CONDITIONS FOR CONVERGENCE OF NUMBER OF CROSSINGS TO THE LOCAL TIME
APPLICATION TO STABLE PROCESSES WITH INDEPENDENT INCREMENTS AND TO GAUSSIAN PROCESSES

BY
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Abstract. Let $X(t)$, $t \in \mathbb{R}$, be a real valued stochastic process admitting a local time and let $X_\varepsilon(t)$, $\varepsilon \in \mathbb{R}^+$, be a family of smooth processes which converge in some sense to $X(t)$. We exhibit sufficient conditions for $L^2$-convergence of the number of crossings of $X_\varepsilon(t)$ to the local time of $X(t)$, after normalization.

Two main cases are considered for $X(t)$, stable processes and Gaussian processes.

Two main cases are considered for $X_\varepsilon(t)$: $X_\varepsilon(t)$ being the convolution of $X(t)$ with a size $\varepsilon$ approximate identity and $X_\varepsilon(t)$ being the size $\varepsilon$ polygonal approximation of $X(t)$.

Such a convergence is shown to hold for both approximations when $X(t)$ is a stable process with independent increments with index $\alpha > 1$.

Convergence of crossings of the polygonal approximation is shown to hold for a Gaussian process under technical conditions.

1. INTRODUCTION

The concept of local time of a stochastic process $X(t)$ was introduced by Levy [19] in the case of the Brownian motion process. The general theory of local time has been developed in three different directions, those of Markov processes, Gaussian processes and semimartingale processes. Even the definition of the local time is different according to the considered class of processes. The paper by Geman and Horowitz [17] gives a general survey of the different approaches.

In this paper we define the local time $L(u, t)$ as the density of the occupation measure of $X(t)$. This definition was used for example by Berman [2] for Gaussian processes, but it can be applied to any other class of processes.

Apart from the work on the definition and study of the properties of local time some effort has been put into the construction of $L(u, t)$ by limiting processes based on the sample path properties of $X(t)$. Particularly, Levy's
constructions [19, sect. 50] for Brownian motion have been extended to general Markov processes (see e.g. [16]).

In 1984, Wschebor [23, 24] proposed a construction which is quite different from those of Fristedt and Taylor. Let $W(t)$ be a Brownian motion, and let $W_\varepsilon$ be the convolution approximation of $W$ defined by

$$W_\varepsilon = W * \psi_\varepsilon,$$

where $\psi_\varepsilon(t) = \frac{1}{\varepsilon} \psi(t/\varepsilon)$, $\psi$ being a non-negative $C^\infty$ function with compact support, and let $N_\varepsilon^u[a, b]$ be the number of crossings of level $u$ by the process $W_\varepsilon(t)$ for $t \in [a, b]$. Wschebor showed that

$$(\pi/2)^{1/2} \|\psi\|_2^{-1} \varepsilon^{-1/2} N_\varepsilon^u[a, b] \xrightarrow{L^p} L(u, [a, b]) \quad \text{as} \quad \varepsilon \to 0.$$ 

Let $X_\varepsilon(t)$ be a family of smooth processes which converge in some sense to $X(t)$. Such a family will here be called an approximating family. The aim of this paper is to exhibit general sufficient conditions for convergence of crossings of $X_\varepsilon$ to the local time of $X(t)$. Two particular cases will be examined in sections 6 and 7:

- the convolution case: $X_\varepsilon(t)$ is defined as for the Wiener process;
- the polygonalization case: $X_\varepsilon(t)$ is the polygonal line going through the points $\{k \Delta, X(k \Delta); k \in \mathbb{N}\}$.

**General notation.** Throughout this paper:
- $p[X_1, \ldots, X_n; x_1, \ldots, x_n]$ — the joint density of random variables $X_1, \ldots, X_n$ at the point $x_1, \ldots, x_n$;
- (const) — an arbitrary positive constant;
- $N_\varepsilon^u$ — the number of crossings of $X_\varepsilon$ defined as
  $$N_\varepsilon^u(I) = \# \{t \in I; X_\varepsilon(t) = u\};$$
- $\bar{N}_\varepsilon(I)$ — the number of local extrema of $X_\varepsilon$ on the interval $I$;
- $X'_\varepsilon$ — the derivative of $X_\varepsilon$ (in the sense of absolute continuity).

**2. MAIN RESULTS**

Let $\mathcal{D}$ be a bounded real interval and let $X(t)$, $t \in \mathcal{D}$, be a stochastic process having 2-dimensional continuous density functions

$$p[X(t_1), X(t_2); x_1, x_2]$$

with respect to the Lebesgue measure. It is assumed that these density functions verify the hypothesis

$H_1$. There exists a function $g$, with $g(\varepsilon) = o(\varepsilon)$, as $\varepsilon$ tends to zero, such that, for every $[a, b]$ included in $\mathcal{D}$,

$$\left| \int \sup_{[a,b]^2 \times (x_1,x_2) \in \mathbb{R}^2} p[X(t_1), X(t_2); x_1, x_2] dt_1 dt_2 \right| \leq g(b-a).$$
The condition of finiteness of the integral in (2.1) with continuity of the density is a little stronger than Geman and Horowitz's [17] condition (23.4). So it implies the existence of local time in the sense that, almost surely, the occupation measure $\mu$ of the paths of $X(t)$ over $[a, b]$ is absolutely continuous with respect to Lebesgue measure: there exists a function $L(u, [a, b])$ a.s. $\mu$ a.e. finite which is the density of $\mu$. Moreover, $L(u, dt)$ is atomless almost surely and for almost every $u$.

These two conditions also provide an easy way to prove that the functions

$$q_t(x_1, x_2) = \int_{I \times I} p[X(t_1), X(t_2); x_1, x_2] dt_1 dt_2$$

are continuous. Arguments used by Berman [3, lemma 2.1] and Geman and Horowitz imply that the random variables

$$\eta_\delta^u[a, b] = (2\delta)^{-1} \sum \{ |X(t) - u| < \delta \} dt$$

converge in $L^2$-norm as $\delta$ tends to zero and that $q_{[a, b]}(u, u)$ is the $L^2$-norm of $L(u, [a, b])$.

Hypothesis $H_1$ is stronger than Geman and Horowitz's condition (23.4) but for non-periodic stationary Gaussian processes it is equivalent to their condition (22.7) which is the weakest known sufficient condition for the existence of the local time.

Hypothesis $H_1$ does not imply the joint continuity of the local time. But Berman [4] has given a set of sufficient conditions depending on the function $q$. For example, if the function $q$ of (2.1) satisfies the additional hypothesis $q(x) = o(x^{1+\delta})$ with $\delta > 0$, then Berman's condition (2.9) is fulfilled for $k = 2$. If, in addition, $q$ satisfies condition (2.8), then by Berman's theorem 2.1, the local time can be chosen jointly continuous.

