MATHEMATICAL EXPECTATION AND MARTINGALES
OF RANDOM SUBSETS OF A METRIC SPACE

BY

WOJIECH HERER (WARSZAWA)

Abstract. Let $F$ be a closed, bounded, non-empty random subset of a metric space $(X, \varrho)$. For some class of metric spaces we define in terms of the metric $\varrho$ (developing an idea of S. Doss) mathematical expectation and conditional mathematical expectation of $F$. We then consider martingales of random subsets of a metric space and prove theorems of convergence for such martingales.

0. Introduction and preliminaries. S. Doss has introduced in [3] a concept of mathematical expectation of a random variable with values in a metric space (see also [1] and [4]). This and other concepts of mathematical expectation were studied by M. Fréchet in [5] and [6].

In this paper we develop an idea of S. Doss and investigate notions of mathematical expectation (Section 1), conditional mathematical expectation (Section 2) and martingale (Section 3) of random subsets of a metric space.

Results of this paper were announced in [8] and [9].

Let $(X, \varrho)$ be a metric space. By $(\hat{X}, \hat{\varrho})$ we denote the metric space of closed, bounded and non-empty subsets of $X$, equipped with the Hausdorff metric $\hat{\varrho}$ defined as

$$\hat{\varrho}(F, F') = \max \{\sup_{x \in F} \varrho(x, F'), \sup_{x' \in F'} \varrho(x', F)\},$$

where $\varrho(x, F) = \inf \{\varrho(x, y) : y \in F\}$ for $x \in X$ and $F \in \hat{X}$.

We put

$$F = \operatorname{Lim} F_n \quad \text{iff} \quad \lim_{n} \hat{\varrho}(F_n, F) = 0.$$ 

For $x \in X$ and $F \in \hat{X}$, we set

$$\varrho(x, F) = \sup \{\varrho(x, y) : y \in F\}.$$

A metric space $(X, \varrho)$ is called finitely compact iff every closed bounded subset of $X$ is compact. Let us note the following known
PROPOSITION 0.1 ([11], Proposition 1.2.5). Let \((X, \rho)\) be a finitely compact metric space and let \(\{F_n\}_{n=1}^\infty\) be a sequence of set elements of \(\mathcal{X}\) such that \(\bigcup_{n=1}^\infty F_n\) is a bounded subset of \(X\). Suppose there exists a dense set \(D \subset X\) such that for every \(x \in D\) the limit \(\lim_{n \to \infty} \rho(x, F_n)\) exists and is finite. Then the sequence \(\{F_n\}_{n=1}^\infty\) converges in \((\mathcal{X}, \rho)\). 

Let \((\Omega, \mathcal{A}, P)\) be a probability space. An event \(A \in \mathcal{A}\) is called negligible iff \(P(A) = 0\). For a collection \(\mathcal{B}\) of subsets of \(\Omega\) we denote by \(\sigma(\mathcal{B})\) the \(\sigma\)-field generated by \(\mathcal{B}\).

A Borel measurable map \(F: \Omega \to \mathcal{X}\) is called an \(\mathcal{X}\)-valued random set (r.s.) and a Borel measurable map \(f: \Omega \to X\) is called an \(X\)-valued random variable (r.v.). We shall frequently identify a random variable \(f\) with a random set \(\{f\}\). An r.s. is called scalarly integrable iff

\[
\int_{\Omega} \delta(x, F(\omega)) dP(\omega) < \infty \quad \text{for every } x \in X.
\]

Throughout this paper \((\Omega, \mathcal{A}, P)\) will be a fixed complete, non-atomic probability space and all random sets will be defined on \((\Omega, \mathcal{A}, P)\).

1. Mathematical expectation.

DEFINITION 1.1. Let \((X, \rho)\) be a metric space and \(F\) an \(\mathcal{X}\)-valued random set. The set \(E[F]\) defined as

\[
E[F] = \{a \in X: \rho(x, a) \leq \int_{\Omega} \delta(x, F(\omega)) dP(\omega) < \infty, \forall x \in X\}
\]

is called a mathematical expectation of \(F\).

For every \(\mathcal{X}\)-valued r.s. \(F\) the set \(E[F]\) is evidently closed. If \(F\) is scalarly integrable, then the set \(E[F]\) is also bounded.

We shall state now the condition imposed on a metric space \((X, \rho)\) in order that for every \(X\)-valued r.s. \(F\) the set \(E[F]\) is non-empty.

DEFINITION 1.2. A metric space \((X, \rho)\) is called convex in the sense of Doss (or \(D\)-convex) iff for any two elements \(x_1, x_2 \in X\) there exists an element \(a \in X\) such that

\[
\rho(x, a) \leq \frac{1}{2}[\rho(x, x_1) + \rho(x, x_2)], \quad \forall x \in X.
\]

Remark 1.1. It is easily checked that every \(D\)-convex metric space is metrically convex in the sense of Menger (see [2], Definition 14.1) but not conversely (e.g., a circle in the Euclidean plane with an arc metric).

Remark 1.2. In ([7], Section 8) the authors have proved that the hyperbolic plane (of Lobachevski) equipped with the geodesic metric is a \(D\)-convex metric space (it can be proved that any simply connected Riemannian manifold of non-positive curvature equipped with geodesic metric is a \(D\)-convex metric space).
Remark 1.3. Suppose $(Y, \| \cdot \|)$ is a Banach space and $q(x, y) = \|x - y\|$ for $x, y \in Y$. Then for every $Y$-valued Bochner-integrable random variable the Bochner integral $\int_{\Omega} f(\omega) dP(\omega) \in E[f]$.

