} \setminus B_{(n+1)^{-1}}, \quad G_n := G|_{I_n}$$

and independent $E$-valued rv's $\xi_n$ with distribution $e(G^{(g)})$, we see that $\sum \xi_n$ converges in $L_1$-norm (cf. Lemma 1). Applying Proposition 1 for $\lambda_1 = \ldots = \lambda_k = 1$ and $m_1 = m_2 = \ldots = m_k = \sum_{n=j}^{l} G_n$, we obtain

$$a \int_E \|x\|(\sum_{n=j}^{l} G_n)^{*k}(dx) \leq \int_E \|x\|(\sum_{n=j}^{l} G_n^{(g)})^{*k}(dx) \quad \text{for } k \in \mathbb{N},$$

where $a$ is the mean-value of $g_1$. Since $G_n^{(g)}(E) = G_n(E)$, we conclude that

$$E \|\sum_{n=j}^{l} \eta_n\| \leq a^{-1} E \|\sum_{n=j}^{l} \xi_n\| \quad \text{for all } j, l \in \mathbb{N},$$

where $\eta_n$ are $E$-valued independent rv's such that $L(\eta_n) = e(G_n)$. Thus $\sum \eta_n$ converges in $L_1$-norm and $G = \sum_G G_n \in \mathcal{M}(E)$, which completes the proof of Proposition 2.

Now we will determine when $G^{(g)}$ is a Lévy measure if so is $G$. However, we should be aware that in full generality the answer may depend on the geometry (the norm) of the Banach space $E$. Let us consider the following example: take $g(dt) = t^{-(p+1)} \, dt$ on $\mathbb{R}^+$ and finite measures $m$ on the unit sphere $S$ in $E$, i.e., for $A \in \mathcal{B}_0$,

$$m^{(g)}(A) = \int_0^\infty \chi_A(tu) t^{-(p+1)} \, dt \, m(du).$$

It is known that $m^{(g)}$ is a Lévy measure (an exponent of $p$-stable distribution) for all finite $m$'s on $S$ if and only if $E$ is of stable type $p$ (cf. [1], Theorem 7.9, p. 165, or, for a partial answer, [6], Theorem 2). In view of this example, sufficient conditions for $G^{(g)}$ to belong to $\mathcal{M}(E)$ will be given for some measures $g$ only.

**Proposition 3.** (1) If $g$ is finite and concentrated on $(0, T]$, then $G^{(g)}$ is a Lévy measure if so is $G$.

(2) Let $g$ be concentrated on $(0, T]$ such that, for some sequence $a_n \downarrow 0$ in $\mathbb{R}^+$, we have

$$a_1 = T, \quad c := \sum a_n < \infty \quad \text{and} \quad b := \sup_n g(a_{n+1}, a_n) < \infty.$$  

Then, for $G \in \mathcal{M}(E)$ concentrated on $B_1$, we have $G^{(g)} \in \mathcal{M}(E)$.  

Let $G$ be a finite measure concentrated on $B_1$ and $g$ be a measure on $R^+$. If

(i) $\int_{B_1'} g(s: s > \|x\|^{-1}) G(dx) < \infty$

and

(ii) $\int_{B_1'} \|x\|^{-1} \int_0^t g(dt) G(dx) < \infty,$

then $G^{(g)}$ is a Lévy measure on $E$.

Proof. In view of (2.2) and (1.7) we assume that all $G$'s are symmetric measures. We will prove each of these cases separately.

Case (1). By (2.6) we may assume additionally that $g$ is a probability measure. Since $G|_{B_1'}$ is finite (see (1.5)), and, for finite $G$, the measure $G^{(g)}$ is also finite, we restrict our consideration to $G \in M(E)$ and concentrated on $B_1$.