Our main result can heuristically be stated as follows. Let $(X_\varepsilon(t), \varepsilon \in R^+)$ be a smooth almost sure approximation of $X(t)$, and let $n(\varepsilon)$ be a normalization such that

$$X, X_\varepsilon, n(\varepsilon)X_\varepsilon \overset{\mathcal{D}}{\longrightarrow} (X, X) \otimes \mathcal{I} \quad \text{as } \varepsilon \to 0,$$

where $\overset{\mathcal{D}}{\longrightarrow}$ denotes the convergence of all finite distributions and $\mathcal{I}$ is a white noise independent of $X$ with $E[|\mathcal{I}(t)|] = 1$; then we get theorem 2:

$$n(\varepsilon)N_\varepsilon^u(I) \overset{L^2}{\longrightarrow} L(u, I).$$

Formally,

$$L(u, I) = \lim_{\delta \to 0} (2\delta)^{-1} \times \int \{ |X(t) - u| < \delta \} dt,$$

and by Kac's formula [18], with appropriate conditions, we have

$$N_\varepsilon^u(I) = \lim_{\delta \to 0} (2\delta)^{-1} \times \int \{ |X_\varepsilon(t) - u| < \delta \} |X_\varepsilon(t)| dt.$$
Condition (2.2) implies the 2-dimensional cylindrical convergence of $n(\varepsilon)\bar{X}_e(t)$ to a white noise independent of $X$ or $X_e$. Our theorems state that, when you integrate a function of $X_e(t)$, $n(\varepsilon)|\bar{X}_e(t)|\,dt$ converges in $L^2$-norm to the Lebesgue measure.

Sections 6 and 7 present applications for a given class of processes. Theorem 4 of section 6 proves the convergence in the convolution and polygonalization cases where $X(t)$ is a stable process with independent increments. Theorem 5 of section 7 proves the convergence in the polygonalization case for a general class of Gaussian processes. Using our method, Florens-Zmirou [15] has shown the convergence in the polygonalization case for recurrent diffusions. At last, our method could be applied to the convolution of stationary Gaussian processes, but in this last case this would need stronger conditions than those of Azaïs and Florens-Zmirou [1] which are obtained by explicit calculations.

$L^2$-convergence of the polygonal approximation crossings (theorem 5) seems to be a new result even for the Brownian motion. Using results by Borodin [9] or Csörgő and Révész [12], we can define a probability space on which the polygonal approximation crossings converge strongly to the local time of another Brownian motion. In our context this implies only weak convergence of the crossings of $X_e$ to the local time of $X$ itself.

Almost sure convergence seems very hard to establish since the only calculations that can be done about crossings are moment calculations. In some cases, like convolution of particular stationary Gaussian processes, it can be proved that the $L^2$-norm between the normalized number of crossings and the local time is bounded by a power of $\varepsilon$. By the Borel–Cantelli lemma this implies almost sure convergence along any sequence $\varepsilon_n$ decreasing to zero at geometrical rate. Unfortunately, nothing is known about the variation of $N_e$ as a function of $\varepsilon$, except that in the polygonalization case we have $N^\omega_{3,n}(I) \geq N^\omega_{4,n}(I)$. Anyway it is not sufficient to control the variation of the normalized number of crossings between two terms of the almost surely converging sequence. The problem remains open even for the Brownian case.

Section 3 gives a new proof of Rice’s formulae for a given class of stochastic processes.

3. RICE’S FORMULAE

Let $Y(t), t \in \mathcal{D}$, be a stochastic process. We say that it satisfies Rice’s formulae of order $n$ for $(t_1, \ldots, t_n)$ belonging to $\mathcal{D}_n \subset \mathcal{D}^n$ iff the following three conditions hold:

(3.1) $Y(t)$ has almost surely absolutely continuous sample paths.

(3.2) For $(t_1, \ldots, t_n)$ belonging to $\mathcal{D}_n$, $Y(t_1), \ldots, Y(t_n), \dot{Y}(t_1), \ldots, \dot{Y}(t_n)$ have
a joint density and the function

\[ A_{t_1,\ldots,t_n}(y_1,\ldots,y_n) = \int_{\mathbb{R}^n} [\dot{y}_1|] \ldots [\dot{y}_n| p[Y(t_1), Y(t_2), \ldots, Y(t_n), \dot{Y}(t_1), \ldots, \dot{Y}(t_n)];\]

\[ y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n \] is continuous in \((y_1, \ldots, y_n)\); this function will be called the order \(n\) Rice function of process \(Y\).

\[(3.3)\] Let

\[ M^n[N^{u_1}(I_1), N^{u_2}(I_2), \ldots, N^{u_n}(I_n)] = \# \left\{ \prod_{i=1}^{n} [t_i \in I_i; Y(t_i) = u_i] - E \right\},\]

where \(E = \{t_1, \ldots, t_n \in \mathcal{D} \mid \exists i, j; t_i = t_j\}\). Then, for each set of levels \(u_i\) and intervals \(I_i\) such that \(\prod_{i=1}^{n} I_i \subset \mathcal{D}_n\), we have

\[ EM^n[N^{u_1}(I_1), N^{u_2}(I_2), \ldots, N^{u_n}(I_n)] = \int_{\mathbb{R}^n} A_{t_1,\ldots,t_n}(u_1, \ldots, u_n) dt_1 \ldots dt_n.\]

Let us remark that:

1. When all \(I_i\)'s and \(u_i\)'s are equal, \(E(M^n)\) is the factorial moment of order \(n\).
2. The restriction "\(t\) belong to \(\mathcal{D}_n\)" and not the whole \(\mathcal{D}\) has been introduced for two reasons. First, if \(Y\) is the polygonal approximation, \(Y\) and \(\dot{Y}\) may not have joint densities for \(|t_1 - t_2| < \varepsilon\). Second, in the same case \((|t_1 - t_2| < \varepsilon)\) we do not know how to prove the existence of joint densities for \(Y\) or \(\dot{Y}\), when \(Y\) is the convolution approximation.
3. The existence of joint densities is not really needed; it has just been supposed, for simplicity, only the existence and continuity of \(A_{t_1,\ldots,t_n}\) is needed [5].

Such formulae have been first established for a given class of Gaussian processes by Rice [21] in 1945. The non-Gaussian case has received contributions by Besson [5], Marcus [20], and Wschebor [22, 23]. These authors have given very general proofs of Rice formulae using weak but technical hypotheses. Here we give a proof for processes with not "too many" crossings — hypothesis \(K_2\) — which prevents us from studying most of the pathological cases of the above-mentioned papers.

