Abstract. A nondegenerate probability measure $v$ on $\mathbb{R}^n$ is an $n$-dimensional version of a symmetric measure $\mu$ on $\mathbb{R}$ if there exists $c : \mathbb{R}^n \to [0, \infty)$ such that $\hat{v}(ta) = \hat{\mu}(|t|c(a))$, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$. If the function $c$ is an $L_p$-norm on $\mathbb{R}^n$, we call the measure $v$ $p$-elliptically contoured. The main result of this paper is that if $\mu$ has an $e$-order for $e > 0$, then every its $n$-dimensional version is $p$-elliptically contoured for some $p \in (0, 2]$. We show also that $\text{supp}(\mu) = \mathbb{R}$ if only $\mu$ has an $n$-dimensional version which is not 2-elliptically contoured.

Distributions on $\mathbb{R}^n$ having all one-dimensional projections the same up to a scale parameter play a particular role in statistics and probability theory. For example, symmetric Gaussian measures and symmetric stable measures have this property. The investigation of this class of measures was started by Eaton [4] in 1981 and continued by Cambanis et al. [2] in 1983. It is still unknown however how large this class is, and this paper is devoted to the investigation of some its properties.

By a nondegenerate distribution on $\mathbb{R}^n$ we will understand a distribution for which the linear support is equal to $\mathbb{R}^n$. By $\mathcal{L}(X)$ we denote the distribution of a random vector $X$.

**DEFINITION 1.** The nondegenerate distribution $v$ of a symmetric random vector $(X_1, \ldots, X_n) \in \mathbb{R}^n$ is said to be an $n$-dimensional version of a symmetric distribution $\mu$ of a random variable $X \in \mathbb{R}$ if for every $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ there exists $c(a) \geq 0$ such that

$$\mathcal{L}\left(\sum a_i X_i\right) = \mathcal{L}\left(c(a) X\right)$$

or, equivalently,

$$\hat{v}(ta) = \hat{\mu}(c(a)t), \quad a \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where $\hat{v}$ and $\hat{\mu}$ are the corresponding characteristic functions.

In general, we know very little about the function $c$. It is known (see [4]) that $c(ta) = |t||c(a)$ for every $t \in \mathbb{R}$ and $a \in \mathbb{R}^n$. It is almost evident also that $c$ is a continuous function on $\mathbb{R}^n$, so it is equivalent to any norm on $\mathbb{R}^n$.

There are very close connections between measures $\mu$ having $n$-dimensional versions and symmetric stable measures on $\mathbb{R}$, as the above definition is almost the same as the definition of stable distribution. The only difference is
that we do not assume here the independence of $X_i$'s. It is trivial then that if at least two of $X_i$'s are independent, then all $X_i$'s and $X$ are symmetric and stable on $\mathbb{R}^n$.

If $\nu$ is a symmetric $p$-stable measure on $\mathbb{R}^n$, then (see, e.g., [6]) its characteristic function is of the form

$$\hat{\mu}(a) = \exp\{-c(a)^p\}, \quad a \in \mathbb{R}^n,$$

where

$$c(a)^p = \int \ldots \int \langle a, x \rangle^p \lambda(dx), \quad a \in \mathbb{R}^n,$$

for some finite measure $\lambda$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. This means that every symmetric $p$-stable measure $\nu$ on $\mathbb{R}^n$ is an $n$-dimensional version of the symmetric $p$-stable measure $\gamma_p$ on $\mathbb{R}$ with the characteristic function $\exp\{-|t|^p\}$. Moreover, every $n$-dimensional version of the measure $\gamma_p$ is symmetric and $p$-stable as stable is a distribution having all one-dimensional projections symmetric and $p$-stable.

As we can see it will not be surprising if it turns out that every function $c: \mathbb{R}^n \to [0, \infty)$, appearing in Definition 1, is given by the formula (*') for some $p > 0$ and a finite measure $\lambda$ on $S^{n-1} \subset \mathbb{R}^n$. In fact, as far as we know, there exists no example of a measure $\nu$ on $\mathbb{R}^n$ being an $n$-dimensional version of some measure $\mu$ on $\mathbb{R}$ with the function $c$ which cannot be written in the form (*) for any $p \in (0, 2]$ and any finite measure $\lambda$. That is why we introduce the following

**DEFINITION 2.** A symmetric measure $\nu$ on $\mathbb{R}^n$ is called $p$-elliptically contoured, $p > 0$, if its characteristic function is of the form

$$\hat{\nu}(ta) = f(t c(a)), \quad a \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where $f: [0, \infty) \to \mathbb{R}$ is a continuous function and $c(a)$ is given by the formula (*) for some finite measure $\lambda$ on $S^{n-1}$.