Let $I_n := B_{1/n} \setminus B_{1/(n+1)}$, $G_n := G|_{I_n}$ and let $\eta_n$ be independent $E$-valued rv's with distributions $e(G_n)$ for $n = 1, 2, \ldots$ From Lemma 1, $\sum_{n=1}^{\infty} \eta_n$ converges in $L_1$-norm. Let $\xi_n, n \in N,$ be independent, $E$-valued rv's with distributions $e(G_n^{(g)})$. Applying Proposition 1 to $\lambda_1 = \ldots = \lambda_k = g$ and $m_1 = m_2 = \ldots = m_k = \sum_{n=1}^{\infty} G_n$, we obtain

$$\int_E \|x\| \left( \sum_{n=1}^{\infty} G_n^{(g)}(dx) \right) \leq 2T \int_E \|x\| \left( \sum_{n=1}^{\infty} G_n^{(g)}(dx) \right).$$

Since $G_n^{(g)}(E) = G_n(E)$, hence summing over $k$ we get

$$\| \sum_{n=1}^{\infty} \xi_n \| \leq 2T \sum_{n=1}^{\infty} \| \eta_n \|$$

for all $j, l \in N,$

i.e., $\sum \xi_n$ converges in $L_1$-norm to an infinitely divisible rv with Lévy measure $\sum_{n=1}^{\infty} G_n^{(g)} = G^{(g)}$, which completes the proof of case (1).

Remark 1. An alternative proof for the case (1) is also possible by a random integral approach. Note that the random integral $\int_0^T t dY(\tilde{g}(t))$ exists for $D_x[0, T]$-valued rv $Y$ with stationary independent increments, $Y(0) = 0$ a.s. and $\tilde{g}(t) := g(s: s \leq t)$. Its Lévy measure equals $G^{(g)}$ (see [4, 5]).

Case (2). Let $L_n := (a_{n+1}, a_n]$ and $g_n := g|_{L_n}, n \in N$. Assuming additionally that $G$ is a finite measure we get

$$\int_E \|x\| e(G^{(g_n)})(dx) < e^{-G(E)(L_n)} \sum_{k=1}^{\infty} \frac{c_k}{k!} \int_E \|x\| G^{*k}(dx)$$
from Proposition 1 and formula (1.1). Since

$$c_k = 2g_k(L_n) \int_0^{a_n} \left[ 1 - (g(s:s \leq t)/g(L_n))^k \right] dt \leq 2a_n g_k(L_n),$$

we obtain (cf. [1], Lemma 2.7, p. 103)

$$\int_0^{a_n} \|x\| e(G^{(\eta)}) (dx) \leq 2a_n \int_0^{a_n} \|x\| e(G(L_n)) (dx) \leq 2a_n \int_0^{a_n} \|x\| e(bG) (dx).$$

Taking $\eta_n$ to be $E$-valued independent rv's with distribution $e(G^{(\eta)})$, we obtain

$$E \left\| \sum_{n=1}^I \eta_n \right\| \leq 2 \sum_{n=1}^I a_n \int_0^{a_n} \|x\| e(bG) (dx).$$

Hence $\sum_{n=1}^I \eta_n$ converges in $L_1$-norm, $\sum_{n=1}^I G^{(\eta)} = G^{(\eta)} \in M(E)$, and

$$\int_0^{a_n} \|x\| e(G^{(\eta)}) (dx) \leq 2c \int_0^{a_n} \|x\| e(bG) (dx),$$

for $G$ finite and concentrated on $B_1$.

If $G \in M(E)$ and is concentrated on $B_1$, then, taking $G_n := G_{|I_n}(I_n := B_{n-1} \setminus B_{n+1} - 1)$, we have $G_n^{(\eta)} \in M(E)$ and the above inequality holds for $G_n^{(\eta)}$. Hence and from Lemma 1 we conclude that $\sum_{n=1}^I G_n^{(\eta)} = G^{(\eta)} \in M(E)$ and

$$\int_0^{a_n} \|x\| e(G^{(\eta)}) (dx) \leq 2c \int_0^{a_n} \|x\| e(bG) (dx),$$

which completes the proof of case (2).