**Proposition.** Let \(Y(t)\) satisfy the following hypotheses.

\(K_1\). Conditions (3.1) and (3.2) hold and for any compact sets \(K_1 \subset \mathcal{D}_n\) and \(K_2 \subset \mathbb{R}^n\) we have

\[ \int_{K_1 (x_1,\ldots,x_n)} \sup_{K_2} A_{t_1,\ldots,t_n}(x_1,\ldots,x_n) dt_1 \ldots dt_n < \infty.\]

\(K_2\). The moment of order \(n\) of the number of local extrema \(\hat{N}(\mathcal{D})\) of \(Y(t)\) over \(\mathcal{D}\) is finite.

Then \(Y(t)\) satisfies Rice’s formulae of order \(n\) for \((t_1, \ldots, t_n)\) belonging to \(\mathcal{D}_n\).
Remark. Hypothesis \( K_2 \) is much stronger than those of Besson, Marcus or Wschebor, but it is trivially satisfied in the polygonalization case. In the convolution case, \( K_2 \) is a consequence of lemma 1.

In the general case it must be considered as a condition of smoothness on the family \( X_x \).

Proof. Since both members of (3.3) are additive, it is sufficient to prove it for \( \text{dist} \left( \prod_{i} I_i, E \right) > 0 \). \( M^n \) is thus an ordinary product. By \( K_2 \) the number \( \hat{N} \) of local extrema of \( Y(t) \) for \( t \) belonging to any subinterval of \( \mathcal{D} \) is almost surely finite; we are allowed to use "the Kac's crossings counter" [18]: original Kac's proof can be extended without problems to absolutely continuous functions. As \( Y(t) \) is supposed to have 1-dimensional densities, it does not take — with probability 1 — any of the values \( u_i \) at extremities of intervals and is not constantly equal to any \( u_i \) on any interval. Thus

\[
\prod_{i=1}^{n} N^{u_i}(I_i) = \lim_{\delta \to 0} (2\delta)^{-n} \prod_{I_i} \left\{ |Y(t_i) - u_i| < \delta \right\} \times |\hat{Y}(t_1)| \ldots |\hat{Y}(t_n)| dt_1 \ldots dt_n \text{ a.s.}
\]

Kac's result can be improved by noting that, for every \( \delta > 0 \),

\[
(2\delta)^{-n} \prod_{I_i} \left\{ |Y(t_i) - u_i| < \delta \right\} \times |\hat{Y}(t_1)| \ldots |\hat{Y}(t_n)| dt_1 \ldots dt_n
\]

\[
\leq \prod_{i} \{N^{u_i}(I_i) + \hat{N}(I_i)\} \leq (\text{const}) \{\hat{N}(\bigcup_{i=1}^{n} I_i)\}^{n} + (\text{const}).
\]

The last inequality in (3.4) is an easy application of the Rolle Theorem. The first inequality may be proved as follows.

It is sufficient to prove it for \( n = 1 \). Let \( J = \{ t \in I : |Y(t) - u| < \delta \} \), where \( J \) is the union of finitely many open intervals:

\[
J = \bigcup_{k=1}^{K} L_k.
\]

Divide \( J \) according to local extrema of \( Y \). Since \( Y \) is monotonous within each subinterval of the partition, the integral of \( |\hat{Y}| \) over such a subinterval is less than \( 2\delta \) in the general case and less than \( \delta \) when \( L_k \) contains no crossing.

Now, by \( K_2 \) and the dominated convergence theorem we get

\[
E \left\{ \prod_{i=1}^{n} N^{u_i}(I_i) \right\}
\]

\[
= \lim_{\delta \to 0} (2\delta)^{-n} \prod_{i} \left\{ \frac{u_1 + \delta}{u_1 - \delta} \ldots \frac{u_n + \delta}{u_n - \delta} \right\} A_{\delta,\ldots,\delta}(y_1, \ldots, y_n) dy_1 \ldots dy_n dt_1 \ldots dt_n.
\]
Convergence of number of crossings

Since \( A_{t_1, \ldots, t_n} \) is continuous and dominated by \( K_1 \), a second application of the dominated convergence theorem gives

\[
E \left\{ \prod_{i=1}^{n} N^{u_i}(I_i) \right\} = \int_{I_1}^{\cdots} A_{t_1, \ldots, t_n}(u_1, \ldots, u_n) dt_1 \cdots dt_n.
\]

4. SUFFICIENT CONDITIONS FOR CONVERGENCE OF NUMBER OF CROSSINGS TO THE LOCAL TIME

**Theorem 1.** Let \( X(t), t \in \mathbb{R}, \) satisfy \( H_1 \), and let \( X_\varepsilon(t), \varepsilon \in \mathbb{R}^+, \) be a family of processes defined on the same probability space. Suppose there exists a normalization \( n(\varepsilon) \) such that \( (X, X_\varepsilon) \) satisfies the following hypotheses \( H_2, H_3, \) and \( H_4. \)

\( H_2. \) Rice formulae. (a) For every \( \varepsilon, X_\varepsilon(t) \) satisfies order two Rice's formulae for \( (t_1, t_2) \) belonging to \( |\mathbb{R}^2 - \{|t_1 - t_2| < 2\varepsilon\}|. \)

(b) For every pair of intervals \( I_1, I_2 \) with \( \text{dist}(I_1, I_2) > 2\varepsilon \) and for every level \( u \) we have

\[
E \left\{ \int_{I_1} \{ |x(t) - u| < \delta \} N^{u}(I_2) \right\} = \int_{I_1} \int_{I_2} \int_{u-\delta}^{u+\delta} \int_{\mathbb{R}} \int_{\mathbb{R}} p [X(t_1), X(t_2), \dot{X}_\varepsilon(t_2); v, u, x] d\dot{x} dv dt_1 dt_2.
\]

The last quantity will be denoted by

\[
\int_{I_1 \times I_2} B_{t_1, t_2}^\varepsilon (u - \delta, u + \delta, u) dt_1 dt_2.
\]

\( H_3. \) Convergence and domination of Rice functions. Let \( A_1^\varepsilon(u) \) \( A_{t_1, t_2}^\varepsilon(u_1, u_2) \) be the order one and two Rice functions of \( X_\varepsilon. \) We have, for all considered \( u, \)

(a) \( (n(\varepsilon))^2 A_{t_1, t_2}^\varepsilon(u, u) \rightarrow p [X(t_1), X(t_2); u, u] \) as \( \varepsilon \rightarrow 0, \) the convergence being pointwise for all different \( t_1 \) and \( t_2; \)

(b) \( \sup_{\varepsilon < |t_1 - t_2|/2} (n(\varepsilon))^2 A_{t_1, t_2}^\varepsilon(u, u) \leq H_u(t_1, t_2) \) with \( \int \int_{\mathbb{R}^2} H_u(t_1, t_2) < \infty; \)

(c) for all \( \delta > 0 \)

\[
n(\varepsilon) B_{t_1, t_2}^\varepsilon (u - \delta, u + \delta, u) \rightarrow \int_{u-\delta}^{u+\delta} \int \int_{\mathbb{R}} p [X(t_1), X(t_2); v, u] dv \quad \text{as} \quad \varepsilon \rightarrow 0,
\]

the convergence being pointwise;

(d) there exists an \( \varepsilon_0 > 0 \) such that

\[
\sup_{t \in \mathbb{R}, \varepsilon < \varepsilon_0} \{ n(\varepsilon) A_1^\varepsilon(u) \} < \infty.
\]

\( H_4. \) For each considered \( u \)

\[
\sup_{I \in \mathbb{R}, l(I) < \varepsilon} E [n(\varepsilon) N^{u}(I)]^2 = o(\varepsilon),
\]

\( l(I) \) being length of the interval \( I. \)
Then for all considered levels \( u \) and every interval \( I \), the random variables \( n(\varepsilon) N^u_\varepsilon(I) \) and
\[
(1/2\delta) \int_I 1 \{|X(t) - u| < \delta\} \, dt
\]
converge in \( L^2 \)-norm to the same limit as \( \varepsilon \) and \( \delta \) tend to zero.