There is a full characterization of $p$-elliptically contoured measures in finite and infinite dimensional spaces for $p = 2$ and $p = 1$ (see [2], [10]–[12]). In [1] one can find a full characterization of measures on $\mathbb{R}$ having $n$-dimensional $p$-elliptically contoured version for every $n \in \mathbb{N}$. But we know very little about $p$-elliptically contoured measures on $\mathbb{R}^n$ if $p \notin \{1, 2\}$ and $n \in \mathbb{N}$ is fixed.

Now let $c: \mathbb{R}^n \to [0, \infty)$. We define $M(c, n)$ as the set of all probability measures $\mu$ on $\mathbb{R}$ having an $n$-dimensional version $\nu$ on $\mathbb{R}^n$ with a given function $c$, i.e., such that

$$\hat{\nu}(ta) = \hat{\mu}(tc(a)), \quad a \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

It is easy to see that the set $M(c, n)$ is convex, weakly sequentially closed and closed with respect to convolution. The following theorem asserts that every $n$-dimensional version of a measure having $p$-th order, $p \in (0, 2]$, is $p$-elliptically contoured.
THEOREM 1. Assume that there exists \( \varepsilon > 0 \) and \( \mu_0 \in \mathcal{M}(c, n) \), \( \mu_0 \neq \delta_0 \), such that \( \int |x|^p \mu_0(dx) < \infty \). Then there exists \( p \in (0, 2] \) such that \( c(a) \) can be given by the formula (*) for some finite measure \( \lambda \) on \( S^{n-1} \). Moreover, if \( p_0 \) is the greatest such \( p \), then \( \gamma_q \in \mathcal{M}(c, n) \) for every \( q \leq p_0 \).

Proof. Without loss of generality we can assume that \( \varepsilon \leq 2 \) and that \( \int |x|^p \mu_0(dx) = 1 \). If the measure \( \nu \) on \( \mathbb{R}^n \) is the \( n \)-dimensional version of the measure \( \mu_0 \), then for every \( a \in \mathbb{R}^n \) we have

\[
 c(a)^p = \int |c(a)x|^p \mu_0(dx) = \int \cdots \int |\langle a, x\rangle|^p \nu(dx).
\]

Now, in the usual way (see, e.g., [6]) we construct an infinitely divisible probability measure \( \exp \{ m \} \) on \( \mathbb{R}^n \) as the weak limit of measures \( \exp \{ m_\delta \} \) when \( \delta \to 0 \), where

\[
 m_\delta(A) = \int_\delta^\infty \nu(A/s)s^{-\varepsilon-1}ds, \quad A \in \mathcal{B}(\mathbb{R}^n).
\]

We obtain

\[
 [\exp \{ m \}]^*(ta) = \exp \left\{ -\int \cdots \int_0^\infty (1 - \cos \langle ta, sx\rangle)s^{-\varepsilon-1}ds\nu(dx) \right\}
 = \exp \left\{ -\int \cdots \int_{\mathbb{R}^n} |t\langle a, x\rangle|^p \nu(dx) \right\}
 = \exp \left\{ -|t|^p c(a)^p \right\}.
\]

We see then that \( \exp \{-c(a)^p\} \) is a positive definite function on \( \mathbb{R}^n \) (as the characteristic function of the measure \( \exp \{ m \} \)), so the function \( c(a)^p \) is negative definite on \( \mathbb{R}^n \). We define

\[
 p = \sup \{ \varepsilon \in (0, 2] : c(a)^p \text{ is negative definite on } \mathbb{R}^n \}.
\]

As the limit of negative definite functions is also negative definite, it follows that \( c(a)^p = \lim c(a)^p \varepsilon \to p \) is negative definite on \( \mathbb{R}^n \), and \( \exp \{ -c(a)^p \} \) is the characteristic function of some probability measure \( \nu_p \) on \( \mathbb{R}^n \).

Observe that all one-dimensional projections of \( \nu_p \) are symmetric, \( p \)-stable and belong to \( \mathcal{M}(c, n) \). Hence (see [8] and [9]) \( \nu_p \) is \( p \)-stable, so there exists a finite measure \( \lambda \) on \( S^{n-1} \) such that

\[
 c(a)^p = \int \cdots \int_{S^{n-1}} |\langle a, x\rangle|^p \lambda(dx), \quad a \in \mathbb{R}^n.
\]

Now \( \gamma_p \in \mathcal{M}(c, n) \). To see that \( \gamma_q \in \mathcal{M}(c, n) \) for every \( 0 < q \leq p \) notice that the following measure is an \( n \)-dimensional version of the measure \( \gamma_q \) with the same
function $c$ as for $v_p$:

$$v_p \circ \gamma^+_{q/p}(A) := \int v_p(A s^{-1/p}) \mu_{q/p}(ds), \quad A \in \mathcal{B}(\mathbb{R}^n),$$

where $\gamma^+_{q/p}$ is the $(q/p)$-stable measure on $(0, \infty)$ with the Laplace transform $\exp\{-t^{q/p}\}$. Indeed, we have

$$v_p \circ \mu_{q/p} (a) = \int v_p(s^{1/p}a) \gamma^+_{q/p}(ds)$$

$$= \int \exp\{-sc(a)^p\} \gamma^+_{q/p}(ds) = \exp\{-c(a)^q\}. \quad \blacksquare$$