Case (3). Note that, for $A \in \mathcal{B}_0$,

$$G^{(\eta)}(A) = \int_{B_1} \int_{\{x\| = 1 \}} 1_A (tx) g (dt) G (dx) + \int_{B_1} \int_{\{x\| \leq 1 \}} 1_A (tx) g (dt) G (dx)$$

is a sum of two measures, $\nu_1$ and $\nu_2$, say. Because of assumption (i), the measure $\nu_1$ is finite and $\nu_1 \in M(E)$. The measure $\nu_2$ is concentrated on $B_1$ and, by (ii),

$$\int_0^{a_n} \|x\| \nu_2 (dx) = \int_0^{a_n} \|x\| \int_0^{a_n} \|x\| t g (dt) G (dx) < \infty.$$

Consequently, by (1.2) and (1.4), $G^{(\eta)} \in M(E)$, which completes the proof of case (3) and Proposition 3.

4. EXAMPLES

A. For $\alpha > 0$, let $v_\alpha (dt) = (\log t^{-1})^{\alpha-1} t^{-1} dt$ be a measure on $(0, 1]$. Taking $a_n := \exp(-n^{1/\alpha})$, $n = 0, 1, 2, \ldots$, we get

$$\sum_{n=0}^\infty a_n < \infty \quad \text{and} \quad v_\alpha (a_{n+1} , a_n ] = \alpha^{-1} < \infty.$$
Condition (i) in case (3) of Proposition 3 or (a) in Proposition 2 for \( \nu_a \) means the following:

\[
\int_{B_1^a} \int \frac{1}{\|x\|^a} (\log t^{-1})^{a-1} t^{-1} dt G(dx) = \alpha^{-1} \int_{B_1^a} \log \|x\| G(dx) < \infty.
\]

With the above restriction on \( G \), condition (ii) in case (3) of Proposition 3 is fulfilled because

\[
\lim_{\|x\| \to \infty} \|x\|/\log \|x\| \int_{B_1^a} (\log t^{-1})^{a-1} dt = \lim_{u \to \infty} u^{-a} e^u \int_0^\infty e^{-t} t^{a-1} dt = 0.
\]

This and Propositions 2 and 3 give

**Corollary 1.** Let \( \alpha > 0 \) and

\[
G_a(A) = \int_{B_1^a} \int_{E} 1_A(tx) (\log t^{-1})^{a-1} t^{-1} dt G(dx)
\]

\[
= \int_{B_1^a} \int_{E} 1_A(e^{-t} x) t^{a-1} dt G(dx) \quad \text{for } A \in \mathcal{B}_0.
\]

Then \( G_a \) is a Lévy measure on \( E \) iff so is \( G \) and

\[
\int_{B_1^a} \log \|x\| G(dx) < \infty.
\]

**Remark 2.** Thu ([8], Theorem 4.3) claims the result as above. The proof is a combination of random integral arguments from [7] and property (1.3) of \( \mathcal{M}(E) \). However, inequality (4.12) in [8] needs a correction and applying Corollary 4.2 (in Cases 1 and 2) one requires that \( G \in G_{[a]+1}(X) \), not only \( G \in G_a(X) \).

**Remark 3.** Taking \( a(t) = \exp(-t^{1/\alpha}) \) in Theorem 4 of Hong [3], one gets Corollary 1 for Hilbert spaces. Since a part of Hong's proof depends on the Three-Series-Theorem, it is not obvious that his arguments can be extended to arbitrary Banach spaces.