If \( X(t) \) admits a.s. a space continuous local time \( L(u, I) \), then it is a.s. the limit of the last variable and we have the following

**Theorem 2.** Let \( X \) and \( X_\varepsilon \) satisfy \( H_1-H_3 \), and suppose that \( X(t) \) admits a space continuous local time \( L(u, I) \). Then, for every interval \( I \) in \( D \),
\[
n(\varepsilon) N^u_\varepsilon(I) \xrightarrow{L^2} L(u, I) \quad \text{as} \quad \varepsilon \to 0.
\]

Put
\[
\zeta^u_\varepsilon(I) = n(\varepsilon) N^u_\varepsilon(I) \quad \text{and} \quad \eta^u_\varepsilon(I) = \frac{1}{2\delta} \int_I 1 \{|X(t) - u| < \delta\} \, dt.
\]

**Remarks.**

A. Hypothesis \( H_2 \) seems very difficult to avoid, because the only tool for studying crossings numbers are Rice formulae.

B. Hypothesis \( H_3 \) seems very difficult to avoid, because in the general case it is impossible to prove the Rice formulae validity without excluding a strip around the diagonal.

C. \( n(\varepsilon) \) may depend additionally on time \( t \) — as in section 7 — and even on the level \( u \). The proof of the theorem is then the same, but the statement of conditions is more complicated. In the most general case, where \( n(\varepsilon) = n(\varepsilon, t, u) \), the main modifications are the following:

1° \( n(\varepsilon) X_\varepsilon(t) \) must be replaced, in (2.2), by \( n(\varepsilon, t, X_\varepsilon(t)) \cdot X_\varepsilon(t) \);

2° \( \zeta^u_\varepsilon(I) \) must be written as
\[
\sum_{t \in I; X_t = u} n(\varepsilon, t, u);
\]

3° \( H_3 \) must be written as \( E \{\zeta^u_\varepsilon(I)\} = o(\varepsilon) \).

**Proof of theorem 1.** It is a direct consequence of the following result:

\[(4.1) \lim_{\delta \to 0} \lim_{\varepsilon \to 0} E |\zeta^u_\varepsilon(I) - \eta^u_\varepsilon(I)|^2 = 0,\]

\[(4.2) \quad E |\zeta^u_\varepsilon(I) - \eta^u_\varepsilon(I)|^2 = E [\zeta^u_\varepsilon(I)]^2 + 2E \eta^u_\varepsilon(I) \zeta^u_\varepsilon(I) + E [\eta^u_\varepsilon(I)]^2.\]

The quantity \([\zeta^u_\varepsilon(I)]^2\) is divided in two terms:
\[
\xi_A = (n(\varepsilon))^2 \# \{(t_1, t_2) \in I^2; \; |t_1 - t_2| > 2\varepsilon; \; X_\varepsilon(t_1) = X_\varepsilon(t_2) = u\},
\]
\[
\xi_B = (n(\varepsilon))^2 \# \{(t_1, t_2) \in I^2; \; |t_1 - t_2| \leq 2\varepsilon; \; X_\varepsilon(t_1) = X_\varepsilon(t_2) = u\}.
\]

\(E(\xi_A)\) can be calculated by Rice formulae. The convergence and domination of Rice functions and the dominated convergence theorem imply

\[(4.3) \quad \lim_{\varepsilon \to 0} E(\xi_A)^2 = \lim_{\varepsilon \to 0} \int_I [ p(X(t_1), X(t_2); u, u) \, dt_1 \, dt_2].\]
$E(\xi_B)$ can be bounded as follows. Let $n$ be the integer part of $(b-a)/\varepsilon + 1)$. Divide $I$ into $n$ ordered equal intervals $I_1, \ldots, I_n$ and put $\xi_i = n(\varepsilon) N^u_x(I_i)$, $i = 1, \ldots, n$. Then

$$E(\xi_B) \leq \sum_{|i-j| \leq 3} E(\xi_i, \xi_j) \leq 7n \sup_{i=1,n} E(\xi_i^2).$$

By $H_4$, $\sup_{i=1,n} E(\xi_i^2) = o(\varepsilon)$, thus

(4.4) \hspace{1cm} E(\xi_B) \to 0.$$

The quantity $\eta^u(x)(I) \xi^2(x)(I)$ is divided into the same two parts as $[\xi^2(x)(I)]^2$:

$$\eta_A = \sum_{t_1 \in I, t_2 \in I_1} \frac{1}{2\delta} \int \int \mathbf{1}\{|X(t_2) - u| < \delta\} dt_1 dt_2,$$

$$\eta_B = \sum_{t_1 \in I, t_2 \in I_1} \frac{1}{2\delta} \int \int \mathbf{1}\{|X(t_2) - u| < \delta\} dt_1 dt_2.$$

$E(\eta_A)$ can be calculated using the “Rice like” formula $H_2$ (b); its limit in $\varepsilon$ is given through the dominated convergence theorem by $H_3$ (c) and $H_3$ (d):

$$\lim_{\varepsilon \to 0} E(\eta_A) = \int \int p[X(t_1), X(t_2); u, v] dv dt_1 dt_2,$$

and now, using $H_1$,

(4.5) \hspace{1cm} \lim_{\delta \to 0, \varepsilon \to 0} E(\eta_A) = \int \int p[X(t_1), X(t_2), u, u] dt_1 dt_2.$$

$E(\eta_B)$ can be bounded by the same proof as $E(\xi_B)$, since $H_1$ implies

$$\sup_{i=1, \ldots, n} E[\eta^u(x)(I_i)]^2 = o(\varepsilon),$$

so

(4.6) \hspace{1cm} \lim_{(\delta, \varepsilon) \to 0} E(\eta_B) = 0.$$

The expectation $E[\eta^u(x)(I)]^2$ can be calculated by the Fubini theorem:

$$E[\eta^u(x)(I)]^2 = \left(\frac{1}{2\delta}\right)^2 \int \int \int \int p[X(t_1), X(t_2); u_1, u_2] du_1 du_2 dt_1 dt_2 dt_1 dt_2,$$

and now, by $H_1$,

(4.7) \hspace{1cm} \lim_{\delta \to 0} E[\eta^u(x)(I)]^2 = \int \int p[X(t_1), X(t_2); u, u] dt_1 dt_2.$
Now, relations (4.2) to (4.7) imply (4.1).

