TESTS OF FIT FOR COX'S REGRESSION MODEL

BY

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Abstract. An omnibus test of fit for Cox's proportional hazards regression model is proposed for continuous data. The procedure is extended to a random censorship model. Density estimation methods are used.

1. Introduction. One of the principal models of failure time data analysis is the proportional hazards model of Cox [6], [7]. This semiparametric model has been assumed as the underlying structure in numerous instances, and so it is important to have a test which can be used to determine whether it is appropriate in a given situation or not. Bednarski [3] shows how the Cox estimator can misbehave if the model is not correct. We propose an omnibus test procedure in which the test statistic is asymptotically normal under the null hypothesis that Cox's model is true. The results are generalized to the random censorship case in Section 4 (cf. Theorem 4.1).

Our approach is based on density function estimates. Although their convergence rate is slower than that of the sample distribution function, they enjoy the desirable property that their limiting distribution does not depend on the fact that parameters of the model must be estimated. For an approach based on the sample distribution function, the results of Durbin [8] and Burke et al. [4] indicate that the limiting behavior would depend on the parametric family of distribution functions underlying the model and possibly on the values of the unknown parameters.

Previous approaches are mostly based on data analytic techniques (e.g., Kay [12], Andersen [1] and Schoenfeld [15]). Schoenfeld [14] proposed a class of chi-squared tests where $p$-dimensional Euclidean space is partitioned into a finite number of classes. His approach, thus, discretizes the data and, by choosing different partitions, one arrives at different tests in the continuous case. While there are many ingredients in the density approach which can be varied (kernel function, bandwidth), this approach seems more natural in view of the model's definition in the continuous case. The monograph by Prakasa Rao [13] gives a good survey of density estimation results. Horváth [10] obtained asymptotic normality for $L_p$-norms of multivariate densities.

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We define the hazard rate function of a random variable $T$ given $Z$ as

$$\lambda(t, z) = \lim_{At \to 0} (At)^{-1} P\{t \leq T < t + At \mid T \geq t, Z = z\},$$

where $T$ denotes the failure time and $Z$ is the $(p-1)$-dimensional covariate or regressor variable. Our null hypothesis is that Cox's model is true:

$$H_0: \lambda(t, z) = \lambda_0(t)e^{\beta Z},$$

where $\beta$ is an unknown $(p-1)$-vector of regression parameters and $\lambda_0(t)$ an unknown base-line hazard function. Our results will also be true if we replace $e^{\beta Z}$ by a known function $\eta(z, \beta)$ for which $\eta(0, \beta) = 1$.

Our test procedure will be based on the fact that $\lambda(t, z)e^{-\beta z} = \lambda_0(t)$, being a function of $t$ only.

Let $F(t, z)$ denote the joint survival function of $(T, Z)$, that is,

$$F(t, z) = P\{T \geq t, Z \leq z\}.$$ 

We assume that the corresponding density $f(t, z)$ exists. Hence $\lambda(t, z)$ of (1.1) can be written as

$$\lambda(t, z) = f(t, z)[g(t, z)]^{-1},$$

where

$$g(t, z) = (\partial^{p-1}/\partial z_1, \ldots, \partial z_{p-1})F(t, z).$$

Statement (1.5) is well defined if the denominator is not zero. Our approach is to estimate $\beta$ by $\hat{\beta}_n$, Cox's [7] partial likelihood estimator, the density $f$ by $f_n$, a $p$-variate kernel estimate, and the derivative $g$ by the estimator $g_n$ of (1.10) below. We then arrive at the process

$$X_n(t, z, w) = f_n(t, z)[g_n(t, z)]^{-1}\exp\{-\hat{\beta}_n z\} - f_n(t, w)[g_n(t, w)]^{-1}\exp\{-\hat{\beta}_n w\}.$$ 

Under $H_0$ and in view of (1.3), each term in the difference (1.7) is an estimate of the base-line hazard rate $\lambda_0(t)$. We will establish the asymptotic normality of

$$W_n^2 = \iint_D X_n^2(t, z, w)dtdzdw,$$

where $D = (0, Q) \times M^2$ (cf. Condition 2.1 (a)).