Doss has proved in ([3], Théorème 1) that

$$E[f] = \left\{ \int_{\Omega} f dP \right\} \text{ if dim } Y = 1.$$  

In ([7], Theorem 1) the authors have proved (answering the question of Fréchet [6]) that $E[f] = \left\{ \int_{\Omega} f dP \right\}$ for any two-valued random variable $f$.

Remark 1.4. Suppose $X$ is a closed, bounded, convex subset of a Banach space $Y$. Then the metric space $(X, q)$ is $D$-convex and the Bochner integral $\int_{\Omega} f dP \in E[f]$ for any Bochner-integrable $X$-valued random variable $f$.

The following example shows that the Bochner integral is not necessarily the only element of $E[f]$.

**Example 1.1.** Let

$$X = \{[\alpha_1, \alpha_2]: \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 \leq 1\}$$

and

$$q([\alpha_1, \alpha_2], [\beta_1, \beta_2]) = |\alpha_1 - \beta_1| + |\alpha_2 - \beta_2|.$$  

Let $f$ be an $X$-valued r.v. satisfying

$$P(f = [1, 0]) = P(f = [0, 1]) = \frac{1}{2}.$$  

One checks easily that $E[f] = \{[\alpha_1, \alpha_2] \in X: \alpha_1 = \alpha_2\}$. 

**Theorem 1.1.** Let $(X, q)$ be a finitely compact metric space. Then for every $X$-valued random set $F$ the set $E[F]$ is non-empty iff $(X, q)$ is a $D$-convex metric space.

**Proof.** The necessity of $D$-convexity of a metric space $(X, q)$ is evident, since “$D$-convexity” means precisely that for every $X$-valued r.v. $f$ satisfying $P(f = x_1) = P(f = x_2) = \frac{1}{2}$, one has $E[f] \neq \emptyset$.

We shall prove now that if a metric space $(X, q)$ is $D$-convex, then for every $X$-valued r.s. $F$ the set $E[F]$ is non-empty.

If a random set $F$ is not scalarly integrable, then $E[F] = X \neq \emptyset$.

If $F$ is a scalarly integrable r.s., then any measurable selection $f$ of $F$ (which always exists by [10]) is a scalarly integrable r.v. and $E[f] \subset E[F]$.

It is thus sufficient to prove that if a metric space $(X, q)$ is $D$-convex, then for every scalarly integrable $X$-valued r.v. $f$ the set $E[f]$ is non-empty.

This will be proved in several steps.

(1) If $f$ is an $X$-valued r.v. with $\text{card} f(\Omega) \leq 2$, then $E[f] \neq \emptyset$. 

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We have to prove that for every \( x_1, x_2 \in X \) and any \( p \in [0, 1] \) there is an element \( a \in X \) such that

\[
g(x, a) \leq pg(x, x_1) + (1 - p)g(x, x_2), \quad \forall x \in X.
\]

We shall prove this first for dyadic rationals \( p = k/2^n \) \((k = 1, \ldots, 2^n; n = 1, 2, \ldots)\). We shall proceed by induction; for \( n = 1 \) our statement is true by the definition of D-convexity of a metric space \((X, g)\). If for some \( n \geq 1 \) and \( 1 \leq k, l \leq 2^n \) one has

\[
g(x, a) \leq \frac{k}{2^n}g(x, x_1) + \left(1 - \frac{k}{2^n}\right)g(x, x_2), \quad \forall x \in X,
\]

and

\[
g(x, b) \leq \frac{l}{2^n}g(x, x_1) + \left(1 - \frac{l}{2^n}\right)g(x, x_2), \quad \forall x \in X,
\]

then there exists \( c \in X \) such that

\[
g(x, c) \leq \frac{1}{2}[g(x, a) + g(x, b)] = \frac{k + l}{2^{n+1}}g(x, x_1) + \left(1 - \frac{k + l}{2^{n+1}}\right)g(x, x_2),
\]

which completes the induction.

Let \( p \in [0, 1] \) be arbitrary and let \( \{p_n\}_{n=1}^{\infty} \) be a sequence of dyadic rationals converging to \( p \). For every \( n = 1, 2, \ldots \) there are elements \( a_n \in X \) such that

\[
g(x, a_n) \leq p_ng(x, x_1) + (1 - p_n)g(x, x_2), \quad \forall x \in X.
\]

Since the sequence \( \{a_n\}_{n=1}^{\infty} \) is bounded and \((X, g)\) is finitely compact, we can extract from \( \{a_n\}_{n=1}^{\infty} \) a subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \) converging to some \( a \in X \). One evidently has

\[
g(x, a) \leq pg(x, x_1) + (1 - p)g(x, x_2), \quad \forall x \in X.
\]

(2°) If \( f \) is an \( X \)-valued r.v. with \( f(\Omega) \) finite, then \( E[f] \neq \emptyset \).

We shall proceed by induction. By (1°), our statement is true if \( \text{card } f(\Omega) \leq 2 \).

Let \( f(\Omega) = \{x_1, x_2, \ldots, x_n, x_{n+1}\} \) and \( P(f = x_i) = p_i \) for \( i = 1, 2, \ldots, n+1 \).