Theorem 1 holds by noting that (4.1) implies that \( \varepsilon_{\varepsilon}^*(I) \) is a Cauchy sequence and, again by (4.1), \( \eta_{\varepsilon}^*(I) \) converges to the same limit.

5. APPROXIMATING FAMILIES WITH NOT "TOO MANY" LOCAL EXTREMA

In this section we assume that the approximating family \( \{X_\varepsilon, \varepsilon \in \mathbb{R}^+ \} \) satisfies the following hypothesis:

**H_5.** For every integer \( m \) there exists a constant \( c_m \) such that, for all intervals of size less than \( \varepsilon \) included in \( \mathcal{D} \), we have

\[
E [\tilde{N}_\varepsilon(I)]^m \leq c_m [\varepsilon^{-1} n(\varepsilon) + 1].
\]

**H_5** can be interpreted as a condition of "smoothness" of the approximating family. It is trivially met for polygonal approximation and is a consequence of lemma 1 below for convolution approximation. Note that \( H_5 \) implies that, for every \( \varepsilon \), \( X_\varepsilon \) satisfies hypothesis \( K_2 \) of the proposition of section 3 about Rice formulæ.

**Theorem 3.** The hypothesis \( H_5 \) on \( X_\varepsilon \) and the following hypothesis \( H_6 \) imply the hypotheses \( H_2, H_3, \) and \( H_4 \) of theorem 1.

- **H_6 (a) to H_6 (c) are \( H_3 \) (a) to \( H_3 \) (c).**
- **H_6 (d).** For every \( \varepsilon \) and every compact set \( K_2 \) in \( \mathbb{R}^2 \)

\[
\int \int \sup_{(t_1,t_2) \in \mathbb{R}^2} A_{t_1,t_2}^\varepsilon(x_1, x_2) dt_1 dt_2 < \infty.
\]

**H_6 (e).** There exists an \( \varepsilon_0 \) such that, for every compact \( Q_2 \subset \mathbb{R}^2 \)

\[
\sup_{\varepsilon < \varepsilon_0, (x_1, x_2) \in Q_2} n(\varepsilon) A_{t_1,t_2}^\varepsilon(x) < \infty.
\]

**H_6 (f).** \( A_{t_1,t_2}^\varepsilon, A_t^\varepsilon \) and \( B_{t_1,t_2}^\varepsilon(u - \delta, u + \delta, \cdot) \) are continuous for given \( \varepsilon \) if \( |t_1 - t_2| > 2\varepsilon \).

**H_6 (g).** There exists \( \beta > 0 \) such that \( n(\varepsilon) = O(\varepsilon^\beta) \).

Consequently, if \( X \) satisfies \( H_1 \), conclusions of theorem 1 hold.

**Proof of theorem 3.** \( H_6 \) (a) and the continuity of \( A_t^\varepsilon(t_1, t_2) \) on \( \mathcal{D}_2^\varepsilon = \{(t_1, t_2) \in \mathcal{D}^2; |t_1 - t_2| > 2\varepsilon \} \) imply \( K_1 \) of the proposition of section 3.

Since \( H_5 \) implies \( K_2 \), \( X_\varepsilon \) satisfies Rice formulæ of order 2 on \( \mathcal{D}^\varepsilon_2 \).

**B_{t_1,t_2}^\varepsilon(u - \delta, u + \delta, x) \) is less than \( A_{t_1}^\varepsilon(x) \) which is bounded by \( H_6 \) (e); the proof of the proposition implies that \( X_\varepsilon \) satisfies the "Rice like" formulæ \( H_2 \) (b). Thus \( H_2 \) hold.

**H_6 (e) is stronger than \( H_3 \) (d), thus \( H_3 \) hold.**

We now prove \( H_4 \). Let \( u \) be fixed; \( H_6 \) (e) and \( H_5 \) imply that \( X_\varepsilon \) satisfies Rice formulæ of order 1 on \( \mathcal{D} \) and that, for every \( [a, b] \subset \mathcal{D} \), every \( u \), and every \( \varepsilon < \varepsilon_0, n(\varepsilon) E(N_{u}^\varepsilon[a, b]) \leq (\text{const})(b - a). \)
Let now \((b-a) < \varepsilon\). By Bienayme’s inequality,

\[
P \left[ N^e [a, b] > 0 \right] \leq (\text{const}) \varepsilon [n(\varepsilon)]^{-1}.
\]

By the Rolle theorem,

\[
E \left[ (N^e [a, b])^2 \right] \leq E \left[ (N^e [a, b] + 1)^2 \mathbf{1} \{ N^e [a, b] > 0 \} \right].
\]

Take now \(q\) an integer and \(p\) its conjugate: \(p^{-1} + q^{-1} = 1\). Then apply the Hölder inequality. From H5 and (5.1) we get

\[
E (n(\varepsilon))^2 (N^e [a, b])^2 \leq c_q' [\varepsilon^{1/p - 1/q} n(\varepsilon)^{2 + 1/q - 1/p} + \varepsilon^{1/p} n(\varepsilon)^2]^{-1/p},
\]

where \(c_q'\) is a constant depending on \(q\).

An elementary calculation shows that \(q\) can be chosen large enough for the right-side term of (5.2) being bounded — by \(H_6\) (g) — by \(\varepsilon^{1+\delta}\) with \(\delta\) positive. This proves \(H_4\) and theorem 3.

The following lemma gives sufficient conditions for \(H_5\).

**Lemma 1.** Suppose that for every \(\varepsilon\) the process \(X_\varepsilon(t)\) verifies the following hypotheses:

(a) \(X_\varepsilon(t)\) has a.s. \(C^\infty\) sample paths;
(b) for \(t\) belonging to \(\varnothing\), \(X_\varepsilon(t)\) and \(X_\varepsilon'(t)\) have a joint density bounded by \((\text{const})\varepsilon [n(\varepsilon)]^2\), where \(n(\varepsilon)\) is a given function;
(c) for \(p > 1\) we have \(E [\sup |X_\varepsilon^{(p)}(t)|] < c_p \varepsilon^{-p}\), \(X_\varepsilon^{(p)}\) denoting the \(p\)th derivative, and \(c_p\) a constant depending on \(p\).

Then \(H_5\) holds.

In the convolution case (a) holds if \(\psi\) is \(C^\infty\), and (c) holds if \(E [\sup |X(t)|] < \infty\), where \(\varnothing^+\) is a domain larger than \(\varnothing\), but (b) has to be proved in each case.

