ADMISSIBLE ESTIMATORS OF VARIANCE COMPONENTS IN NORMAL MIXED MODELS

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Abstract. A sufficient condition for an invariant quadratic estimator of a linear function of the vector of variance components to be admissible under the mean square error among all translation invariant estimators is given.

1. Introduction. Throughout the paper $Y$ will stand for a random $n$-vector normally distributed with expectation $A\beta$ and covariance matrix $\sum_{i=1}^{p} \sigma_i V_i$, i.e., let

$$Y \sim N(A\beta, \sum_{i=1}^{p} \sigma_i V_i),$$

where $A$ is a known $(n \times k)$-matrix, $V_1, \ldots, V_p$ are known nonnegative definite $(n \times n)$-matrices, while $\beta \in \mathbb{R}^k$ and $\sigma_1 \geq 0, \ldots, \sigma_p \geq 0$ are the unknown parameters. Assume that

$$\mathcal{R}(A) + \mathcal{R}(\sum_{i=1}^{p} V_i) = \mathbb{R}^n,$$

where $\mathcal{R}(\cdot)$ denotes the range of the matrix argument.

We concentrate on estimation of a linear function $F'\sigma$, where $F'$ is the transpose of $(p \times s)$-matrix $F$ $(s \leq p)$, while $\sigma = (\sigma_1, \ldots, \sigma_p)'$ is the vector of variance components. The regression vector is treated as a nuisance parameter.

We consider a class $\mathcal{F}_p$ of estimators based on $MY$, where $M$ is the orthogonal projection matrix on the null space of $A'$. These estimators are invariant with respect to the translations $Y \rightarrow Y + A\beta$, $\beta \in \mathbb{R}^k$, and $MY$ is a maximal invariant for this group of translations. Clearly, $MY \sim N(\theta_n, M\sigma)$, where $\theta_n$ denotes the zero vector in $\mathbb{R}^n$, while

$$M_\sigma = \sum_{i=1}^{p} \sigma_i M_i, \quad M_i = MV_i M, \quad i = 1, \ldots, p.$$

To compare estimators we shall use the mean square error defined for any estimator $\delta = \delta(MY)$ of $F'\sigma$ by

$$R(\delta, \sigma) = E(\delta - F'\sigma)'(\delta - F'\sigma).$$
Let \( \Theta \) be a subset of \( \mathbb{R}^p \) defined by
\[
\Theta = \{ \sigma \in \mathbb{R}^p : \sigma \geq 0, \mathcal{R}(M_\sigma) = \mathcal{R}(M) \},
\]
where the expression \( \sigma \geq 0, (\sigma > 0) \) means that all coordinates of \( \sigma \) are nonnegative (positive). Consider a subset \( 2_F \subset \mathcal{I}_F \) of the form
\[
q_u = q_u(Y) = \frac{Y'M_u^+ Y}{2 + r} F'u, \quad u \in \Theta,
\]
where \( r = \text{rank}(M) \), while \( M_u^+ \) denotes the Moore-Penrose g-inverse of \( M_u \). The estimators in \( 2_F \) have the following property. For a given \( u \in \Theta \) the estimator \( q_u \) minimizes the risk at each point \( \sigma = \lambda u, \lambda > 0 \), among all invariant quadratic estimators, i.e., among estimators of the form
\[
(Y'M A_1 MY, \ldots, Y'M A_p MY),
\]
where \( A_1, \ldots, A_p \) can be arbitrary symmetric \((n \times n)\)-matrices.

Note that if \( M_1, \ldots, M_p \) commute, as in the case of balanced models, then there exist idempotent nonzero matrices \( Q_1, \ldots, Q_m \), say, with their ranges contained in \( \mathcal{R}(M) \), such that \( Q_i Q_j \) is zero matrix for \( i \neq j = 1, \ldots, m \), and that
\[
M_i = \sum_{j=1}^{m} h_{ij} Q_j, \quad i = 1, \ldots, p.
\]
In this case \( M_u^+ \) can be represented as
\[
M_u^+ = \sum_{j=1}^{m} (1/\theta_j) Q_j,
\]
where \( (\theta_1, \ldots, \theta_m)^\prime = H'u \), while \( H = (h_{ij}) \).