Let us consider the r.v. distributed as follows:

\[
P(g = x_i) = p_i \sum_{j=1}^{n} p_j \quad \text{for } i = 1, 2, \ldots, n.
\]

Supposing that (2°) is true for an \( n \)-valued r.v. \( g \) we have \( E[g] \neq \emptyset \). Let \( a \in E[g] \) and let us consider the r.v. \( h \) distributed as follows:

\[
P(h = a) = \sum_{j=1}^{n} p_j, \quad P(h = x_{n+1}) = p_{n+1}.
\]

Then by (1°) we infer that \( E[h] \neq \emptyset \). It is easily checked that \( E[h] \subset E[f] \), and thus \( E[f] \neq \emptyset \).
(3') If \( f \) is an \( X \)-valued scalarly integrable r.v. with \( f(\Omega) \) countable, then 
\( E[f] \neq \emptyset \).

Let \( f(\Omega) = \{x_1, x_2, \ldots\} \) and \( P(f = x_i) = p_i \) for \( i = 1, 2, \ldots \) By (2'), for every \( n = 1, 2, \ldots \) there are elements \( a_n \in X \) such that
\[
q(x, a_n) \leq \sum_{i=1}^{n} q(x, x_i) q_i^n, \quad \forall x \in X,
\]
where
\[
q_i^n = p_i / \sum_{j=1}^{n} p_j, \quad i = 1, 2, \ldots, n.
\]

For every \( x \in X \) we have
\[
\lim_{n} \sum_{i=1}^{n} q(x, x_i) q_i^n = \sum_{i=1}^{\infty} q(x, x_i) p_i < \infty.
\]
This implies that the sequence \( \{a_n\}_{n=1}^{\infty} \) is bounded, and since the metric space 
\( (X, \rho) \) is finitely compact, we can extract from \( \{a_n\}_{n=1}^{\infty} \) a subsequence \( \{a_{k_n}\}_{n=1}^{\infty} \)
convergent to some \( a \in X \). Then for every \( x \in X \) we have
\[
q(x, a) = \lim_{n} q(x, a_{k_n}) \leq \lim_{n} \sum_{i=1}^{k_n} q(x, x_i) q_i^{k_n} = \sum_{i=1}^{\infty} q(x, x_i) p_i,
\]
which means that \( a \in E[f] \).

(4') If \( f \) is an arbitrary scalarly integrable \( X \)-valued r.v., then \( E[f] \neq \emptyset \).

Since the metric space \( (X, \rho) \) is separable, for every \( n = 1, 2, \ldots \) there exists an 
\( X \)-valued r.v. \( f_n \) such that \( f_n(\Omega) \) is countable and
\[
\rho(f_n(\omega), f(\omega)) \leq 1/n, \quad \forall \omega \in \Omega,
\]
which implies that
\[
\rho(x, f_n(\omega)) \leq \rho(x, f(\omega)) + 1/n, \quad \forall \omega \in \Omega, \ \forall x \in X, \ \forall n \geq 1.
\]
By (3') there exists a sequence \( \{a_n\}_{n=1}^{\infty} \) of elements of \( X \) such that
\[
\rho(x, a_n) \leq \int_{\Omega} \rho(x, f_n) dP \leq \int_{\Omega} \rho(x, f) dP + 1/n, \quad \forall x \in X, \ \forall n \geq 1.
\]
Thus the sequence \( \{a_n\}_{n=1}^{\infty} \) is bounded and we can extract from it a subsequence \( \{a_{k_n}\}_{n=1}^{\infty} \)
convergent to some \( a \in X \). Since
\[
\rho(x, f_n(\omega)) \leq \rho(x, f(\omega)) + 1/n \quad \text{for} \ x \in X, \ \omega \in \Omega, \ n = 1, 2, \ldots,
\]
by Lebesgue's bounded convergence theorem we have
\[
\rho(x, a) = \lim_{n} \rho(x, a_{k_n}) \leq \lim_{n} \int_{\Omega} \rho(x, f_n) dP = \int_{\Omega} \rho(x, f) dP, \quad \forall x \in X,
\]
which means that \( a \in E[f] \).
2. Conditional mathematical expectation. Throughout this section we shall assume that \((X, g)\) is a finitely compact, \(D\)-convex metric space and \(F\) is an \(X\)-valued scalarly integrable random set.

Suppose \(\mathcal{F}\) is a finite subfield of \(\mathcal{A}\) with non-negligible atoms (throughout this paper we shall always assume that finite subfields of \(\mathcal{A}\) have non-negligible atoms). Let us define the following random set:

\[
E^{\mathcal{F}}[F](\omega) = E[F|A] \quad \text{for } \omega \in A, \text{ an atom of } \mathcal{F},
\]

where

\[
E[F|A] = \left\{ a \in X : \quad g(x, a) \leq \frac{1}{P(A)} \int \delta(x, F) dP, \quad \forall x \in X \right\}.
\]

**Lemma 2.1.** Let \(\{\mathcal{F}_n\}_{n=1}^\infty\) be an increasing sequence of finite subfields of \(\mathcal{A}\). Then:

1. \(\bigcup_{n=1}^{\infty} E^{\mathcal{F}_n}[F](\omega)\) is a bounded subset of \(X\) for almost every \(\omega \in \Omega\).
2. For every \(x \in X\) the sequence of reals \(\{g(x, E^{\mathcal{F}_n}[F](\omega))\}_{n=1}^\infty\) converges to a finite limit for almost every \(\omega \in \Omega\).