The maximal $p$ we have found in Theorem 1 is a characterizing constant of the set $M(c, n)$ or of the function $c$ on $\mathbb{R}^n$. Therefore, let us define

$$p(c) = \sup\{p \in (0, 2): \exists \mu \in M(c, n), \mu \neq \delta_0, \|x\|^p \mu(dx) < \infty\}$$

or, equivalently,

$$p(c) = \sup\{p \in (0, 2): c(a)^p \text{ is negative definite on } \mathbb{R}^n\},$$

where $\sup \emptyset = 0$. Now, if $p(c) > 0$, then every $n$-dimensional version of any measure from $M(c, n)$ has to be $p(c)$-elliptically contoured. So only in the case $p(c) = 0$ maybe we would be able to find $c$ which is not any $L^p$-norm for any $p \in (0, 2]$. In 1985 Kuritsyn and Schestiakov [7] showed that the function $\exp\{-|x|^p + |y|^{p/2}\}$ is a characteristic function for every $p > 2$. They expressed in this way the fact that every two-dimensional normed space embeds isometrically into some $L^1$-space or, equivalently, that every norm on $\mathbb{R}^2$ is negative definite. The two-dimensional measures obtained in [7] are special cases of 1-elliptically contoured measures. So the problem whether or not there exists an $n$-dimensional version of a symmetric measure on $\mathbb{R}$ other than $p$-elliptically contoured, $p \in (0, 2]$, remains open.

The following result gives us some more information about measures having an $n$-dimensional version.

**Theorem 2.** Let $\mu \in M(c, n)$, $\mu \neq \delta_0$, $n \geq 2$, and let $v$ be an $n$-dimensional version of $\mu$. Then either $\text{supp}(\mu)$ is a compact set (and then $v$ is $2$-elliptically contoured) or $\text{supp}(\mu) = \mathbb{R}$.

**Proof.** It is easy to see that if $\mu \in M(c, n)$, $n \geq 2$, then $\mu \in M(c', 2)$, where $c'(a) = c((a_1, a_2, 0, \ldots, 0))$ for $a = (a_1, a_2) \in \mathbb{R}^2$. Assume then, without loss of generality, that $\mu \in M(c, 2)$, $c(1, 0) = 1$, and $v$ is a two-dimensional version of $\mu$. Since

$$\int \exp\{i <ta, x>\} v(dx) = \int \exp\{ic(ta)x\} \mu(dx) = \hat{\mu}(te(a)),$$
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for every \( t, s \in \mathbb{R} \), \( t < s \), we have

\[
\nu \left\{ t < \frac{\langle a, x \rangle}{c(a)} < s \right\} = \mu \{ t < x < s \}.
\]

Suppose now that \( \text{supp}(\mu) \neq \mathbb{R} \); then there exist \( t, s \in \mathbb{R} \), \( t < s \), such that \( \mu \{ t < x < s \} = 0 \) (by symmetry of \( \mu \) we can assume that \( t > 0 \)). The sets

\[
A(a) = \left\{ t < \frac{\langle a, x \rangle}{c(a)} < s \right\}, \quad a \in \mathbb{R}^2,
\]

are open cylinders in \( \mathbb{R}^2 \) and it is easy to see that

\[
\{ x \in \mathbb{R}^2 : \|x\| > Mt \} \subseteq \bigcup_a A(a),
\]

where \( M = \sup \{ c(a) : \|a\| = 1, a \in \mathbb{R}^2 \} \) and \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^2 \).

Now let \( K \subseteq \{ x \in \mathbb{R}^2 : \|x\| > Mt \} \) be a compact set. There exists a finite set \( a_1, \ldots, a_k \in \mathbb{R}^2 \) such that \( K \subseteq \bigcup A(a_i) \) and we obtain

\[
\nu(K) \leq \sum_v \nu(A(a_i)) = 0.
\]

This means that \( \mu \) as well as \( \nu \) have compact supports, so they in particular have the second moment, and then

\[
\int \int |\langle a, x \rangle|^2 \nu(dx) = \int |c(a)x|^2 \mu(dx) = c(a)^2 \int |x|^2 \mu(dx) < \infty.
\]

Consequently, the function \( c(a) \) is given by an \( L^2 \)-norm on \( \mathbb{R}^2 \), i.e., the measure \( \nu \) is 2-elliptically contoured. \( \blacksquare \)

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