Karlin [3] has proved that for \( p = 1 \) the set \( 2_F, F \in \mathcal{R} \), contains exactly one estimator, which is the only invariant quadratic estimator admissible for \( \sigma \) among \( \mathcal{I}_F \). For \( p > 1 \) and under the assumption that matrices \( M_1, \ldots, M_p \) commute Farrell et al. [2] have shown that each estimator in \( 2_F \) is admissible among \( \mathcal{I}_F \). Moreover, they have also proved that \( 2_F \), where \( I \) denotes the identity \((p \times p)\)-matrix, represents the class of all invariant quadratic estimators admissible for \( \sigma \) among \( \mathcal{I}_F \). Dey and Gelfand [1] have established the admissibility of estimators in \( 2_F, F \in \mathcal{R}^p \), under more restrictive conditions.

In this paper we drop the assumption that matrices \( M_1, \ldots, M_p \) commute and prove that each estimator in a subset \( 2_F \) of \( 2_F \) consisting of \( q_u \) with \( u > 0 \) is admissible for \( F^\prime \sigma \) among \( \mathcal{I}_F \).

2. Results. We shall use an idea of Farrell et al. [2] to establish the admissibility of estimators in \( 2_F \) also in the case where matrices \( M_1, \ldots, M_p \) do not commute.

**Theorem.** All estimators in \( 2_F \) are admissible for \( F^\prime \sigma \) among the class \( \mathcal{I}_F \) of invariant estimators.
Proof. According to a lemma due to Shinozaki (see, e.g., [4]) it is sufficient to prove the theorem for $F = I$.

First note that since

$$M_{\sigma} M_{\mu}^+ M_{\sigma} M_{\mu}^+ M_{\sigma} = \frac{\lambda}{2} M_{\sigma} M_{\mu}^+ M_{\sigma}$$

for $\sigma = \sigma_\lambda = (\lambda/2) u$, $\lambda > 0$, and since $\text{rank}(M_{\mu}) = r$ for $u > \theta_p$, it follows that when $\sigma = \sigma_\lambda$, the random variable $Y' M_{\mu}^+ Y$ has the gamma distribution with the shape parameter $r/2$ and the scale parameter $\lambda$. Thus, by Karlin's theorem,

$$q = \frac{2}{2 + r} Y' M_{\mu}^+ Y$$

is admissible for $\lambda$ among all estimators based on $Y' M_{\mu}^+ Y$.

The risk of any estimator $\delta = (\delta_1, \ldots, \delta_p)'$ at $\sigma_\lambda$ can be written as

$$R(\delta, \sigma_\lambda) = \frac{1}{4} \mathbb{E} \left( \sum_{i=1}^p (2\delta_i - \lambda u_i)^2 \right) = \frac{a}{4} \mathbb{E} \left( \sum_{i=1}^p \frac{u_i^2}{a} \left( \frac{2\delta_i - \lambda}{u_i} \right)^2 \right),$$

where $a = \sum_{i=1}^p u_i^2$. Applying Jensen's inequality to the expression in brackets, we obtain the inequality

$$R(\delta, \sigma_\lambda) \geq \frac{a}{4} \mathbb{E} \left( \frac{2}{a} \sum_{i=1}^p u_i \delta_i - \lambda \right)^2$$

which is strict unless $\delta_i/u_i = \delta_j/u_j$ for all $i, j = 1, \ldots, p$.

Since the random variable $Y' M_{\mu}^+ Y$ is a sufficient statistics for $\lambda$ when $\sigma = \sigma_\lambda$, there exists an estimator $\delta^*$ of $\lambda$ based on $Y' M_{\mu}^+ Y$ as good as $2a^{-1} \sum_{i=1}^p u_i \delta_i$. Moreover, since, as we have already noted, $q$ is admissible for $\lambda$ and since the mean square error of $q_u$ and $q$ are related at $\sigma = \sigma_\lambda$ by

$$R(q_u, \sigma_\lambda) = \frac{a}{4} R(q, \sigma_\lambda),$$

it follows that if, say, $\delta$ dominates $q_u$, then

$$R(q, \lambda) = \mathbb{E} \left( \frac{2}{a} \sum_{i=1}^p u_i \delta_i - \lambda \right)^2.$$

Consequently, $\delta_i = u_i q$ for all $i$ with probability 1, so that $\delta = q_u$ with probability 1. But this contradicts the assumption that $\delta$ dominates $q_u$ and concludes the proof of the Theorem.

It is an open problem whether there exist alternative invariant quadratic estimators to (1.1) admissible for $\sigma$ in the case where matrices $M_1, \ldots, M_p$ do not commute.
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Received on 12.6.1990