Let $(T_1, Z_1), (T_2, Z_2), \ldots, (T_n, Z_n)$ be independent random vectors with survival function (1.4) and let

$$F_n(u, v) = n^{-1} \sum_{i=1}^{n} I\{T_i > u, Z_i \leq v\}$$
denote the empirical survival function, \( u \in R, v \in R^{p-1} \). For the kernel function \( K(u, v) \) satisfying Condition 2.1 (c), we define

\[
(1.9) \quad f_n(t, z) = - \{b^{-p} K [b^{-1} ((t, z) - (u, v))]) dF_n(u, v) \n\]

\[
= -(nb^n)^{-1} \sum_{i=1}^{n} K [b^{-1} (t - T_i, z - Z_i)],
\]

where the “bandwidth” sequence of constants \( \{b = b_n \} \) satisfies Condition 2.1 (e). Next, with the kernel function \( K_2 \) satisfying Condition 2.1 (d), we define

\[
(1.10) \quad g_n(t, z) = \int b^{-p-1} K_2 [b^{-1} (z, v)] dF_n(t, v)
\]

\[
= (nb)^{-p-1} \sum_{i=1}^{n} K_2 [b^{-1} (z - Z_i)] I \{ T_i > t \}.
\]

Lastly, let \( \beta_n \) be the sequence of estimators obtained by maximizing the partial likelihood (Cox [6], [7]):

\[
L(\beta) = \prod_{i \in S} \{ \exp \{ \beta Z_i \} \} \left( \sum_{j \in R(t_i)} \exp \{ \beta Z_i \} \right)^{-1},
\]

where \( S \) is the set of indices 1, 2, ..., \( n \) corresponding to individuals who died (failed), \( t_i \) is the failure time of the \( i \)-th individual, and \( R(t_i) \) is the set of indices corresponding to individuals who survived until time \( t_i \).

In Section 2 we give the main results for the uncensored case. The proofs are indicated in Section 3. Although these results may be considered as preliminary to the results on randomly censored data (Section 4), they are of interest in their own right. (The behavior of Cox’s partial likelihood estimator under a sequence of local alternatives is treated in Burke and Gombay [5].) We follow the approach of Hall [9] in our handling of density-type estimators.

2. The uncensored case. We will assume the following conditions:

CONDITION 2.1. (a) Let \( \mathcal{W} = (0, Q) \times M \) be the support of \( (T, Z) \), where \( M \) is a bounded subset of \( R^{p-1} \) having (finite) Lebesgue measure \( \lambda_M \).

(b) Let \( f \) be the joint density of \( (T, Z) \). Assume that all partial derivatives of order 2 are bounded and uniformly continuous on \( R^p \).

(c) Let \( K \) be a \( p \)-variate density function satisfying

\[
\int u_i K(u) du = 0, \quad \int u_i u_j K(u) du = C \delta_{ij} < \infty
\]

for each \( i, j = 1, 2, \ldots, p \), where the constant \( C \) does not depend on \( i \) and \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) otherwise.

(d) Let \( K_2 \) be a \( (p-1) \)-variate density function satisfying

\[
\int u_i K_2(u) du = 0, \quad \int u_i u_j K_2(u) du = C \delta_{ij} < \infty
\]

for each \( i, j = 1, 2, \ldots, p - 1 \), where \( C \) is independent of \( i \).
(c) \( b = b_n \) is a nonincreasing sequence of positive numbers such that
\[
 nb^p \to \infty \quad \text{and} \quad nb^{p+4} \to 0 \quad \text{as} \quad n \to \infty.
\]

The main result of this paper is

**Theorem 2.2.** Under Conditions 2.1 and the null hypothesis \( H_0 \) defined by (1.2), we have

\[
 nb^{p/2} \sigma^{-1} (W_n^2 - \mu) \xrightarrow{D} N(0, 1),
\]

where \( W_n^2 \) is defined by (1.8), \( \mu = \mu_1 + \mu_2 + \mu_3 \), with
\[
 \mu_1 = (nb^p)^{-1} \cdot 2 \lambda_m \int_0^{\infty} K^2(v) a^2(x) f(x - bv) dv dx,
\]
\[
 \mu_2 = (nb^{p-1})^{-1} \cdot 2 \lambda_m \int_{\mathcal{P}^{-1}} \int_0^{\infty} K^2(v_2) g(t, z - bv) dv_2 dzdt,
\]
\[
 \mu_3 = -4(nb^{p-1})^{-1} \int a(x) r(x) \int_{-\infty}^{\infty} K(v) K_2(v_2) f(x - bv) dv_2 dv_1 dx,
\]
and
\[
 \sigma^2 = 8 \lambda_m \int_0^{\infty} \int K^2(t, z) a^2(t, z) \int [K(u) K(u - v)]^2 dv du,
\]

with
\[
 a(t, z) = [g(t, z) e^\beta]^2.
\]