**Proof.** Apply Besson and Wschebor’s condition [6], [24] for finiteness of moments of crossings. Consider the process \(Y_\varepsilon(t) = n(\varepsilon) X_\varepsilon(t)\).

Apply Wschebor’s [24] corollary 4, section 3.2, with \(p = 2m + 2\) to get that the order \(m\) moment of \(Y_\varepsilon\) level zero crossings over an interval of length less than 1 is bounded by \((\text{const})\varepsilon^{-1} n(\varepsilon) + (\text{const})\), where both constants depend on \(m\). Returning to \(X_\varepsilon(t)\), \(H_5\) holds.

**6. APPLICATION TO A STABLE PROCESS WITH INDEPENDENT INCREMENTS**

**Theorem 4.** Let \(X(t)\) be a stable process with independent increments and index \(\alpha > 1\) defined by

\[
X(0) = 0, \quad E \exp [i \lambda (X(a) - X(b))] = \exp \{ -(b-a) s(\lambda) \},
\]

where \(s(\lambda) = r [1 - i \text{sign}(\lambda) \beta \arctan(\pi a/2)|\lambda|^\alpha] \), \(r \in \mathbb{R}^+, \beta \in [-1, 1], \alpha \in (1, 2)\) (see [14]).

When \(\alpha = 2\) and \(\beta = 0\), \(X(t)\) is a Brownian motion.
Let $\psi$ be a $C^\infty$ function with support in $[-1/2, 1/2]$, $\psi_\varepsilon(t) = (1/\varepsilon)\psi(t/\varepsilon)$ and $X_\varepsilon$ obtained by convolution with $\psi_\varepsilon$. Then for every level $u$ and every interval $I$ included in $\mathcal{D} = [a_0, b_0]$, with $a_0 > 0$, we have

$$ (E[X(1)])^{-1} \|\psi\|^{-1}_a \varepsilon^{-1} N_\varepsilon(u, I) \overset{L^2}{\to} L(u, I). $$

Let $X_\Delta$ be the size $\Delta$ polygonal approximation of $X$ and $N_\Delta$ the number of crossings of $X_\Delta$. Then

$$ (E[X(1)])^{-1} \Delta^{-1/\alpha} N_\Delta(u, I) \overset{L^2}{\to} L(u, I). $$

**Proof.** Boylan [10] has shown that such processes admit a bicontinuous local time $L(u, I)$. We know that $X(1)$ has a bounded continuous density and that $X(\lambda t)$ has the same distribution as $\lambda^{1/\alpha} X(t)$, hence $H_1$ is met. We give the proof only for $X_\varepsilon$, which is the more complicated case, the proof for $X_\Delta$ being very similar except that we do not use lemma 1. The following lemma gives the $n$-dimensional distribution of $(X_\varepsilon(t), \dot{X}(t))$:

**Lemma 2.** Let $g$ be a continuous function with compact support in $\mathbb{R}^+$ and let

$$ G(t) = \int_u^t g(t) \, dt, \quad Y = \int_{\mathbb{R}^+} g(t) X(t) \, dt. $$

Then

$$ E[e^{sY}] = \exp[-\int_{\mathbb{R}} s[G(t)] \, dt]. \quad (6.1) $$

**Proof.** Since $gX$ is right continuous with left limits, $\int g(t) X(t) \, dt$ is equal to the limit of its Riemann's sums, so it suffices to prove (6.1) for $g(t)$ being a sum of "Dirac functions". In this case, by a triangular operation (which is in fact a hidden integration by parts), $\int g(t) X(t) \, dt$ can be written as a linear combination of increments of $X$ over disjoint intervals. Since these increments are independent, both members of (6.1) are additive and it suffices to prove this for $g(t)$ defining an increment over an interval, say $(a, b]$. In this case $g(t) = \delta_\varepsilon - \delta_a$, $\delta$ being the Dirac distribution, $G(t) = 1 \{|a, b]\}$, and (6.1) is trivial.

**Continuation of the proof of Theorem 4.** $\varepsilon$ will be supposed to be less than $a_0$. Take $g(t) = \sum \lambda_i \psi_\varepsilon(t_i - t) + \sum \mu_i \psi_\varepsilon(t_i + t)$. Then lemma 2 gives the characteristic function of $X_\varepsilon(t_1), \ldots, X_\varepsilon(t_n)$, $\dot{X}_\varepsilon(t_1), \ldots, \dot{X}_\varepsilon(t_n)$.

If every two $t_i$'s differ from each other more than $2\varepsilon$, we can see that this characteristic function is integrable over $R$; thus the density of the $2n$-tuple exists, is bounded and continuous. From the expression of the characteristic function we can also see that (2.2) holds with

$$ n(\varepsilon) = (E[X(1)])^{-1} \|\psi\|^{-1}_a \varepsilon^{-1} N_\varepsilon(u, I). \quad (6.2) $$

We prove now that the hypotheses of lemma 1 are met.

(a) is trivial.

(b) comes from the fact that $(\dot{X}_\varepsilon(t), \dot{X}_\varepsilon(t))$ has the same law as $\varepsilon^{1/\alpha - 1} \dot{X}_1(t),\varepsilon^{1/\alpha - 2} \ddot{X}_1(t)$, which has a bounded density by lemma 2.
(c) $X(t) - tE[X(1)]$ is a martingale with finite order $\beta$ ($1 < \beta < \infty$) moments. Apply the Doob inequalities [13] to get

$$E[ \sup_{t \in [0,b_0+a_0]} |X(t)|] < \infty.$$ 

Since $\varepsilon < a_0$, (c) follows and $H_5$ is proved.

We prove now that the hypotheses of theorem 3 are met. (6.2) implies $H_6 (g)$.

$X(t)$ is a Markov process. Let $p$ denote densities for the distribution $X(0) = z$ and define

$\varepsilon B_t^\varepsilon = n(\varepsilon) \int |\dot{x}| \sup_{\varepsilon} \int p [X_\varepsilon(t), \dot{X}_\varepsilon(t); x, \dot{x}] dx.$

Then

$$\sup_{x \in [0, \varepsilon]} \varepsilon B_t^\varepsilon \leq (\text{const}) t^{-1/\alpha}.$$ 

To prove (6.3) use a conditioning by $X(t/2)$:

$\varepsilon B_t^\varepsilon = \int |\dot{x}| \sup_{\varepsilon} \int p [X_\varepsilon(t/2), w] \int p [X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x, \dot{x}] dw dx.$

$\varepsilon p [X(t/2), w]$ is bounded by $(\text{const} t/2)^{-1/\alpha}$.