**Proof.** (1) Let \(x\) be some element of \(X\). For every \(\omega \in \Omega\) and every \(a \in E^{\mathcal{F}_n}[F](\omega)\) we have

\[
g(x, a) \leq \frac{1}{P(A_n)} \int \delta(x, F) dP, \quad \text{where } \omega \in A_n, \text{ an atom of } \mathcal{F}_n.
\]

The real martingale

\[
\left\{ \frac{1}{P(A_n)} \delta(x, F) dP, \mathcal{F}_n \right\}_{n=1}^\infty
\]

converges almost surely to a finite limit ([12], Proposition II - 2 - 11). Thus for almost every \(\omega \in \Omega\)

\[
\sup_n \sup_{a \in F_n} g(x, a) < \infty, \quad \text{where } F_n = E^{\mathcal{F}_n}[F](\omega),
\]

which proves that the set \(\bigcup_{n=1}^{\infty} E^{\mathcal{F}_n}[F](\omega)\) is bounded for almost every \(\omega \in \Omega\).

(2) Let \(x\) be some element of \(X\). Denote by \(\{\xi_n\}_{n=1}^\infty\) the sequence of real random variables defined as

\[
\xi_n(\omega) = g(x, E^{\mathcal{F}_n}[F](\omega)) \quad \text{for } \omega \in \Omega \quad (n = 1, 2, \ldots).
\]

It is sufficient to prove that \(\{\xi_n, \mathcal{F}_n\}_{n=1}^\infty\) is a submartingale satisfying Doob's condition ([12], Theorem IV.1.2):

\[
\sup_n \int_{\Omega} \xi_n dP < \infty.
\]
For every $A$, an atom of $\mathcal{F}_n$, we have
\[
\int_A \xi_n dP = \int_A q(x, E^{\mathfrak{F}*}[F]) dP = P(A) q(x, E[F|A]) \leq \int_A \delta(x, F) dP.
\]
Hence
\[
\int_{\Omega} \xi_n dP \leq \int \delta(x, F) dP < \infty
\]
and Doob’s condition is satisfied.

For every $n = 1, 2, \ldots, \xi_n$ is evidently $\mathcal{F}_n$-measurable. Thus we have to check that for every $n = 1, 2, \ldots$ and every atom $A$ of $\mathcal{F}_n$ the inequality
\[
\int_A \xi_n dP \leq \int \xi_{n+1} dP
\]
holds, that is
\[
(2.1) \quad P(A) q(x, E[F|A]) \leq \int_A q(x, E^{\mathfrak{F}*}[F]) dP \quad (n = 1, 2, \ldots).
\]

Let $A_1, \ldots, A_k$ be (disjoint) atoms of $\mathcal{F}_{n+1}$ such that
\[
A = \bigcup_{i=1}^k A_i.
\]
Since every set $E[F|A_i]$ is non-empty and compact, we can find the elements $a_i \in E[F|A_i]$ such that
\[
q(x, a_i) = q(x, E[F|A_i]) \quad \text{for } i = 1, \ldots, k.
\]
Let $g$ be an $X$-valued r.v. distributed as follows:
\[
P(g = a_i) = \frac{P(A_i)}{P(A)} \quad \text{for } i = 1, \ldots, k.
\]
It is easily checked that $E[g] \subseteq E[F|A]$. Hence for every $a \in E[g]$ we have
\[
q(x, E[F|A]) \leq q(x, a).
\]
Thus, taking an arbitrary element $a \in E[g]$, we obtain
\[
P(A) q(x, E[F|A]) \leq P(A) q(x, a) \leq P(A) \sum_{i=1}^k \frac{P(A_i)}{P(A)} q(x, a_i) = \sum_{i=1}^k P(A_i) q(x, E[F|A_i]) = \int_A q(x, E^{\mathfrak{F}*}[F]) dP,
\]
which proves (2.1) and completes the proof of the lemma.

**Theorem 2.1.** Let $\{\mathcal{F}_n\}_{n=1}^\infty$ be an increasing sequence of finite subfields of $\mathfrak{F}$. Then the sequence $\{E^{\mathfrak{F}*}[F]\}_{n=1}^\infty$ of $\hat{X}$-valued random sets converges almost surely.
Proof. Let \( D \) be a countable dense set in \((X, \mathcal{Q})\). By Lemma 2.1 there exists a negligible event \( N \) such that for every \( \omega \in \Omega \setminus N \) the set \( \bigcup_{n=1}^{\infty} E^{\mathcal{S}_n}[F](\omega) \) is bounded in \( X \) and \( \lim_n \mathcal{Q}(x, E^{\mathcal{S}_n}[F](\omega)) \) exists and is finite for every \( x \in D \). Thus by Lemma 0.1 it follows that the sequence \( \{E^{\mathcal{S}_n}[F](\omega)\}_{n=1}^{\infty} \) is convergent in \((\hat{X}, \hat{\mathcal{Q}})\) for every \( \omega \in \Omega \setminus N \).

We shall now define conditional mathematical expectation of \( F \) relative to an arbitrary sub-\( \sigma \)-field \( \mathcal{B} \) of \( \mathcal{A} \).

Let \( L_0 \) be a space of (equivalence classes of) \( \hat{X} \)-valued random sets equipped with topology of convergence in probability with respect to the Hausdorff metric \( \hat{\mathcal{Q}} \) in \( \hat{X} \). This topology is metrizable by the metric:

\[
\hat{\mathcal{Q}}_0(F, F') = \inf\{\varepsilon > 0 : P(\hat{\mathcal{Q}}(F, F') > \varepsilon) < \varepsilon\}
\]

and the metric space \((L_0, \hat{\mathcal{Q}}_0)\) is complete.

Let us remark that, as in the real case, almost sure convergence of \( F_n \) to \( F \) implies that \( \hat{\mathcal{Q}}_0(F_n, F) \to 0 \) as \( n \to \infty \).