Remark 2.3. To use the result of Theorem 2.2 as a test of the null hypothesis \( H_0 \) of (1.2), one can estimate \( \mu, \sigma \), of (2.2), (2.3) by \( \hat{\mu}, \hat{\sigma} \), where \( \hat{\mu} \) and \( \hat{\sigma} \) are defined like \( \mu \) and \( \sigma \) but with \( f, g \) and \( \beta \) replaced by \( f_n, g_n \) and \( \beta_n \), respectively. It is easy to show that
\[
 nb^{p/2} \sigma^{-1} (W_n^2 - \hat{\mu}) \xrightarrow{D} N(0, 1)
\]
under \( H_0 \). Hence \( H_0 \) would be rejected if
\[
 nb^{p/2} \sigma^{-1} (W_n^2 - \hat{\mu}) \geq z_{1-a},
\]
where \( z_{1-a} \) is the \((1-a)100\) percentile of the standard normal distribution.

As an alternative to a test based on \( W_n^2 \), one can also consider the vector
\[
 \mathbf{\xi}_n = [X_n(t_1, z_1, w_1), \ldots, X_n(t_k, z_k, w_k)]
\]
and establish

**Theorem 2.4.** Assume that Conditions 2.1 hold and that the support of \( K \) is finite. Then, as \( n \to \infty \),
\[
 (nb^p) \mathbf{\xi}_n \xrightarrow{D} N,
\]
where \( N \) is a \( k \)-variate normal distribution with zero mean and covariance matrix
$\Sigma$ having entries

$$\sigma_{ij} = [a(t_i, z_i) a(t_j, z_j) f(t_i, z_i) - a(t_i, z_i) a(t_j, w_j) f(t_i, z_i)$$
$$- a(t_i, w_i) a(t_j, z_j) f(t_i, w_i)$$
$$+ a(t_i, w_i) a(t_j, w_j) f(t_i, w_i)] \int K^2(v) dv.$$

As a consequence of Theorem 2.4, by replacing $\Sigma$ by its estimator $\hat{\Sigma}$ as in Remark 2.3, we have

$$(nb^n)^{-1} \xi_n^2 \xrightarrow{D} \chi^2(k),$$

where $\chi^2(k)$ is a chi-square distribution with $k$ degrees of freedom. The test: reject $H_0$ of (1.2) if

$$(nb^n)^{-1} \xi_n^2 \geq \chi^2(k),$$

where

$$P(\chi^2(k) \leq \chi^2(k)) = 1 - \alpha$$

is an asymptotically $\alpha$-level test which would detect departures from $H_0$ at a finite number of points.

3. Proof of the uncensored results. We herewith sketch the proofs of the results. Details of the proofs can be found in the technical report of Burke and Gombay [5].

We will consider a closely related statistic to that of $W_n^2$, namely

$$(3.1) \quad [W_n^{(1)}] = \int \int [X_n^{(1)}(t, z, w)]^2 dt dz dw,$$

where

$$(3.2) \quad X_n^{(1)}(t, z, w) = a(t, z)[f_n(t, z) - f(t, z)] - a(t, w)[f_n(t, w) - f(t, w)]$$
$$- r(t, z)[g_n(t, z) - g(t, z)] + r(t, w)[g_n(t, w) - g(t, w)],$$

$a(t, z)$ is defined by (2.4), and $r(t, z) = f(t, z)a(t, z)g(t, z)^{-1}$. We will prove the following

**Theorem 3.1.** Under the conditions of Theorem 2.2,

$$nb^{n/2} \sigma^{-1}([W_n^{(1)}]^2 - \mu) \xrightarrow{D} N(0, 1) \quad \text{as } n \to \infty,$$

where $\mu$ and $\sigma$ are defined by (2.2) and (2.3), respectively, and $W_n^{(1)}$ is defined by (3.1).