Since $X$ has independent increments,

$\int p [X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x, \dot{x}] = \int p [X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x-w, \dot{x}],$

we obtain

$\varepsilon B_t^\varepsilon \leq (\text{const} t^{-1/\alpha} n(\varepsilon) \int |\dot{x}| \sup_{\varepsilon} \int p [X_\varepsilon(t/2), \dot{X}_\varepsilon(t/2); x-w, \dot{x}] dw dx.$

Noting that the integral over $w$ does not depend on $x$, we get

$\varepsilon B_t^\varepsilon \leq (\text{const} t^{-1/\alpha} n(\varepsilon) E[\dot{X}_\varepsilon(t/2)]) = (\text{const} t^{-1/\alpha}).$

Using the Markov property, relation (6.3) gives all the required estimations:

$H_6 (c)$ with $H_u(t_1, t_2) = (\text{const}) a_0^{-1/\alpha} |t_1 - t_2|^{-1/\alpha};$

$H_6 (d)$ and $H_6 (e).$

(6.3) implies also the continuity of Rice functions in $H_6 (f)$ by the dominated convergence theorem.

**Convergence of normalized Rice functions.** A change of variable gives

$$[n(\varepsilon)]^2 A_{t_1,t_2}^\varepsilon (x_1, x_2) \rho,$$

$$= \int |\dot{x}_1 \dot{x}_2| p [X_\varepsilon(t_1), X_\varepsilon(t_2), n(\varepsilon) \dot{X}_\varepsilon(t_1), n(\varepsilon) \dot{X}_\varepsilon(t_2); x_1, x_2, \dot{x}_1, \dot{x}_2] dx_1 dx_2.$$ 

As soon as $\varepsilon \leq \inf (t_1/2, (t_1-t_2)/3)$, conditioning by $X(t_1/2)$ and the increments of $X$ over $[(2t_1+t_2)/3, (t_1+2t_2)/3]$ gives
\[ [n(e)]^2 A^t_{1,2}(x_1, x_2) = \int \int \left[ \hat{X}_e(t_1', t_2) n(e) \hat{X}_e(t_1'), n(e) \hat{X}_e(t_2') \right] \left[ X_e(t_1), X_e(t_2) \right] p[X_e(t_1'), X_e(t_2); x_1 - z_1, x_2 - z_2, x_1, x_2] dz_1 dz_2 dx_1 dx_2 \]

where \( a = t_1/2, b = t_1/2 + (t_2 - t_1)/3, t_1' = t_1 - a, \) and \( t_2' = t_2 - b. \)

Now \( p[X(a), X(b); \ldots] \) is bounded and continuous, and the quadruplet \((X_e(t_1'), X_e(t_2'), n(e) \hat{X}_e(t_1'), n(e) \hat{X}_e(t_2'))\) converges in distribution, but \(|\hat{x}_1'| |\hat{x}_2'|\) is not bounded. Noting that \([n(e) \hat{X}_e(t_1'), n(e) \hat{X}_e(t_2')]\) has the same distribution as \(\mathcal{G} \otimes \mathcal{G}\), we get for \(1 < \gamma < \alpha\)

\[
sup_{x} E |n(e) \hat{x}_e(t_1') n(e) \hat{x}_e(t_2')|^{\gamma} < \infty,
\]

and now, by a uniform integrability argument [7], the limit of (6.4) is equal to

\[
\int \int \left[ p[X(a), X(b); z_1, z_2] p[X(t_1'), X(t_2'); x_1 - z_1, x_2 - z_2] dz_1 dz_2
\]

\[= p[X(t_1), X(t_2); x_1, x_2].\]

Convergence of \(n(e) B^t_{1,2}(u - \delta, u + \delta, u)\) follows by a similar proof. \(H_5\) (a) and \(H_6\) (c), and thus theorem 4, are proved.

7. POLYGONAL APPROXIMATION OF GAUSSIAN PROCESSES

In this section we prove the following

**Theorem 5.** Let \(X(t)\) be a zero-mean Gaussian process with covariance function \(r(t, s).\) We suppose that \(r\) is twice continuously differentiable outside the diagonal \((t = s)\) and satisfies the following hypotheses on a compact set \(\mathcal{D}.

There exist \(\alpha < 2, \beta > \alpha/2\) and \(d\) positive such that, for \(|t - s| < d,

\[
(7.1) \quad r(s, s) r(t, t) - r^2(s, t) > (\text{const}) |t - s|^\alpha,
\]

\[
(7.2) \quad |r(s, t) - r(t, t)| < (\text{const}) |t - s|^{\beta}.
\]

Condition (7.1) implies that \(X(t)\) is non-differentiable in quadratic mean [11].

Outside the diagonal the process satisfies a non-determinism condition:

\[
(7.3) \quad \inf_{|t - s| > d} \{r(s, s) r(t, t) - r^2(s, t)\} = \epsilon > 0.
\]

The variance is bounded:

\[
(7.4) \quad 0 < \inf_{t \in \mathcal{D}} r(t, t) \leq \sup_{t \in \mathcal{D}} r(t, t) < \infty.
\]

Let \(r_\Delta\) be the covariance function of the size \(-\Delta\) polygonal approximation. There exists a constant \(f\) such that in a strip \(|t - s| < (\text{const})\) we have, for \(\Delta < |t - s|/f,

\[
(7.5) \quad r_\Delta(t, t) r_\Delta(s, s) - r_\Delta^2(t, s) > (\text{const}) |t - s|^{\gamma} \quad \text{with} \quad \gamma < 2.
\]
Then the number of crossings over \([a, b] \subset \mathcal{D}\) of the size \(-\Delta\) polygonal approximation \(X(t)\), normalized by

\[
(7.6) \quad n(\Delta, t) = \sqrt{\pi/2} \Delta \left[ r(k\Delta, k\Delta) + r((k + 1)\Delta, (k + 1)\Delta) - 2r(k\Delta, (k + 1)\Delta) \right]^{-1/2}
\]

for \(k\Delta \leq t < (k + 1)\Delta\)

converges in \(L^2\)-norm to the same limit as \(\eta_0^a[a, b]\).

Proof. Let \(r'\) denote the derivative with respect to the second variable and \(r''\) the cross second derivative.

Put \(t_1 = k\Delta + s_1\Delta\) (0 \(s_1 < 1\)), \(t_2 = h\Delta + s_2\Delta\) (0 \(s_2 < 1\)) and suppose that \(\Delta \leq d\). Then

\[
r_d(t_1, t_2) = (1 - s_1)(1 - s_2)r(k\Delta, h\Delta) + (1 - s_1)s_2r(k\Delta, (h + 1)\Delta) +
+ (1 - s_2)s_1r((k + 1)\Delta, h\Delta) + s_1s_2r((k + 1)\Delta, (h + 1)\Delta).
\]

Differentiation gives:

\[
r_d'(t_1, t_2) = E(\tilde{X}_d(t_1)\tilde{X}_d(t_2)) = E(\tilde{X}_d(t_1)\tilde{X}_d(t_2)) =
= A^{-1}\left\{(1 - s_1)[r(k\Delta, (h + 1)\Delta) - r(k\Delta, h\Delta)] +
+ s_1[r((k + 1)\Delta, (h + 1)\Delta) - r((k + 1)\Delta, h\Delta)]\right\},
\]

\[
r_d''(t_1, t_2) = E(\tilde{X}_d(t_1)\tilde{X}_d(t_2)) =
= A^{-2}[r(k\Delta, h\Delta) + r((k + 1)\Delta, (h + 1)\Delta) - r(k\Delta, (h + 1)\Delta) - r((k + 1)\Delta, h\Delta)].
\]

The convergence of the 2-dimensional distribution (2.2) is obtained with the normalization \(\sqrt{\pi/2} \cdot r_d(t, t)\), which explains (7.6).