Let \( \mathcal{B} \) be an arbitrary (not necessarily finite) sub-\( \sigma \)-field of \( \mathcal{A} \) and let \( \mathcal{F}(\mathcal{B}) \) be the collection of all finite subfields of \( \mathcal{B} \) downward directed by inclusion. Theorem 2.1 states that, for any increasing sequence \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) of elements in the directed set \( \mathcal{F}(\mathcal{B}) \), the sequence \( \{E^{\mathcal{F}_n}[F]\}_{n=1}^{\infty} \) converges in a complete metric space \((L_0, \hat{\mathcal{Q}}_0)\). This implies ([12], Lemma V-1-1) that the net \( \{E^{\mathcal{F}_n}[F]\}_{\mathcal{F} \in \mathcal{F}(\mathcal{B})} \) is convergent in \((L_0, \hat{\mathcal{Q}}_0)\).

**Definition 2.1.** Any random set from the equivalence class \( E^{\mathcal{F}[F]} \) is called a (version of the) conditional mathematical expectation of \( F \) relative to \( \mathcal{B} \).

We shall prove now the following metric analogous of a theorem of P. Lévy.

**Theorem 2.2.** Let \( \{\mathcal{B}_n\}_{n=1}^{\infty} \) be an increasing sequence of countably generated sub-\( \sigma \)-fields of \( \mathcal{A} \). Then the sequence \( \{E^{\mathcal{B}_n}[F]\}_{n=1}^{\infty} \) of \( \hat{X} \)-valued random sets converges almost surely and

\[
\lim_n E^{\mathcal{B}_n}[F] = E^{\mathcal{B}_\infty}[F] \text{ a.s., where } \mathcal{B}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{B}_n\right).
\]

Before proving Theorem 2.2 let us state two lemmas.

**Lemma 2.2.** Let \( \{A_n\}_{n=0}^{\infty} \) be a sequence of non-negligible events in \( \mathcal{A} \) such that \( P(A_n \Delta A_0) \to 0 \) as \( n \to \infty \) (\( \Delta \) stands for symmetric difference of sets). If a sequence \( a_n \in E[F | A_n] \) \( (n = 1, 2, \ldots) \) is convergent, then \( \lim_n a_n \in E[F | A_0] \).
Proof. For every $n = 1, 2, \ldots$

$$q(x, a_n) \leq \frac{1}{P(A_n)} \int \delta(x, F) dP,$$ \quad \forall x \in X.

Hence, for every $x \in X$,

$$q(x, \lim_n a_n) = \lim_n q(x, a_n) \leq \frac{1}{P(A_0)} \int \delta(x, F) dP,$$

which means that $\lim_n a_n \in E[F \mid A_0]$.

**Lemma 2.3.** Let $\{\mathcal{F}_n\}_{n=1}^\infty$ be an increasing sequence of finite subfields of $\mathcal{A}$. Then

$$\lim_n E^{\mathcal{F}_n}[F] \subset E^{\mathcal{F}_\infty}[F] \ a.s., \quad \text{where } \mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{F}_n\right).$$

**Proof.** Since $E^{\mathcal{F}_\infty}[F]$ is a limit in $(L_0, \bar{\eta}_0)$ of a net $\{E^{\mathcal{F}_n}[F]\}_{\mathcal{F}_n \in \mathcal{F}(\mathcal{F}_n)}$, there exists an increasing sequence $\{\mathcal{F}'_n\}_{n=1}^\infty$ of finite subfields of $\mathcal{F}_\infty$ such that

(2.2) $\lim_n \bar{\eta}_0(E^{\mathcal{F}'_n}[F], E^{\mathcal{F}_\infty}[F]) = 0$ and $\mathcal{F}_n \subset \mathcal{F}'_n$ for $n = 1, 2, \ldots$

By Theorem 2.1, both sequences $\{E^{\mathcal{F}_n}[F]\}_{n=1}^\infty$ and $\{E^{\mathcal{F}'_n}[F]\}_{n=1}^\infty$ converge almost surely. By Egoroff's theorem (which is just as valid for r.v.'s with values in a metric space as for real r.v.'s) and by the density of $\bigcup_{n=1}^\infty \mathcal{F}_n$ in $\mathcal{F}_\infty$ we infer that for every $\varepsilon > 0$ there is a positive integer $n(\varepsilon)$ and a set $B_\varepsilon \in \mathcal{F}_{n(\varepsilon)}$ such that $P(B_\varepsilon) > 1 - \varepsilon$ and both sequences $\{E^{\mathcal{F}_n}[F]\}_{n=1}^\infty$ and $\{E^{\mathcal{F}'_n}[F]\}_{n=1}^\infty$ converge uniformly on $B_\varepsilon$. Thus there exists a subsequence $\{p_n\}_{n=1}^\infty$ of positive integers with $p_1 \geq n(\varepsilon)$ and such that

(2.3) $\bar{\eta}_0(E^{\mathcal{F}_n}[F](\omega), E^{\mathcal{F}'_n}[F](\omega)) \leq 1/n, \quad \forall \omega \in B_\varepsilon, \forall n \geq 1, \forall i, j \geq n.$

Let us fix arbitrary $\omega_0 \in B_\varepsilon$. We shall prove that

(2.4) $\lim_n E^{\mathcal{F}_n}[F](\omega_0) = \lim_n E^{\mathcal{F}'_n}[F](\omega_0).$

If $x \in \lim_n E^{\mathcal{F}_n}[F](\omega_0)$, then there exists a subsequence $\{p_n\}_{n=1}^\infty$ of $\{p_n\}_{n=1}^\infty$ and a sequence $\{x_n\}_{n=1}^\infty$ of elements $x_n \in E^{\mathcal{F}_{p_n}}[F](\omega_0)$ ($n = 1, 2, \ldots$) such that $x = \lim_n x_n$.