We have the expansion

$$[W_n^{(1)}]^2 = \sum_{i=1}^{4} T_i,$$
where

\[ T_1 = \int \int_D \{ a(t, z)[f_n(t, z) - f(t, z)] - a(t, w)[f_n(t, w) - f(t, w)] \}^2 \, dt \, dz, \]
\[ T_2 = \int \int_D \{ r(t, z)[g_n(t, z) - g(t, z)] - r(t, w)[g_n(t, w) - g(t, w)] \}^2 \, dt \, dz, \]

(3.3)

\[ T_3 = -4\lambda_M \int \int_D a(t, z) r(t, z)[f_n(t, z) - f(t, z)] [g_n(t, z) - g(t, z)] \, dt \, dz, \]
\[ T_4 = 4 \int \int_D a(t, z)[f_n(t, z) - f(t, z)] r(t, w)[g_n(t, w) - g(t, w)] \, dt \, dz. \]

We will first consider \( T_1 \) and write

\[ T_1 = \sum_{i=1}^{4} T_{1i}, \]

where

\[ T_{11} = \int \int_D \{ a(t, z)[f_n(t, z) - Ef_n(t, z)] - a(t, w)[f_n(t, w) - Ef_n(t, w)] \}^2 \, dt \, dz, \]
\[ T_{12} = \int \int_D \{ a(t, z)[Ef_n(t, z) - f(t, z)] - a(t, w)[Ef_n(t, w) - f(t, w)] \}^2 \, dt \, dz, \]

(3.4)

\[ T_{13} = 4\lambda_M \int \int_D a^2(t, z)[f_n(t, z) - Ef_n(t, z)] [Ef_n(t, z) - f(t, z)] \, dt \, dz, \]
\[ T_{14} = 4 \int \int_D a(t, z)a(t, w)[f_n(t, z) - Ef_n(t, z)] [Ef_n(t, w) - f(t, w)] \, dt \, dz. \]

Under Conditions 2.1,

\[ \sup_{\gamma} |Ef_n(t, z) - f(t, z)| \rightarrow 0, \]

where \( \frac{\partial^2 f}{\partial \gamma^2} \) is the Laplacian and \( C \) is a constant. Hence, as \( n \rightarrow \infty \), we obtain

(3.6)

\[ T_{12} = O(b^4). \]

**Lemma 3.2.** Under Conditions 2.1,

\[ T_{13} \xrightarrow{D} (4\lambda_M b^2 n^{-1/2} C \sigma_{13}) \cdot Z, \]

where \( Z \) is a standard normal \((0, 1)\) random variable, \( C \) is a constant, and

\[ \sigma_{13}^2 = \int_{\gamma} a^2(\frac{\partial^2 f}{\partial \gamma^2})^2 f - \left[ \int_{\gamma} a^2(\frac{\partial^2 f}{\partial \gamma^2}) f \right]^2. \]

**Lemma 3.3.** Under Conditions 2.1,

\[ T_{14} \xrightarrow{D} (4b^2 n^{-1/2} C \sigma_{14}) \cdot Z, \]
where $Z$ is a standard normal random variable, $C$ is a constant, and
\[
\sigma_{14}^2 = \int_a \left[ \int a(t, z) m(t) f(t, z) dz \right] dt - \left[ \int a(t, z) m(t) f(t, z) dz \right]^2
\]
with
\[
m(t) = \int_a a(t, w) V^2 f(t, w) dw.
\]

As a consequence of (3.6) and Lemmas 3.2 and 3.3 we have
\[
nb^{1/2}(T_{12} + T_{13} + T_{14}) \xrightarrow{P} 0 \quad \text{as } n \to \infty.
\]
Hence the term $T_{11}$ determines the asymptotic distribution of $T_1$ which is described by

**Lemma 3.4.** Under Conditions 2.1, $nb^{1/2}\mu_1^{-1}(T_1 - \mu_{1n}) \xrightarrow{P} Z$, where $Z$ is a standard normal random variable, $\mu_{1n}$ and $\sigma$ are defined by (2.2) and (2.3), respectively.

We can treat the term $T_2$ of (3.3) in a similar manner to that of $T_1$ and write
\[
T_2 = \sum_{i=1}^4 T_{2i},
\]
where $T_{2i}$ is defined like $T_{1i}$ in (3.4) but with $g$ and $r$ replacing $f$ and $a$, respectively. We then obtain
\[
nb^{1/2}(T_{22} + T_{23} + T_{24}) \xrightarrow{P} 0 \quad \text{as } n \to \infty
\]
in a similar way to (3.8).

