A PROBABILISTIC PROPERTY OF THE SPACE $l_2^m$

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Abstract. It is shown that for every sequence $(x_k)$ of elements of $l_2^m$ the following two properties are equivalent:

(a) $(x_k/k) \in l_2(l_2)$.

(b) $(||S_n/n||, \mathcal{F}_n)$ is a quasimartingale, where $S_n = \sum_{1 \leq k \leq n} e_k x_k$,

$(e_k)$ being a sequence of independent Rademacher r.v. and $\mathcal{F}_n$ denoting the $\sigma$-field generated by $(e_1, \ldots, e_n)$.

In a recent paper [3], we studied a new geometrical property of Banach spaces — property related with Kolmogorov's (non i.i.d.) strong law of large numbers. The purpose of the present note is to show that the Euclidean space $l_2^m$ has this geometrical property.

In the sequel, $(B, ||||)$ will be a real separable Banach space. We will denote by $(e_k)$ a sequence of independent Rademacher r.v.; for every $n$, $\mathcal{F}_n$ will be the $\sigma$-field generated by $(e_1, \ldots, e_n)$. With every sequence $x = (x_n)$ of elements of $B$ we will associate the random sums

$S_n = S_n(x) = \sum_{1 \leq k \leq n} e_k x_k$.

The announced geometrical property is defined as follows:

Definition 1. We say that $(B, ||||)$ has the Kolmogorov quasimartingale property (in short, Kqm-property) if and only if, for every sequence $x = (x_n)$ of elements of $B$, the following two properties are equivalent:

(i) $(||S_n(x)/n||^2, \mathcal{F}_n)$ is a quasimartingale.

(ii) $(x_n/k) \in l_2(B)$.

Remark. In this very special case, the fact that $(||S_n/n||^2, \mathcal{F}_n)$ is a quasimartingale simply reduces to

\[ \sum_{n \geq 1} E \left| E \left( \left( \frac{S_{n+1}}{n+1} \right)^2 | \mathcal{F}_n \right) - \left( \frac{S_n}{n} \right)^2 \right| < + \infty \]
Among the properties proved in [3], let us mention that
(a) the real line has the Kqm-property;
(b) if \((B, || ||)\) has the Kqm-property, then \(B\) is isomorphic to a Hilbert space;
(c) an infinite-dimensional Hilbert space does not have the Kqm-property.

In the conclusion of [3], the following question is asked: "Does a finite-dimensional Euclidean space have the Kqm-property?" The purpose of the present note is to answer this question. The result is as follows:

**Theorem 2.** The space \(l_2^m\) has the Kqm-property.

**Proof.** A straightforward computation shows that in the space \(l_2^m\) property (ii) in Definition 1 implies property (i). To check that the converse implication also holds we recall first an exponential lower bound. That inequality is due independently to Ledoux and Talagrand ([4], Lemma 4.9) and Montgomery-Smith [5]. We give the statement under Montgomery-Smith's form:

**Proposition 3.** There exists a constant \(C > 1\) such that for every \(y \in l_2\) and all \(t > 0\) we have

\[
P\left( \sum_{k \geq 1} \delta_k y_k \geq \frac{t}{C} \sqrt{\sum_{k \geq [t^2] + 1} (y_k^*)^2} \right) \geq \frac{1}{C} \exp\left(-Ct^2\right),
\]

where \([\ ]\) denotes the integer part of a real number and \((y_k^*)\) is the non-increasing rearrangement of the sequence \((|y_k|)\).

Let us define \(\mu = [4Cm + 1]\) and \(M = \mu^2\).

Let now \((x_n)\) be a sequence of elements of \(l_2^m\) such that \((\|S_n/n\|^2, \mathcal{F}_n)\) is a quasimartingale. By arguing as in the scalar case (see [3], Lemma 1.6), we obtain easily the following technical lemma:

**Lemma 4.** For every integer \(n\) we denote by \(z_1(n), \ldots, z_n(n)\) the non-increasing rearrangement of the sequence \((\|x_1\|, \ldots, \|x_n\|)\). Then for every \(n \geq M\) we have

\[
\frac{2n+1}{n^2(n+1)^2} \sum_{1 \leq k \leq M} z_k^2(n) \leq u_n,
\]

where \(u_n\) is the general term of a convergent series.