For every $n = 1, 2, \ldots$ and every $\omega \in B_\varepsilon$ let us denote by $A_n(\omega)$ and $A'_n(\omega)$ the atoms of $\mathcal{F}_{p_n}$ and $\mathcal{F}'_{p_n}$, respectively, such that $\omega \in A_n(\omega)$ and $\omega \in A'_n(\omega)$.
Since \( F_m \) is generated by the \( F_n \)'s, for every \( n = 1, 2, \ldots \) there is a sequence of sets \( \{B_{n,m}\}_{m=1}^{\infty} \) such that \( B_{n,m} \in F_{p_{n,m}} \) for \( m = 1, 2, \ldots \) and

\[
\lim_{m} P(A_n'_{\omega_0}) \Delta B_{n,m} = 0.
\]

Since, by (2.2), \( A_n'(\omega_0) \subset A_n(\omega_0) \), we can and shall assume that \( B_{n,m} \subset A_{n,m}(\omega_0) \) for \( n, m = 1, 2, \ldots \) From (2.3) we have

\[
q(x_n, E[F \mid A_j(\omega_0)]) \leq 1/n, \quad \forall n \geq 1, \forall j \geq n.
\]

Since \( A_j(\omega) = A_j(\omega_0) \) for every \( \omega \in A_j(\omega_0) \) \( (j = 1, 2, \ldots) \), from (2.5) we obtain

\[
q(x_n, E[F \mid A_j(\omega)]) \leq 1/n, \quad \forall n \geq 1, \forall j \geq n, \forall \omega \in A_j(\omega_0).
\]

Since \( F_k \) \( (k = 1, 2, \ldots) \) are finite subfields of \( A \), there are finite sets of indices \( I = I(n, m) \) \( (n, m = 1, 2, \ldots) \) such that

\[
B_{n,m} = \bigcup_{i \in I} A_{n+m}(\omega_i) \quad \text{and} \quad A_{n+m}(\omega_i) \cap A_{n+m}(\omega_j) = \emptyset \quad \text{for} \ i \neq i, i, j \in I.
\]

By (2.6) there exist elements \( d_{n,m} \in E[F \mid A_{n+m}(\omega_i)] \) such that \( q(x_n, d_{n,m}) \leq 1/n \). Since the metric space \( (X, \rho) \) is \( D \)-convex, there exist elements \( b_{n,m} \in X \) \( (n, m = 1, 2, \ldots) \) such that

\[
q(x, b_{n,m}) \leq \sum_{i \in I} \frac{P(A_{n+m}(\omega_i))}{P(B_{n,m})} q(x, d_{n,m}), \quad \forall x \in X.
\]

It is easily checked that

\[
b_{n,m} \in E[F \mid B_{n,m}] \quad \text{and} \quad q(x_n, b_{n,m}) \leq 1/n \quad (n, m = 1, 2, \ldots).
\]

Since \( (X, \rho) \) is finitely compact, for every \( n = 1, 2, \ldots \) the bounded sequence \( \{b_{n,m}\}_{m=1}^{\infty} \) contains a convergent subsequence \( \{b_{n,km}\}_{m=1}^{\infty} \). Put

\[
b_n = \lim_{m} b_{n,km} \quad (n = 1, 2, \ldots).
\]

Since for \( n = 1, 2, \ldots \) we have \( P(B_{n,km} \Delta A_n'(\omega_0)) \rightarrow 0 \) as \( m \rightarrow \infty \), we infer from (2.7) and Lemma 2.1 that

\[
b_n \in E[F \mid A_n'(\omega_0)] \quad \text{and} \quad q(x_n, b_n) \leq 1/n \quad \text{for} \ n = 1, 2, \ldots
\]

Thus

\[
x = \lim_{n} x_n = \lim_{n} b_n \in \text{Lim} E[F \mid A_n'(\omega_0)] = \text{Lim} E^{\omega_n}[F](\omega_0),
\]

which proves (2.4) and completes the proof of Lemma 2.3.

**Proof of Theorem 2.2.** Since each \( B_n \) is countably generated, for each \( n = 1, 2, \ldots \) there exists an increasing sequence \( \{F_{n,m}\}_{m=1}^{\infty} \) of finite subfields of \( B_n \) which generates \( B_n \). Since each \( E^{\omega_n}[F] \) is a limit in \( L_0 \) of a net
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\( \{E^\mathcal{F}[F] \}_{F \in \mathcal{F}(\mathcal{B}_n)} \) and \( \mathcal{B}_n \subset \mathcal{B}_{n+1} \), we can and shall assume that

\[
\lim_{m} \delta_0 (E^{\mathcal{F}_{n,m}}[F], E^{\mathcal{B}_n}[F]) = 0 \quad \text{and} \quad \mathcal{F}_{n,m} \subset \mathcal{F}_{n+1,m} \quad \text{for} \ n, m = 1, 2, \ldots
\]

Let us fix an arbitrary \( \varepsilon > 0 \). By Egoroff's theorem there are sets \( B_i \in \mathcal{B}_n \) such that \( P(B_i) > 1 - \varepsilon/2^n \) (\( n = 1, 2, \ldots \)) and

\[
\lim_m E^{\mathcal{F}_{n,m}}[F] = E^{\mathcal{B}_n}[F] \quad \text{uniformly on} \ B_i \quad (n = 1, 2, \ldots).
\]