**Lemma 3.5.** Under Conditions 2.1,
\[
nb^{1/2}(T_2 - \mu_{2n}) \xrightarrow{P} 0 \quad \text{as } n \to \infty,
\]
where $\mu_{2n}$ is defined by (2.2).

Since $g_n$ is a $(p-1)$-dimensional kernel estimator, the deviation of $T_2$ from its mean is asymptotically negligible as compared to $T_1$. Similarly, we have

**Lemma 3.6.** Under Conditions 2.1,
\[
nb^{1/2}(T_3 - \mu_{3n}) \xrightarrow{P} 0, \quad nb^{1/2} T_4 \xrightarrow{P} 0 \quad \text{as } n \to \infty,
\]
where $T_3$ and $T_4$ are defined by (3.3) and $\mu_{3n}$ is defined by (2.2).

**Proof of Theorem 3.1.** The theorem follows directly from (3.8), (3.9) and Lemmas 3.4, 3.5 and 3.6.

**Proof of Theorem 2.2.** We have
\[
W_n^2 = \int_D \left[ \int a^{(1)}(t, z, w) + R_n(t, z, w) \right]^2 dt dz dw
\]
\[
= \left[ W_n^{(1)} \right]^2 + 2 \int_D \int a^{(1)} R_n + \int_D \int R_n^2,
\]
where
\[ R_n(t, z, w) = \sum_{i=1}^{4} [R_{in}(t, z) - R_{in}(t, w)] \]
and
\[ R_{1n}(t, z) = f_n(t, z)g(t, z)^{-1}[\exp\{-\beta z\} - \exp\{-\beta z\}], \]
\[ R_{2n}(t, z) = [g_n(t, z)g(t, z)\exp\{\beta z\}]^{-1}[f_n(t, z) - f(t, z)][g(t, z) - g_n(t, z)], \]
\[ R_{3n}(t, z) = f(t, z)[g_n(t, z)g(t, z)]^{-1}[\exp\{-\beta z\} - \exp\{-\beta z\}][g(t, z) - g_n(t, z)], \]
\[ R_{4n}(t, z) = f(t, z)[g_n(t, z)g(t, z)\exp\{-\beta z\}]^{-1}[g(t, z) - g_n(t, z)]^2. \]

Under conditions weaker than ours, Tsiatis [16] has shown that \( n^{1/2}(\hat{\beta} - \beta) \) is asymptotically normal with zero mean and finite variance. For another approach, see Andersen and Borgan [2]. Hence, by the mean value theorem,
\[ \exp\{\beta z\} - \exp\{\beta z\} = o_p(n^{-1/2}), \]
uniformly in \( z \in M \). Since \( f_n \) is a uniformly consistent estimator of \( f \), we have
\[ \sup_{z} |R_{1n}(t, z)| = o_p(n^{-1/2}). \]
Consequently,
\[ \int_{D} \int a(t, z)[f_n(t, z) - f(t, z)]R_{1n}(t, z)dtdzdw = o_p(n^{b/2}). \]
Using similar calculations to those above, we obtain
\[ \int_{D} \int X_{n}^{(1)} \xi_n = o_p(n^{b/2}), \quad \int_{D} \int R_{2}^{2} = o_p(n^{b/2}). \]
Hence Theorem 2.2 follows from (3.11) and Theorem 3.1. 

Remark 3.7. We have assumed throughout that \( nb^{p+4} \to 0 \). The cases \( nb^{p+4} \to c \) and \( nb^{p+4} \to \infty \) can also be treated with an asymptotic normal result. However, in these cases the terms \( T_{12} \) and \( T_{13} \) (\( i = 1, 2, 3; 4 \)) are the ones determining the asymptotic behavior of \( [W_n^{(1)}]^2 \) (cf., e.g., Lemmas 3.2 and 3.3). The resulting asymptotic variance would be too complicated for this approach to be practical.

