ERGODICITY AND WEAK MIXING OF SEMISTABLE PROCESSES

PIOTR KOKOSZKA AND KRZYSZTOF PODGÓRSKI (WROCLAW)

Abstract. We prove that ergodicity and weak mixing coincide for the class of stationary symmetric semistable processes and give several characterizations of ergodic semistable processes.

1. Preliminaries. Before we state and prove our results concerning ergodicity and weak mixing of semistable processes we recall basic definitions and properties of semistable laws which will be exploited in this note.

Let us first recall (cf. [6]) that a real random variable $X$ has a symmetric $r$-semistable distribution with index $\alpha$, where $r \in (0, 1)$ and $\alpha \in (0, 2)$, if its characteristic function takes the form

$$\phi_\alpha(t) = \exp\left(-\int_\mathcal{A}|ts|^\alpha k_\alpha(ts) \Gamma(ds)\right),$$

where $\mathcal{A} = \{x \in \mathbb{R}: r^{1/\alpha} < |x| \leq 1\}$, $\Gamma$ is a symmetric measure defined on the $\sigma$-field $\mathcal{B}_\mathcal{A}$ of Borel subsets of $\mathcal{A}$ such that $\int_\mathcal{A} |s|^\alpha \Gamma(ds) < \infty$ and a function $k_\alpha$ is defined for $t \in \mathbb{R}$ by

$$k_\alpha(t) = |t|^\alpha \sum_{n=-\infty}^{\infty} r^{-n}(1 - \cos(r^n|t|)).$$

The function $k_\alpha$ satisfies the following inequalities:

\begin{equation}
0 < c_0 = \inf_{t \in \mathbb{R}\setminus\{0\}} k_\alpha(t) \leq \sup_{t \in \mathbb{R}\setminus\{0\}} k_\alpha(t) = c_1 < \infty,
\end{equation}

and for each $t \in \mathbb{R}$

\begin{equation}
k_\alpha(r^{-1/\alpha}t) = k_\alpha(t).
\end{equation}

Note that in this paper we restrict our attention to symmetric semistable measures, so the abbreviation $r$-SS($\alpha$) will always mean symmetric $r$-semistable with index $\alpha$.

A stochastic process $X = (X_t)_{t \in \mathbb{R}}$ is called $r$-SS($\alpha$) if all finite linear combinations of the form $\sum_{k=1}^n a_k X_{t_k}$ are $r$-SS($\alpha$) random variables or, equivalently, all finite dimensional distributions of $X$ are $r$-SS($\alpha$).
In the sequel, we will exploit the spectral representation of $r$-SS($\alpha$) processes and we need to introduce the notion of an integral with respect to an $r$-SS($\alpha$) random measure. A symmetric random measure $M$ on $\mathcal{B}_d$ is called $r$-SS($\alpha$) if there exists a measure $m$ on $\mathcal{B}_d$ such that $\int_A |s|^\alpha m(ds) < \infty$ and for $A \in \mathcal{B}_d$

$$\phi_{M(A)}(t) = \exp\left(-\int_A |ts|^\alpha k_\alpha(ts)m(ds)\right).$$

A measure $m$ is called the control measure of $M$. Let $m_\alpha$ be a measure on $\mathcal{B}_d$ such that for $s \in A$

$$\frac{dm_\alpha}{dm}(s) = |s|^\alpha.$$

If $f \in L_\alpha(A, m_\alpha)$, then $f$ is integrable with respect to $M$ and

$$\phi_{f|dM}(t) = \exp\left(-\int_A |s f(s)|^\alpha k_\alpha(s f(s))m(ds)\right).$$

Let $X$ be a symmetric stochastically continuous $r$-SS($\alpha$) process. It follows from the spectral representation theorem for $r$-SS($\alpha$) processes (see [6]) that there exists an $r$-SS($\alpha$) random measure $M$ on $\mathcal{B}_d$ with the control measure $m$ and a family $(f_t)_{t \in \mathbb{R}}$ of elements of $L_\alpha(A, m_\alpha)$ such that

(1.3)

$$(X_t)_{t \in \mathbb{R}} = (\int f_t dM)_{t \in \mathbb{R}}$$

and, consequently, for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $f = \sum_{k=1}^n a_k f_{t_k}$

$$E \exp\left(i \sum_{k=1}^n a_k X_{t_k}\right) = \exp\left(-\int_A |s f(s)|^\alpha k_\alpha(s f(s))m(ds)\right).$$

Let $X$ be additionally a stationary process. Stationarity means that for any choice of $t_1, \ldots, t_n \in \mathbb{R}$ the distribution of $(X_{t_1+1}, \ldots, X_{t_n+1})$ does not depend on $t$. Let $\{f_t dM\}_{t \in \mathbb{R}}$ be the spectral representation of $X$. For $t \in \mathbb{R}$ we define $U_t$ by

$$U_t(\sum_{k=1}^n \vartheta_k f_{t_k}) = \sum_{k=1}^n \vartheta_k f_{t_k + t}.$$ 