Thus there exists a subsequence \( \{m_n\} \) of positive integers such that

\[
(2.9) \quad \sup_{\omega \in B} \delta_0 (E^{\mathcal{F}_{n,m_n}}[F](\omega), E^{\mathcal{B}_n}[F](\omega)) \to 0 \quad \text{as} \ n \to \infty,
\]

where

\[
B = \bigcap_{n=1}^{\infty} B_n.
\]

Defining \( \mathcal{F}_n = \mathcal{F}_{n,m_n} \) we obtain an increasing sequence \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) of finite subfields of \( \mathcal{B}_\infty \) which generates \( \mathcal{B}_\infty \). By Lemma 2.3 we have

\[
\lim_n E^{\mathcal{F}_n}[F] \subset E^{\mathcal{B}_\infty}[F] \quad \text{a.s.}
\]

Let \( B_i \in \mathcal{B}_\infty \), \( B_i \subset \mathcal{B}_n \), be a set with \( P(B_i) > 1 - 2\varepsilon \) and such that \( E^{\mathcal{F}_n}[F] \) converges uniformly on \( B_i \). It follows from (2.9) that \( E^{\mathcal{B}_n}[F] \) converges uniformly on \( B_i \) and

\[
\lim_n E^{\mathcal{B}_n}[F](\omega) = \lim_n E^{\mathcal{F}_n}[F](\omega) \subset E^{\mathcal{B}_\infty}[F](\omega), \quad \forall \omega \in B_i,
\]

which completes the proof since \( P(B_i) > 1 - 2\varepsilon \) and \( \varepsilon > 0 \) was chosen arbitrarily.

3. Martingales. Throughout this section \( (X, \mathcal{Q}) \) is a finitely compact, \( D \)-convex metric space and all random sets take values in \( \hat{X} \).

If \( F \) is a random set and \( \mathcal{B} \) a sub-\( \sigma \)-field of \( \mathcal{A} \), then we denote by \( S(F; \mathcal{B}) \) the collection of all \( \mathcal{B} \)-measurable selections of \( F \).

**Definition 3.1.** Let \( \{\mathcal{B}_n\}_{n=1}^{\infty} \) be an increasing sequence of sub-\( \sigma \)-fields of \( \mathcal{A} \) and \( \{F_n\}_{n=1}^{\infty} \) a sequence of scalarly integrable, \( \mathcal{B}_n \)-measurable random sets. We say that \( \{F_n, \mathcal{B}_n\}_{n=1}^{\infty} \) is a martingale iff

\[
E^{\mathcal{B}_n}[f] \subset F_n \quad \text{a.s.} \quad \text{for every} \ f \in S(F_{n+1}; \mathcal{B}_{n+1}), \ n = 1, 2, \ldots
\]

**Lemma 3.1.** Let \( F \) be a scalarly integrable random set, \( \mathcal{B} \) a sub-\( \sigma \)-field of \( \mathcal{A} \) and \( \{\mathcal{F}_m\}_{m=1}^{\infty} \) an increasing sequence of finite subfields of \( \mathcal{B} \) such that

\[
E^{\mathcal{A}}[F] = \lim_m E^{\mathcal{F}_m}[F] \quad \text{a.s.}
\]
Then for every $x \in X$ there exists a negligible event $N$ such that for every $\omega \in \Omega \setminus N$ and $a \in E^\infty[F](\omega)$ we have

$$q(x, a) \leq E^{\mathcal{F}_\omega}[\delta(x, F)](\omega), \quad \text{where } \mathcal{F}_\omega = \sigma\left(\bigcup_{m=1}^{\infty} \mathcal{F}_m\right).$$

**Proof.** Let $N'$ be a negligible event such that

$$E^\infty[F](\omega) = \lim_{m} E^{\mathcal{F}_m}[F](\omega), \quad \forall \omega \in \Omega \setminus N'.$$

Let $a \in E^\infty[F](\omega)$ for some $\omega \in \Omega \setminus N'$. Thus there exists a sequence $\{a_m\}_{m=1}^{\infty}$ of elements of $X$ converging to $a$ and such that $a_m \in E^{\mathcal{F}_m}[F](\omega)$ ($m = 1, 2, \ldots$), which means that for every $x \in X$ the following inequality holds:

$$q(x, a_m) \leq \frac{1}{P(A_m)} \int \delta(x, F) dP, \quad \text{where } \omega \in A_m, \text{ an atom of } \mathcal{F}_m.$$ 

The real martingale

$$\left\{ \frac{1}{P(A_m)} \int \delta(x, F) dP, \mathcal{F}_m \right\}_{m=1}^{\infty}$$

converges to $E^{\mathcal{F}_\omega}[\delta(x, F)]$ outside some negligible event $N''$ ([12], Proposition II.2.11). Thus for every $x \in X$ there is a negligible event $N = N' \cup N''$ such that

$$q(x, a) = \lim_{m} q(x, a_m) \leq E^{\mathcal{F}_\omega}[\delta(x, F)](\omega), \quad \forall \omega \in \Omega \setminus N.$$

**Theorem 3.1.** Let $F$ be a scalarly integrable random set and $\{B_n\}_{n=1}^{\infty}$ an increasing sequence of sub-$\sigma$-fields of $\mathcal{F}$. Then $\{E^{B_n}[F], B_n\}_{n=1}^{\infty}$ is a martingale.