**B.** For \( \beta > 0 \) let us put \( g_\beta(dt) = t^{-(\beta+1)} dt \) on \((0, 1]\). For \( 0 < \beta < 1 \) and \( a_n = n^{-1/\beta} \) we get \( \sum_n a_n < \infty \), \( g_\beta(a_{n+1}, a_n) = \beta^{-1} \). If \( G \) is a Lévy measure concentrated on \( B_1 \), then \( G^{(0)} \in \mathcal{M}(E) \) by case (2) of Proposition 2. If \( G \) is supported by \( B_1^a \) and finite, then from assumptions (i) and (ii) of case (3) and (a) in Proposition 2 if follows that

\[
\int_{B_1^a} \|x\|^\beta G(dx) < \infty.
\]

From this and Propositions 2 and 3 we obtain

**Corollary 2.** Let \( 0 < \beta < 1 \) and

\[
G_\beta(A) = \int_{B_1^a} \int_{E} 1_A(tx) t^{-(\beta+1)} dt G(dx) \quad \text{for } A \in \mathcal{B}_0.
\]
Then \( G_\beta \) is a Lévy measure iff so is \( G \) and

\[
\int_{B^*_1} \|x\|^\beta G(dx) < \infty.
\]

Remark 4. Taking in Corollary 2 a finite measure \( m \) on the unit sphere \( S \) of \( E \) we obtain measures

\[
m_\nu(A) = \int_0^\infty \int_A (tx)^{-(\theta+1)} dtm(dx)
\]

which are always Lévy measures (corresponding to stable distributions with the exponent \( \beta \in (0, 1) \)).

C. For \( \gamma > 0 \) let us put \( \nu_\gamma(dt) = (\log t^{-1})^{\gamma-1} dt \) on \((0, 1] \). Since \( \nu_\gamma \) are finite measures \((\nu_\gamma(0, 1] = \Gamma(\gamma))\), (a) of Proposition 2 is fulfilled. Thus Proposition 3, case (1), and Proposition 2 give the following

**Corollary 3.** Let \( \gamma > 0 \) and

\[
G_\gamma(A) = \int_{\mathbb{E}} \int_A (tx)(\log t^{-1})^{\gamma-1} dtG(dx)
\]

\[
= \int_{\mathbb{E}} \int_A (e^{-1}x)e^{-1}t^{\gamma-1} dtG(dx)
\]

for \( A \in \mathcal{B}_0 \).

Then \( G_\gamma \) is a Lévy measure iff so is \( G \).

Remark 5. Lévy measures \( G_\gamma \) from Corollary 1 correspond to infinitely divisible measures from the class \( L_\alpha \) distributions (cf. [8]). These are subclasses of the class \( L = L_1 \) of selfdecomposable distributions. Similarly, measures \( G_\gamma \) from Corollary 3 are Lévy measures of distributions from classes \( \Upsilon_\beta \). The class \( \Upsilon_1 = \Upsilon \) coincides with limit distributions of non-linearly deformed rv's (s-selfdecomposable distributions; cf. [4], Section 2).

### 5. Final Comments

(1) All results (Propositions 1–3) are also valid if in the definition of \( m^{(x)} \) (see (2.1)) we replace \( 1_A(tx) \) by \( 1_A(df(t)x) \), where \( f \) is a real-valued measurable function on \( R^+ \). Simply, the measure \( \lambda \) should be replaced by the measure \( f\lambda = \lambda f^{-1} \) in (2.1). There is a need to have analogous characterizations for operator-valued functions (cf. [5] for measures from \( \Upsilon_\beta(Q) \) with \( \beta < 0 \). However, some of the present methods of proofs do not cover such a generality.

(2) The integrability of log\(^\gamma\) \((1+\|x\|)(1+\|x\|^\beta)\) over \( B^*_1 \) with respect to \( G \in \mathcal{M}(E) \) is equivalent to

\[
\int_E \log^\gamma(1+\|x\|) \bar{e}(G)(dx) < \infty \quad \text{or} \quad \int_E \|x\|^\beta \bar{e}(M)(dx) < \infty
\]

(cf., for instance, [2], Corollary 3.4).
REFERENCES


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Received on 22. 8. 1989