Now, denote by \((x_1^m, \ldots, x_m^m)\) the coordinates of \(x_k\) and consider the following \(m+1\) sets of positive integers:

\[
\forall j = 1, \ldots, m, \ H_j = \left\{ n : \frac{\mu}{C} \left( \sqrt{\frac{2n+1}{n^2(n+1)^2} \sum_{1 \leq k \leq n} (x_k^j)^2} - \sqrt{u_n} \right) \geq \sqrt{2} \frac{\|x_{n+1}\|}{n+1} \right\}
\]
and

$$H_0 = \mathbb{N}^* - \bigcup_{1 \leq j \leq m} H_j.$$  

For all $n$ belonging to $H_j$ we get

(2)  

$$P\left(\frac{\sqrt{2n+1}}{n(n+1)} \sum_{1 \leq k \leq n} \epsilon_k x_k > \sqrt{2} \frac{\|x_{n+1}\|_{n+1}}{n+1}\right)$$

$$\geq P\left(\frac{\sqrt{2n+1}}{n(n+1)} \sum_{1 \leq k \leq n} \epsilon_k x_k > \sqrt{2} \frac{\|x_{n+1}\|_{n+1}}{n+1}\right)$$

$$\geq P\left(\frac{\sqrt{2n+1}}{n(n+1)} \sum_{1 \leq k \leq n} \epsilon_k x_k > \frac{\mu}{C \sqrt{n^2 (n+1)^2 \sum_{k \geq \lfloor \mu^2 \rfloor + 1} (x_k)^2}} \|x_{n+1}\|_{n+1} \right) \geq \frac{1}{C} \exp(-C\mu^2),$$

where in the last step Proposition 3 has been used. From the quasimartingale property (1) we obtain, by the definition of the norm on $l^m_2$,

(3)  

$$\sum_{n \geq 1} E \left| \frac{(2n+1) \|S_n\|^2}{n^2 (n+1)^2} - \frac{\|x_{n+1}\|^2_{n+1}}{(n+1)^2} \right| < +\infty,$$

and so, by the application of (2),

$$\sum_{n \in H_0^*} \frac{\|x_{n+1}\|^2_{n+1}}{(n+1)^2} < +\infty.$$  

Consider that time an integer $n \in H_0$. By the choice of $\mu$ we have

$$\frac{2n+1}{n^2 (n+1)^2} \sum_{1 \leq k \leq n} \|x_k\|^2 = \frac{2n+1}{n^2 (n+1)^2} \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq m} (x_k)^2$$

$$\leq 2m \left( \frac{2C^2 \|x_{n+1}\|^2_{n+1}}{(n+1)^2} + u_n \right) \leq 2mu_n + \frac{\|x_{n+1}\|^2_{n+1}}{4(n+1)^2}.$$  

Denote by $A$ the set of elements of $H_0$ such that

$$u_n \leq \frac{\|x_{n+1}\|^2_{n+1}}{8m(n+1)^2},$$

and by $B$ the complement of $A$ in $H_0$. By Jensen's inequality, it follows easily from (3) that

$$\sum_{n \in A} \frac{\|x_{n+1}\|^2_{n+1}}{(n+1)^2} < +\infty.$$
Remembering finally that $u_n$ is the general term of a convergent series, we get

$$\sum_{n \in B} \frac{\|x_{n+1}\|^2}{(n+1)^2} < +\infty,$$

and this completes the proof of the implication (i) $\Rightarrow$ (ii).

From Theorem 2 and the fact that an infinite-dimensional Hilbert space does not have the Kqm-property we deduce easily the following:

**Corollary 5.** A real separable Banach space $(B, \| \cdot \|)$ which is isometrically isomorphic to a Hilbert space has the Kqm-property if and only if $B$ is finite dimensional.

**References**


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