Note that $U_t$ extends to a topological automorphism of $\text{lin} \{f_t, t \in \mathbb{R}\}$, where the closure is taken in the norm of the space $L^2(A, m_\alpha)$. Indeed, stationarity implies that for each $f \in \text{lin} \{f_t, t \in \mathbb{R}\}$

$$\int_A |s U_t f(s)|^\alpha k_\alpha(s U_t f(s))m(ds) = \int_A |s f(s)|^\alpha k_\alpha(s f(s))m(ds)$$

and inequalities (1.1) yield

$$c_0 c_1^{-1} \int_A |f|^\alpha dm_\alpha \leq \int_A |U_t f|^\alpha dm_\alpha \leq c_0 c_1 \int_A |f|^\alpha dm_\alpha.$$
Thus \((U_t)_{t \in \mathbb{R}}\) is a group of topological automorphisms of \(\text{lin} \{f_t, t \in \mathbb{R}\}\). By (1.3) the group \((U_t)_{t \in \mathbb{R}}\) corresponds to the group \((T_t)_{t \in \mathbb{R}}\) of shift operators on \(\text{lin} \{X_t, t \in \mathbb{R}\}\).

For more information on semistable measures and processes see [2], [6], [7] and references therein.

2. Ergodicity weak mixing and mixing of \(r\)-SS(\(\alpha\)) processes. The following lemma will play a decisive role in this section. It will allow us to characterize ergodic \(r\)-SS(\(\alpha\)) processes in terms of their spectral representations and to prove that ergodic \(r\)-SS(\(\alpha\)) processes are also weak mixing.

**Lemma 2.1.** For a bounded Borel function \(\phi: \mathbb{R} \to \mathbb{R}\) the following conditions are equivalent:

(i) for two different positive numbers \(c_1, c_2\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T \exp(c_1 \phi(t)) dt = 1.
\]

and

\[
\lim_{T \to \infty} T^{-1} \int_0^T \exp(c_2 \phi(t)) dt = 1;
\]

(ii) for each \(\varepsilon > 0\)

\[
\lim_{T \to \infty} T^{-1} \text{Leb}\{\{t \in [0, T]: |\phi(t)| > \varepsilon\}\} = 0;
\]

(iii) for each positive number \(c\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T |e^{c \phi(t)} - 1| dt = 0;
\]

(iv) \(\lim_{T \to \infty} T^{-1} \int_0^T |\phi(t)| dt = 0\).

**Proof.** The equivalence of (i), (ii) and (iv) is the contents of Lemma 2 of [5]. Clearly, (iii) implies (i), and (iii) follows from (ii) by the inequality

\[
\int_0^T |e^{c \phi(t)} - 1| dt \leq (e^c \|\phi\|_{\infty} + 1) \lambda(\{t \in [0, T]: |\phi(t)| > \varepsilon\})
\]

\[
+ (e^{ce} - 1) \lambda(\{t \in [0, T]: |\phi(t)| \leq \varepsilon\}),
\]

which holds for each \(c, \varepsilon > 0\).

From now on, let \(X\) be a measurable and stationary stochastic process. Let us recall here (cf. [3], [8]) that the process \(X\) is ergodic if and only if for each \(Y, Z \in \text{lin}_C\{X_t, t \in \mathbb{R}\}\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T E((T_t Y) Z) dt = EYEZ;
\]
X is weak mixing if and only if for each $Y, Z \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} T^{-1} \int_0^T |E((T_s Y)Z) - EYEZ| dt = 0; \]
and $X$ is mixing if and only if for each $Y, Z \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} E((T_s Y)Z) dt = EYEZ. \]

It is well known (see [9]) that ergodicity can be characterized by the condition saying that for each $Y \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} T^{-1} \int_0^T E((T_s Y) Y) dt = |EY|^2. \]

Standard approximation arguments (for details see [5]) imply that the process $X$ is ergodic if and only if for each $Y \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} T^{-1} \int_0^T E \exp[i(T_s Y - Y)] dt = |E \exp(iY)|^2. \] (2.1)

Similarly, the process $X$ is weak mixing if and only if for each $Y \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} T^{-1} \int_0^T |E \exp[i(T_s Y - Y)] - |E \exp(iY)|^2| dt = 0, \]
and $X$ is mixing if and only if for each $Y \in \text{lin}_c \{X_t: t \in \mathbb{R}\}$
\[ \lim_{T \to \infty} E \exp[i(T_s Y - Y)] = |E \exp(iY)|^2. \]

Now, let $X$ be a stationary and stochastically continuous $r$-SS($\alpha$) process with spectral representation $(\bigcup f_0 dM)_{\alpha \in \mathbb{R}}$.