Boundedness and continuity of Rice functions. As soon as \(\Delta \leq |t - s|/f\), the distribution of \(X_d(t_1)\) and \(X_d(t_2)\) is non-degenerated and the Rice functions are trivially continuous. Thus \(H_6\) (f) holds by putting for example \(\varepsilon = f/\Delta/2\).

Conditions \(H_6\) (d) and \(H_6\) (e) are easy to prove.

We turn now to give bounds to \(A_{11,12}(u, u)\) (hypothesis \(H_6\) (b)):

\[
A_{11,12} = E([\tilde{X}_d(t_1)\tilde{X}_d(t_2)]/X_d(t_1) = u; X_d(t_2) = u) p[X_d(t_1), X_d(t_2); u, u].
\]

By the Schwarz inequality we see that in order to prove \(H_6\) (b) we have to bound

\[
E[[\tilde{X}_d(t_1)]^2/X_d(t_1) = X_d(t_2) = u] p[X_d(t_1), X_d(t_2); u, u]
= \{\text{Var}[\tilde{X}_d(t_1)/X_d(t_1), X_d(t_2)] + [E[\tilde{X}_d(t_1)/X_d(t_1) = X_d(t_2) = u]]^2\} \times
\times p[X_d(t_1), X(t_2); u, u].
\]

Let \(t_1, t_2\) and \(\Delta\) be such that (7.5) holds. We have

\[
\text{Var}[\tilde{X}_d(t_1)/X_d(t_1), X_d(t_2)] \leq \text{Var}[\tilde{X}_d(t_1)] \leq (\text{const})[n(\Delta, t_1)]^{-2}
\]
and

\[ E [\dot{X}_d(t_1)/X_d(t_1) = X_d(t_2) = u]^2 = a^2 u^2. \]

Put \((Y, S) = (\dot{X}_d(t_1), X_d(t_1) + X_d(t_2))\) under the conditional distribution \(X_d(t_1) - X_d(t_2) = 0\). Then \(\text{Var}(Y) \leq (\text{const})(n(A, t_1))^{-2}\).

Let \(\{\sigma_{ij}; i = 1, 2; j = 1, 2\}\) be the variance matrix of \(X_d(t_1)\) and \(X_d(t_2)\).

Then

\[ \text{Var}(S) = 4 \frac{\sigma_{11} \sigma_{22} - \sigma_{12}^2}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}. \]

Now

\[ a^2 = \left[ \frac{\text{Cov}(Y, S/2)}{\text{Var}(S/2)} \right]^2 \leq \frac{4 \text{Var}(Y)}{\text{Var}(S)}; \]

then

\[ a^2 u^2 p [X_d(t_1), X_d(t_2); u, u] \]

\[ \leq (\text{const})(n(A, t_1))^{-2} u^2 \frac{\sigma_{11} + \sigma_{22} - 2\sigma_{12}}{[\sigma_{11} \sigma_{22} - \sigma_{12}^2]^{3/2}} \exp \left[ - \frac{u^2 \sigma_{11} + \sigma_{22} - 2\sigma_{12}}{2 \sigma_{11} \sigma_{22} - \sigma_{12}^2} \right] \]

\[ \leq (\text{const})(n(A, t_1))^{-2} |t_1 - t_2|^{-n/2} \]

because of (7.5) and the fact that \(ze^{-z^2/2}\) is bounded. Gathering the pieces we get

\[ E [\dot{X}_d(t_1)^2/X_d(t_1) = X_d(t_2) = u]^2 p [X_d(t_1), X_d(t_2); u, u] \]

\[ \leq (\text{const})(n(A, t_1))^{-2} (t_1 - t_2)^{-n/2}, \]

which proves \(H_6\) (b) with \(\varepsilon = (fA)/2\) and \(H_6(t_1, t_2) = (\text{const})|t_1 - t_2|^{-n/2}\).

We turn now to the proof of the convergence of normalized Rice functions (hypotheses \(H_6\) (a) and \(H_6\) (c)). Formulae (7.2) and (7.7) imply

\[ r_d'(t_1, t_1) = E(\dot{X}_d(t_1), \dot{X}_d(t_1)) < (\text{const})A^{\beta - 1}. \]

Let \(t_1\) and \(t_2\) be any time points. Suppose that \(A\) is less than \((t_1 - t_2)/2\),

\[ r_d'(t_1, t_2) = (1-s) r'(kA, \xi) + s r'(k+1)A, \xi \]

with \(s \in [hA, (h+1)A]; \xi \in [kA, (k+1)A]\),

but the derivative \(r'\) is bounded on the compact set \{\(s, t \in \mathcal{D}; |s-t| > (t_1 - t_2)/2\}\), so \(r_d'(t_1, t_2)\) is bounded by a constant as \(A\) varies.

We have \(r_d''(t_1, t_2) = r''(\xi_1, \xi_2), \xi_1 \in [kA, (k+1)A], \xi_2 \in [hA, (h+1)A]\), so that \(r_d''(t_1, t_2)\) is bounded by a constant as a function of \(A\). On the other hand, (7.4) and (7.2) imply that \(r_d\) converges uniformly to \(r\).

The relations above imply the convergence of the variance matrix of \(X_d(t_1), X_d(t_2), n(A, t_1) \dot{X}_d(t_1), n(A, t_2) \dot{X}_d(t_2)\) to the limit of (2.2). This implies \(H_6\) (a). \(H_6\) (c) follows by a similar proof, and theorem 5 is proved.

The following are two examples of processes satisfying the hypotheses of theorem 5.
Convergence of number of crossings

1. The fractional Wiener process defined by

\[ r(t, s) = \frac{|t|^\alpha + |s|^\alpha - |t-s|^\alpha}{2} \quad \text{with } 0 < \alpha < 2. \]

The limit is actually the local time.
2. Any stationary process with

\[ r(t) = 1 - |t|^\alpha + O(|t|^\alpha) \quad \text{and} \quad 0 < \alpha < 2. \]

If the process satisfies the hypothesis of theorem 7.1 of [3] the limit is the local time again.

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