**Proof.** Let $n \geq 1$ be fixed and $f \in S(E^{B_n+1}[F]; B_{n+1})$. Let $\{F_m\}_{m=1}^{\infty}$ and $\{\mathcal{F}_{m+1}\}_{m=1}^{\infty}$ be two increasing sequences of finite subfields of $B_n$ and $B_{n+1}$, respectively, such that $\mathcal{F}_m \subset \mathcal{F}_{m+1}$ for $m = 1, 2, \ldots$ and satisfying

$$E^{B_n}[f](\omega) = \lim_{m} E^{\mathcal{F}_m}[f](\omega), \quad E^{B_n}[F](\omega) = \lim_{m} E^{\mathcal{F}_m}[F],$$

$$E^{B_{n+1}}[F] = \lim_{m} E^{\mathcal{F}_{m+1}}[F](\omega), \quad \forall \omega \in \Omega \setminus N,$$

for some negligible event $N$.

Let $a \in E^{B_n}[f](\omega)$ for some $\omega \in \Omega \setminus N$. There is then a sequence $\{a_m\}_{m=1}^{\infty}$ of elements of $X$ converging to $a$ such that $a_m \in E^{\mathcal{F}_m}[f](\omega)$ for $m = 1, 2, \ldots$, which means that for every $x \in X$ the following inequality holds:

$$q(x, a_m) \leq \frac{1}{P(A_m)} \int q(x, f) dP, \quad \text{where } \omega \in A_m, \text{ an atom of } \mathcal{F}_m.$$
Since \( f(\omega) \in E^{\mathfrak{m}+1}[F](\omega) \) for every \( \omega \in \Omega \), by Lemma 3.1 we have
\[
\varrho(x, a_m) \leq \frac{1}{P(A_m) A_m} \int \mathbb{E}^{\mathfrak{m}+1}[\delta(x, F)] dP, \quad \forall x \in X,
\]
where
\[
\mathfrak{f}^{n+1} = \sigma(\bigcup_{m=1}^{\infty} \mathfrak{f}^n).
\]
But \( \mathfrak{f}^n \subset \mathfrak{f}^{n+1} \) for \( m = 1, 2, \ldots \), and thus we have
\[
\varrho(x, a_m) \leq \frac{1}{P(A_m) A_m} \int \delta(x, F) dP, \quad \forall x \in X, \ m \geq 1,
\]
which means that \( a_m \in E^{\mathfrak{m}+1}[F](\omega) \) for \( m = 1, 2, \ldots \), and thus
\[
a = \lim_m a_m \in E^{\mathfrak{m}}[F](\omega).
\]

The theorem is proved.

**Theorem 3.2.** Let \( \{F_n, \mathfrak{B}_n\}_{n=1}^{\infty} \) be a martingale and suppose that:

(a) The set \( \bigcup_{n=1}^{\infty} F_n(\omega) \) is a bounded subset of \( X \) for almost every \( \omega \in \Omega \).

(b) \( \sup_{\omega} \int \varrho(x, F_n) dP < \infty, \ \forall x \in X. \)

(c) The \( \sigma \)-fields \( \mathfrak{B}_n \) are countably generated for \( n = 1, 2, \ldots \).

Then the sequence \( \{F_n\}_{n=1}^{\infty} \) of random sets converges almost surely.

**Proof.** We shall show first that \( \{\varrho(x, F_n), \mathfrak{B}_n\}_{n=1}^{\infty} \) is a (real) submartingale for every \( x \in X. \)

Let \( n \geq 1 \) be fixed and let \( f \in S(F_{n+1}; \mathfrak{B}_{n+1}) \) satisfy
\[
\varrho(x, f(\omega)) = \varrho(x, F_{n+1}(\omega)), \quad \forall \omega \in \Omega.
\]

Since \( \{F_n, \mathfrak{B}_n\}_{n=1}^{\infty} \) is a martingale, for every \( x \in X \) we have
\[
\varrho(x, F_n) \leq \varrho(x, E^{\mathfrak{B}_n}[f]) \text{ a.s.}
\]
Thus for every \( A \in \mathfrak{B}_n \) we have
\[
\int_A \varrho(x, F_n) dP \leq \int_A \varrho(x, E^{\mathfrak{B}_n}[f]) dP.
\]
Since the \( \sigma \)-field \( \mathfrak{B}_n \) is countably generated, from Lemma 3.1 we obtain
\[
\int_A \varrho(x, E^{\mathfrak{B}_n}[f]) dP \leq \int_A E^{\mathfrak{B}_n} \varrho(x, f) dP.
\]
But
\[
\int A E^\mathbb{B}[\mathbb{Q}(x, f)] dP = \int A \mathbb{Q}(x, f) dP = \int A \mathbb{Q}(x, F_{n+1}) dP,
\]
which proves that \( \{\mathbb{Q}(x, F_n), \mathbb{B}_n\}_{n=1}^\infty \) is a submartingale.

By (b), the submartingale \( \{\mathbb{Q}(x, F_n), \mathbb{B}_n\}_{n=1}^\infty \) converges almost surely ([12], Theorem IV.1.2) for every \( x \in X \). Let \( D \) be a countable dense subset of \( X \). There exists a negligible event \( N \) such that for every \( \omega \in \Omega \setminus N \) the set \( \bigcup_{n=1}^\infty F_n(\omega) \) is a bounded subset of \( X \) and the sequence of reals \( \{\mathbb{Q}(x, F_n(\omega))\}_{n=1}^\infty \) converges for every \( x \in D \). Hence, by Proposition 0.1, the sequence \( \{F_n(\omega)\}_{n=1}^\infty \) converges in \( (X, \mathcal{F}) \) for every \( \omega \in \Omega \setminus N \).

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Institute of Mathematics
Technical University of Warsaw
pl. Jedności Robotniczej 1
00-661 Warsaw, Poland

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