Proof of Theorem 2.4. The proof follows as in the proof of Theorem 2.2 above. We can replace \( \xi_n \) by
\[ \xi_n^{(1)} = [X_{n}^{(1)}(t_1, z_1, w_1), \ldots, X_{n}^{(1)}(t_k, z_k, w_k)], \]
that is,
\[ (nb^{p})^{1/2} \| \xi_n - \xi_n^{(1)} \|_{F} \to 0. \]
The vector $z_n^{(i)}$ is a sum of independent random vectors with zero mean and covariance matrix $(nb^p)^{-1} \Sigma + o((nb^p)^{-1})$. Note that

$$(nb^p)^{1/2} r(t, z)[g_n(t, z) - g(t, z)] \to 0.$$ 

On applying a central limit theorem the theorem is proved. $\square$

4. The censored case. Suppose that the survival times $T_1, T_2, \ldots, T_n$ of $n$ individuals are subject to random censoring by the random variables $C_1, C_2, \ldots, C_n$, which are assumed to be independent. Moreover, $T_i$ and $C_i$ are assumed to be conditionally independent given the covariate vector $Z_i$ (cf. Tsiatis [16]). The observable time until death will be denoted by $Y_i = \min\{T_i, C_i\}$ and let $\delta_i = I\{Y_i = T_i\}, i = 1, 2, \ldots, n$.

Let $F^*$ denote a joint "survival" function of $Y$ and $Z$,

$$F^*(t, z) = P\{Y \geq t, Z \leq z\},$$

where $0 \leq t \leq Q$ and $z \in M \subset \mathbb{R}^{p-1}$ (cf. Condition 2.1). Let

$$g^*(t, z) = (\partial^{p-1}/\partial z_1, \ldots, \partial z_{p-1}) F^*(t, z).$$

Then, if $f_Z$ is the marginal density of $Z$ and if

$$F(t|z) = P\{T_i > t \mid Z_i = z\}, \quad G(t|z) = P\{C_i > t \mid Z_i = z\},$$

we have

$$g^*(t, z) = f_Z(z) F(t|z) G(t|z)$$

by the conditional independence of $T_i$ and $C_i$, given $Z_i$. Also,

$$\lambda(t, z) = f(t, z)[g^*(t, z)]^{-1},$$

where

$$f(t, z) = -(\partial^p/\partial t, \partial z_1, \ldots, \partial z_{p-1}) P\{Y_i > t, \delta_i = 1, Z_i \leq z\}
= f(t, z) G(t|z)$$

is the joint subdensity of $Y_i$ and $Z_i$ with $Y_i = T_i$ (uncensored), and $f$ is the joint density of $(T_i, Z_i)$.

To proceed with our test of $H_0$ of (1.2) in this random censorship case, we estimate $\lambda$ by

$$\lambda_n^*(t, z) = f_n(t, z)[g_n^*(t, z)]^{-1},$$

where

$$f_n(t, z) = -b^{-p} \int K(b^{-1} [(t, z) - (u, v)]) dF_n(u, v),$$

$$g_n^*(t, z) = b^{-(p-1)} \int K_2(b^{-1} [z - v]) d\nu F^*_n(t, v)$$

for
and

\[ \tilde{F}_n(u, v) = n^{-1} \sum_{i=1}^{n} I\{Y_i > u, Z_i \leq v, \delta_i = 1\}, \]

\[ F_n^*(u, v) = n^{-1} \sum_{i=1}^{n} I\{Y_i > u, Z_i \leq v\}. \]

Note that both \( \tilde{F}_n \) and \( F_n^* \) are based on the observed data \((Y_i, Z_i, \delta_i), i = 1, 2, \ldots, n\).

Let \( \hat{\beta}_n \) denote the Cox estimator for the \((p-1)\)-vector \( \beta \) (cf. Tsiatis [16]). We arrive at the process corresponding to (1.7):

\[ X_n^*(t, z, w) = \tilde{f}_n(t, z)[g_n^*(t, z)]^{-1} \exp\{-\beta z\} \]

\[ -\tilde{f}_n(t, w)[g_n^*(t, w)]^{-1} \exp\{-\beta w\}, \]

and to the statistic corresponding to (1.8):

\[ (W_n^*)^2 = \int \int_D [X_n^*(t, z, w)]^2 dtdzdw, \]

where \( \tilde{f}_n \) and \( g_n^* \) are defined by (4.4).

We have

**Theorem 4.1.** Assume Conditions 2.1 hold with \( f \) and \( g \) replaced by \( \tilde{f} \) and \( g^* \), respectively. Then the conclusions of Theorems 2.2 and 2.4 and Remark 2.3 hold for \((W_n^*)^2\) and \( X_n^* \) with \( f \) and \( g \) replaced by \( \tilde{f} \) and \( g^* \), respectively.

The proof of Theorem 4.1 follows from the arguments in Section 3 and on noting (4.2).

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