**Theorem 2.1.** Let $X$ be a stationary and stochastically continuous $r$-SS($\alpha$) process. Then the following conditions are equivalent:

1. $X$ is ergodic;
2. for each $f \in \text{lin}_c \{U_t f_0: t \in \mathbb{R}\}$
   \[ \lim_{T \to \infty} T^{-1} \int_0^T e^{\phi(f, t)} dt = 1; \]
3. for each $\varepsilon > 0$ and each $f \in \text{lin}_c \{U_t f_0: t \in \mathbb{R}\}$
   \[ \lim_{T \to \infty} T^{-1} \text{Leb}\{t \in [0, T]: |\phi(f, t)| > \varepsilon\} = 0; \]
4. for each $c > 0$ and each $f \in \text{lin}_c \{U_t f_0: t \in \mathbb{R}\}$
   \[ \lim_{T \to \infty} T^{-1} \int_0^T |e^{c\phi(f, t)} - 1| dt = 0; \]
(iv) for each \( f \in \text{lin}\{U_t f_0: \ t \in \mathbb{R}\}\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T |\phi(f, t)|dt = 0,
\]

where

\[
\phi(f, t) = \int a |sf(s)|^\alpha k_\alpha(sf(s))m(ds) - |s(f - U_t f)(s)|^\alpha k_\alpha(s(f - U_t f)(s))m(ds).
\]

**Proof.** Direct verifications show that \( X \) is ergodic if and only if for each \( f \in \text{lin}\{U_t f_0: \ t \in \mathbb{R}\}\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T e^{\phi(f, t)}dt = 1.
\]

By (1.2) and the above condition, \( X \) is ergodic if and only if for each \( f \in \text{lin}\{U_t f_0: \ t \in \mathbb{R}\}\)

\[
\lim_{T \to \infty} T^{-1} \int_0^T \exp[r^{-1} \phi(f, t)]dt = 1.
\]

Thus we have condition (i) of Lemma 2.1 with \( c_1 = 1 \) and \( c_2 = r^{-1} \), and the equivalence of all conditions follows immediately from this lemma.

**Remark 2.1.** In [1] a characterization of ergodic \( \alpha \)-SS (short for symmetric \( \alpha \)-stable) processes in terms of their spectral representation was given. By means of Lemma 2.1 this characterization has been extended in [5] to yield conditions similar to (i)-(iv) of Theorem 2.1.

While, in general, weak mixing is a stronger property than ergodicity, they coincide for Gaussian processes (cf. [3]). We show here that, in fact, they coincide for all \( r \)-SS(\( \alpha \)) processes.

**Corollary 2.1.** Let \( X \) be a stationary and stochastically continuous \( r \)-SS(\( \alpha \)) process. If \( X \) is ergodic, then it is weak mixing.

**Proof.** Suppose that \( X \) is ergodic and let \((\int U_t f_0 dM)_{t \in \mathbb{R}}\) be a spectral representation of \( X \). Then

\[
\exp[i(T, Y - Y)] - |\exp(iY)|^2 = \exp[-\int a |sf(s)|^\alpha k_\alpha(sf(s))m(ds)][e^{\phi(f, t)} - 1],
\]

where \( f \in \text{lin}\{U_t f_0: \ t \in \mathbb{R}\} \) and \( Y = \int f dM \). Thus by (2.2) it suffices to prove that for \( f \in \text{lin}\{U_t f_0: \ t \in \mathbb{R}\} \)

\[
\lim_{T \to \infty} T^{-1} \int_0^T |e^{\phi(f, t)} - 1|dt = 0,
\]

but this follows at once from condition (iii) of Theorem 2.1.
Since a random variable is \( S \alpha S \) if and only if it is \( r \)-SS(\( \alpha \)) for each \( r \in (0,1) \), and a measurable \( S \alpha S \) process is stochastically continuous (Theorem 0 of [1]), we immediately get the following result (see also [4]):

**Corollary 2.2.** A measurable stationary \( S \alpha S \) process is ergodic if and only if it is weak mixing.

**Remark 2.2.** As symmetric Gaussian processes are exactly \( S2S \) processes, the proof of Theorem 2.1 provides an elementary proof of the equivalence under discussion for Gaussian processes.

Since for an \( r \)-SS(\( \alpha \)) process \( X \) with the spectral representation \( (\int U_t f_t dM)_{t \in \mathbb{R}} \) we have

\[
\mathbb{E} \exp(i(TY - Y)) = e^{\phi(f, T)}, \quad \text{where } f \in \text{lin}\{U_t f_t: t \in \mathbb{R}\} \text{ and } Y = \int f dM,
\]

we obtain at once the following characterization of the mixing property for these processes.

**Corollary 2.3.** A stationary and stochastically continuous \( r \)-SS(\( \alpha \)) process \( X \) is mixing if and only if for each \( f \in \text{lin}\{U_t f_t: t \in \mathbb{R}\} \)

\[
\lim_{T \to \infty} \phi(f, T) = 0.
\]

**REFERENCES**


Institute of Mathematics
Technical University of Wroclaw
Wybrzeże Wyspiańskiego 27
50-370 Wroclaw, Poland

Received on 27.12.1990; revised version on 16.6.1991